Orbital Mechanics

- Energy and velocity in orbit
- Elliptical orbit parameters
- Orbital elements
- Coplanar orbital transfers
- Noncoplanar transfers
- Time in orbit
- Interplanetary trajectories
- Planetary launch and entry overview
- Relative orbital motion ("proximity operations"

Newton's Law of Gravitation

- Inverse square law

$$F = \frac{GMm}{r^2}$$

- Since it's easier to remember one number,

$$\mu = GM$$

- If you're looking for local gravitational acceleration,

$$g = \frac{\mu}{r^2}$$

Some Useful Constants

- Gravitation constant $\mu = GM$
  - Earth: 398,604 km^3/sec^2
  - Moon: 4667.9 km^3/sec^2
  - Mars: 42,970 km^3/sec^2
  - Sun: 1.327×10^{11} km^3/sec^2
- Planetary radii
  - r_{Earth} = 6378 km
  - r_{Moon} = 1738 km
  - r_{Mars} = 3393 km

Energy in Orbit

- Kinetic Energy

$$K.E. = \frac{1}{2}mv^2 \Longrightarrow \frac{K.E.}{m} = \frac{v^2}{2}$$

- Potential Energy

$$P.E. = -\frac{\mu m}{r} \Longrightarrow \frac{P.E.}{m} = -\frac{\mu}{r}$$

- Total Energy

$$\text{Constant} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

<--Vis-Viva Equation

Implications of Vis-Viva

- Circular orbit (r=a)

$$v_{circular} = \sqrt{\frac{\mu}{r}}$$

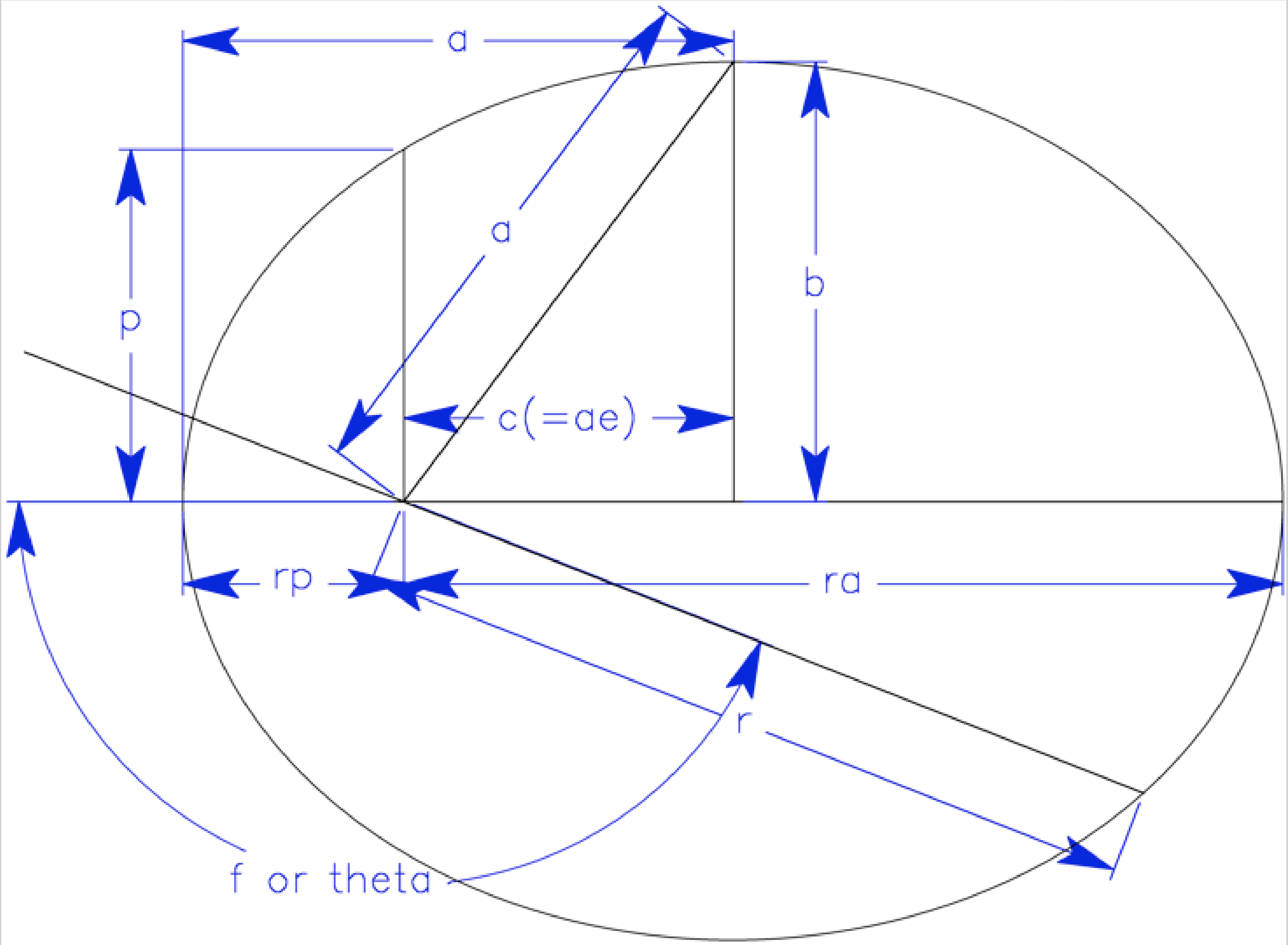
- Parabolic escape orbit (a tends to infinity)

$$v_{escape} = \sqrt{\frac{2\mu}{r}}$$

- Relationship between circular and parabolic orbits

$$v_{escape} = \sqrt{2}v_{circular}$$

Classical Parameters of Elliptical Orbits



Basic Orbital Parameters

- Semi-latus rectum (or parameter)

$$p = a(1 - e^2)$$

- Radial distance as function of orbital position

$$r = \frac{p}{1 + e\cos\theta}$$

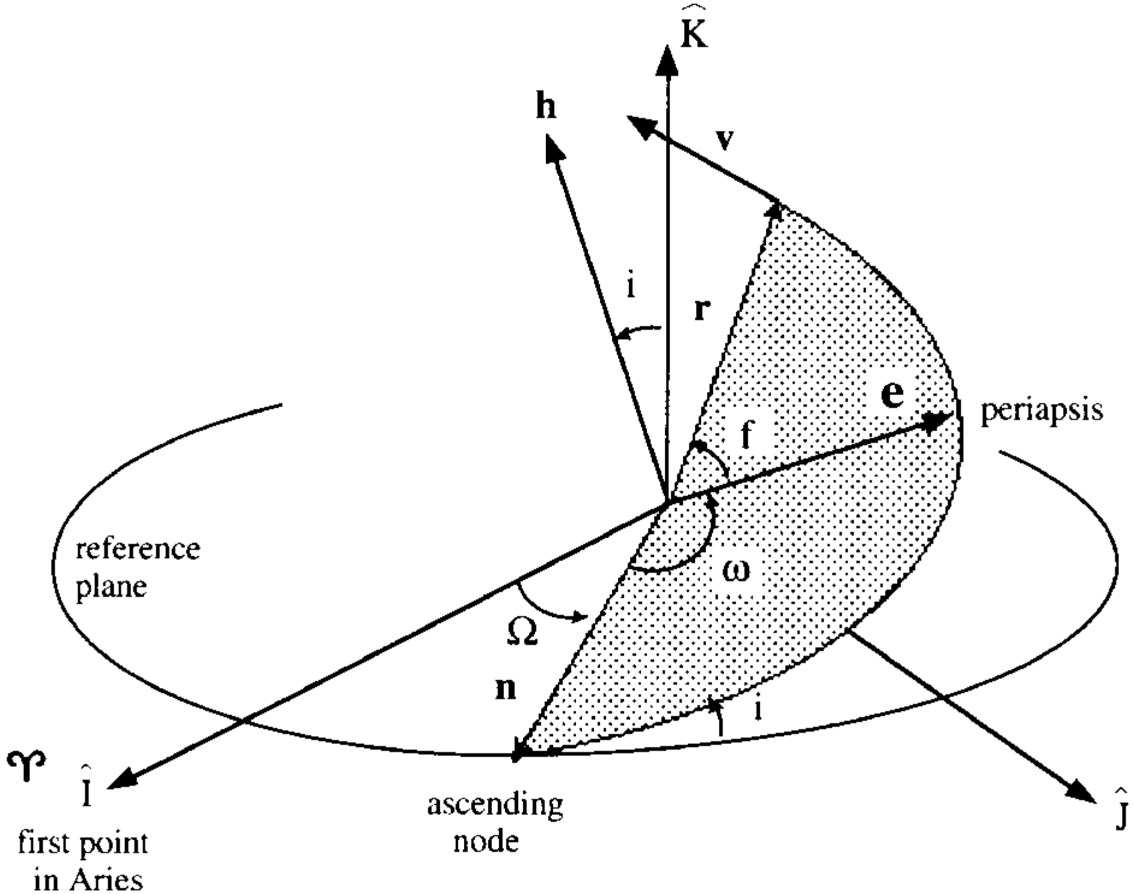
- Periapse and apoapse distances

$$r_p = a(1 - e) \qquad r_a = a(1 + e)$$

- Angular momentum

$$\vec{h} = \vec{r} \times \vec{v} \qquad h = \sqrt{\mu p}$$

The Classical Orbital Elements



Ω : longitude of the ascending node

ω : argument of periapsis

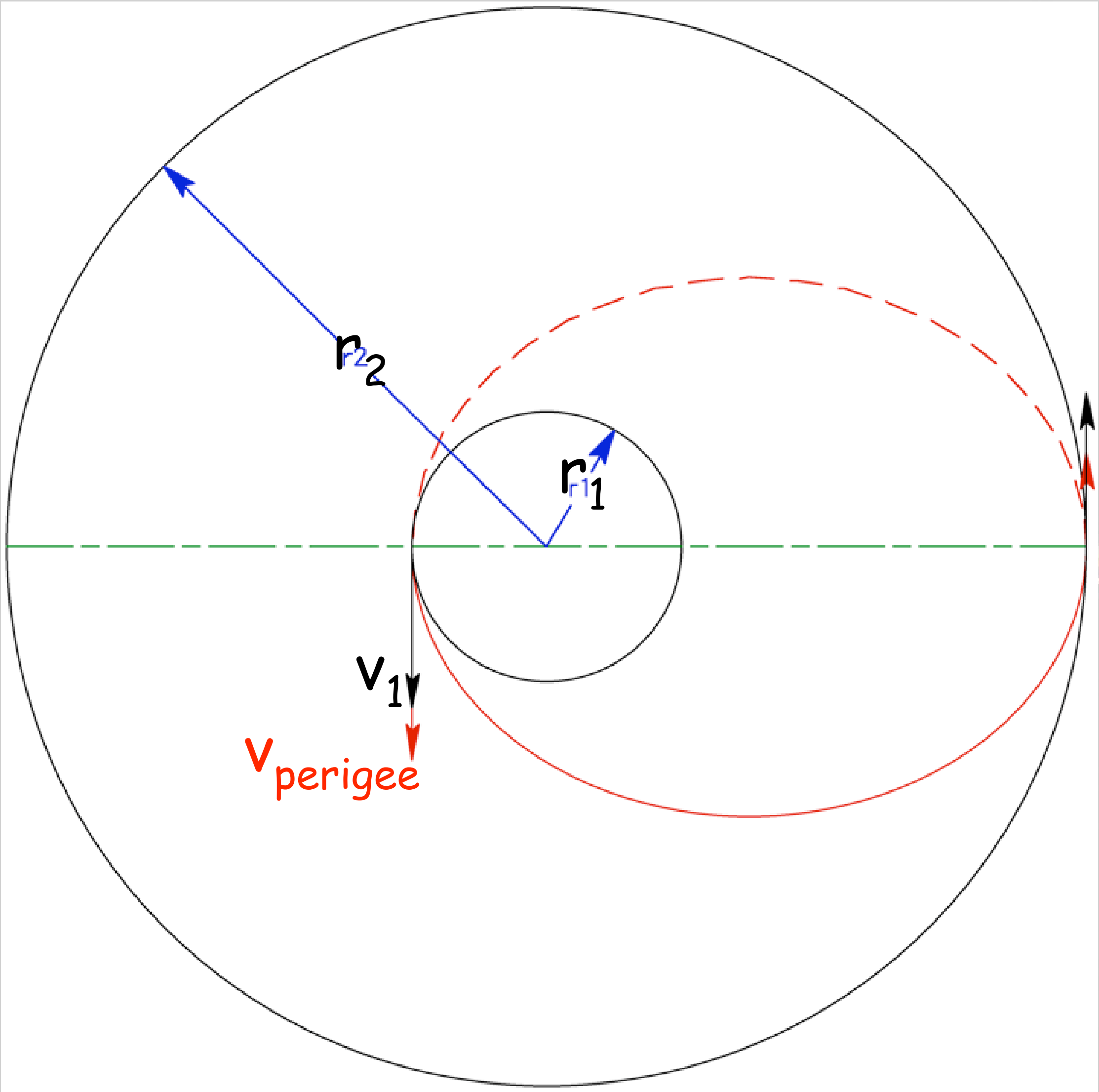
$\tilde{\omega} = \Omega + \omega$: longitude of periapsis

f : true anomaly

$L = \tilde{\omega} + f$: true longitude

Ref: J. E. Prussing and B. A. Conway, Orbital Mechanics Oxford University Press, 1993

The Hohmann Transfer



First Maneuver Velocities

- Initial vehicle velocity $$v_1 = \sqrt{\frac{\mu}{r_1}}$$
- Needed final velocity $$v_{perigee} = \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{2r_2}{r_1+r_2}}$$
- Delta-V $$\Delta v_1 = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1+r_2}}-1\right)$$

Second Maneuver Velocities

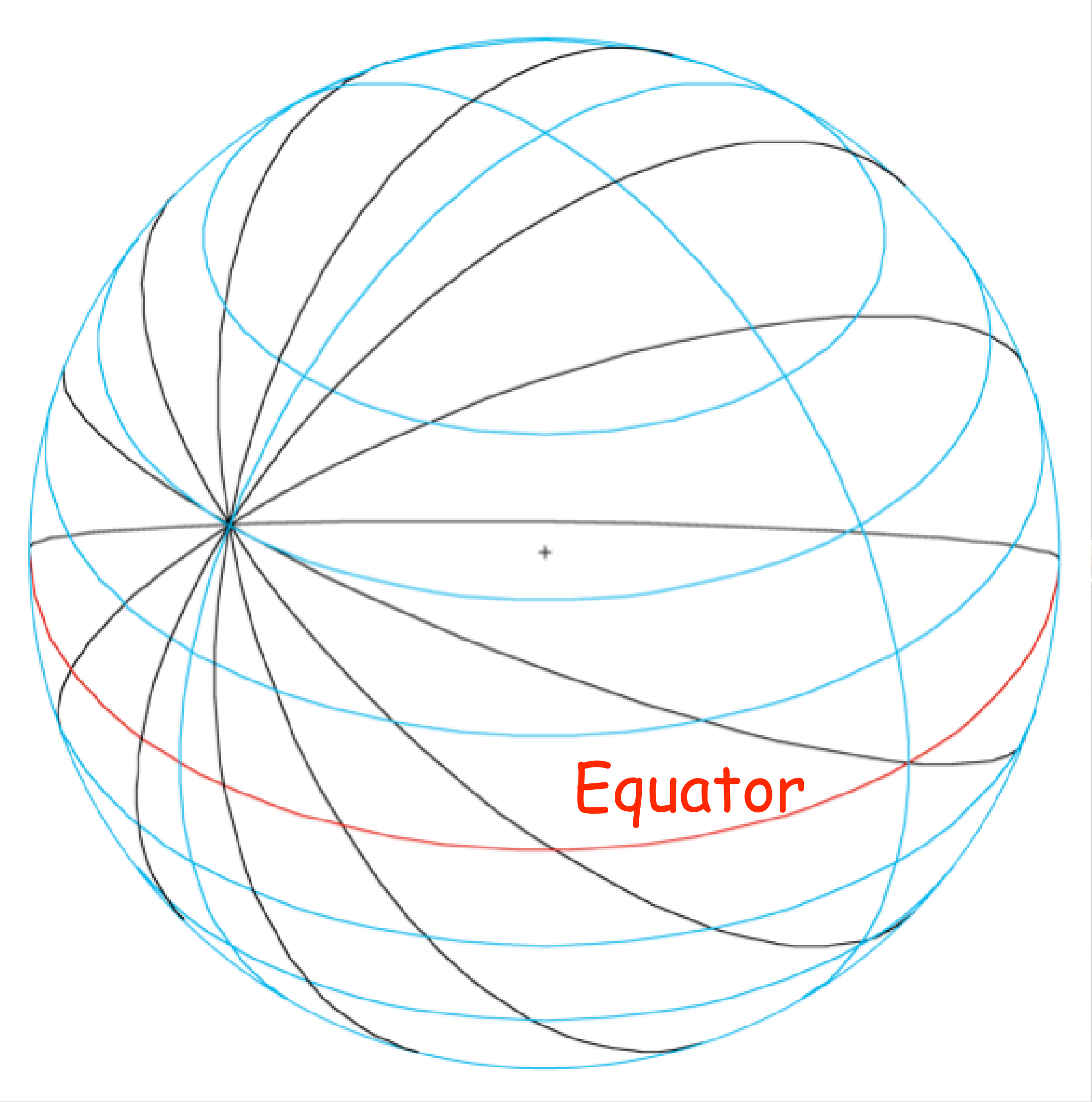
- Initial vehicle velocity $v_{apogee} = \sqrt{\frac{\mu}{r_2}}\sqrt{\frac{2r_1}{r_1 + r_2}}$

- Needed final velocity $v_2 = \sqrt{\frac{\mu}{r_2}}$

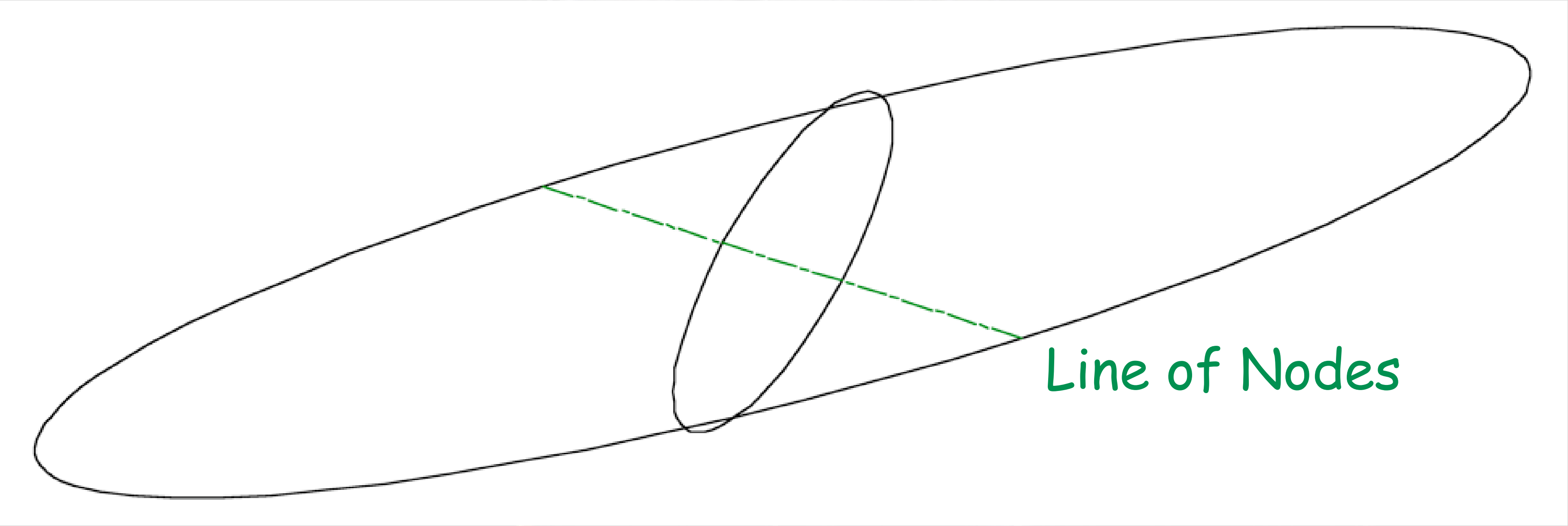
- Delta-V

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right)$$

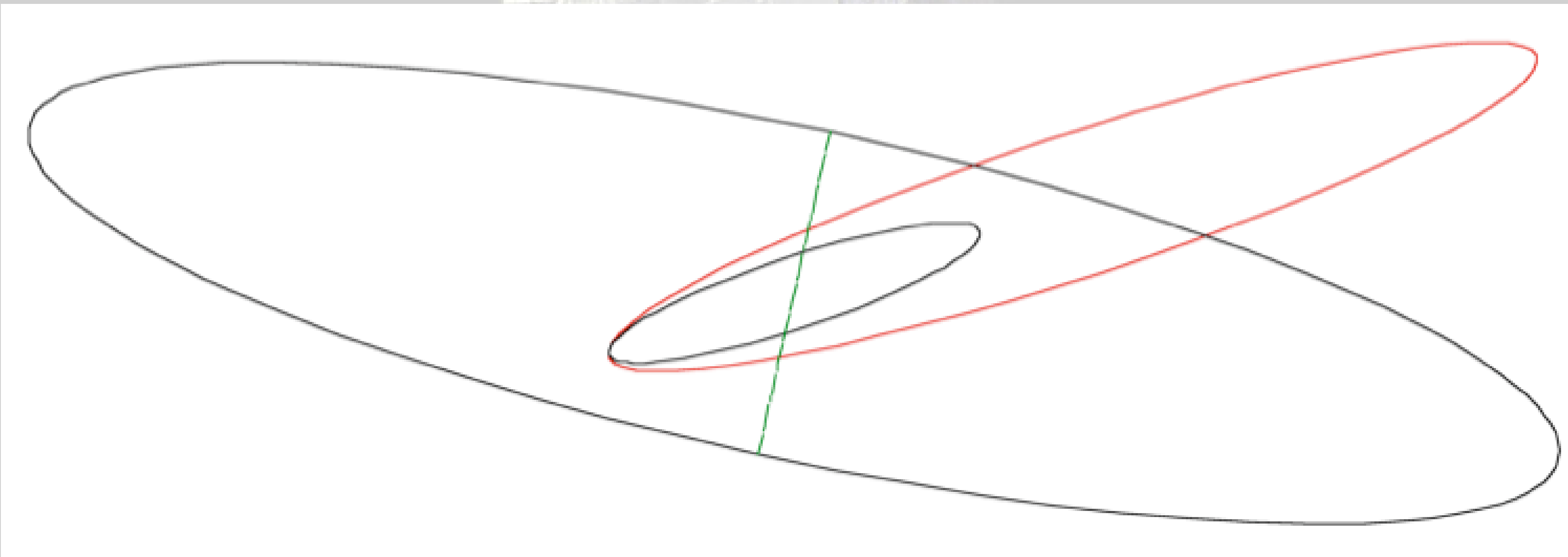
Limitations on Launch Inclinations



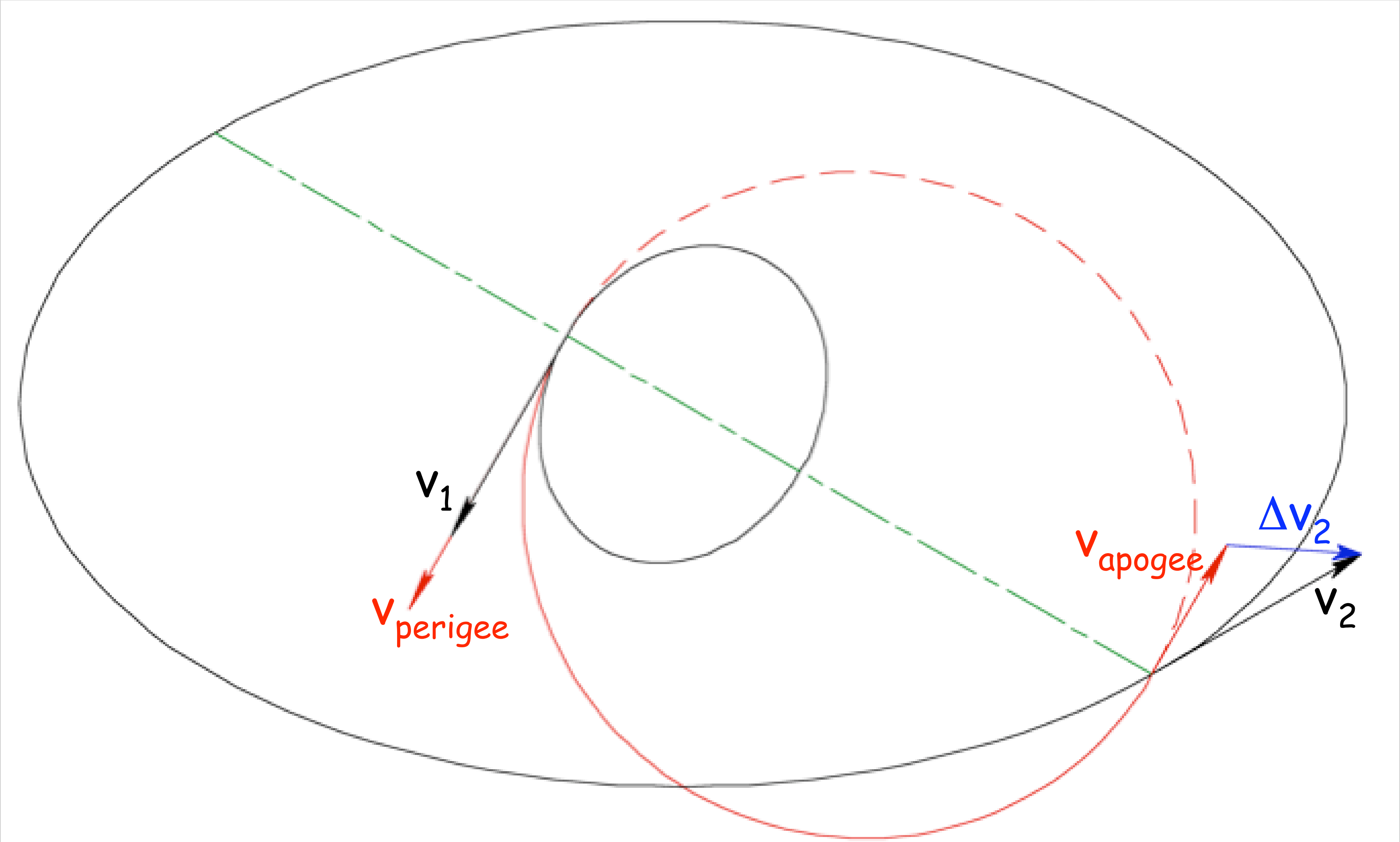
Differences in Inclination



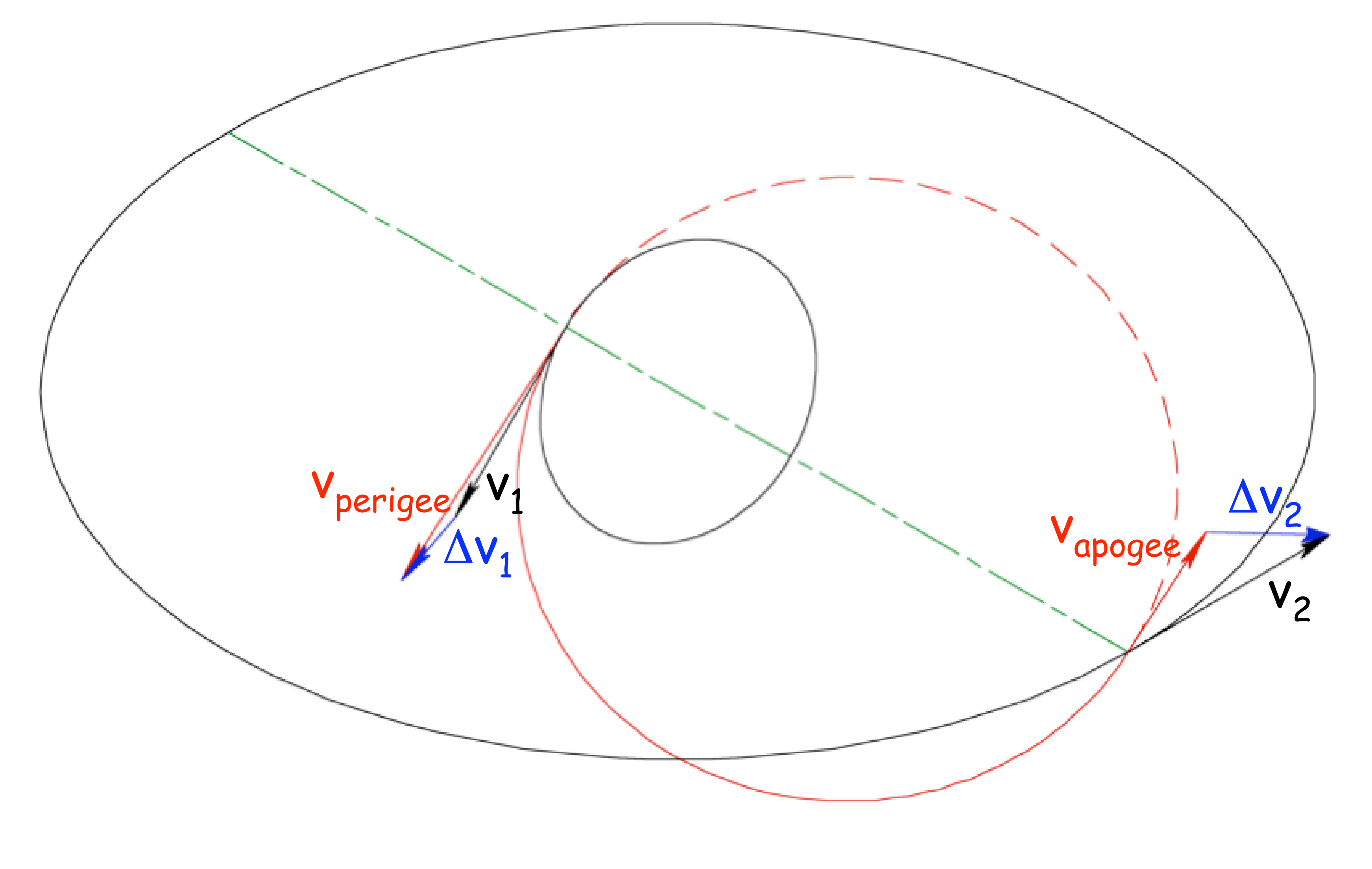
Choosing the Wrong Line of Apsides



Simple Plane Change



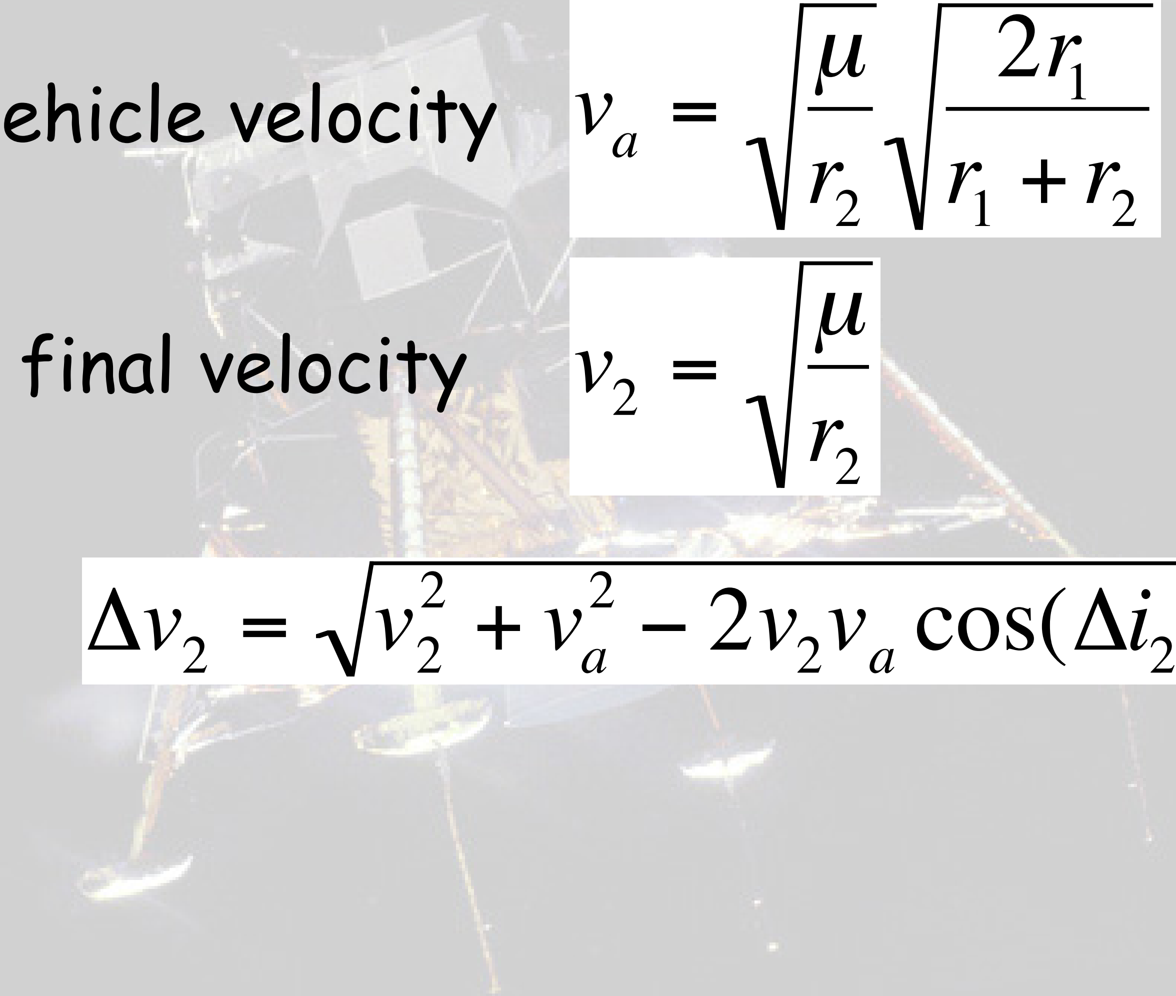
Optimal Plane Change



First Maneuver with Plane Change Δi_1

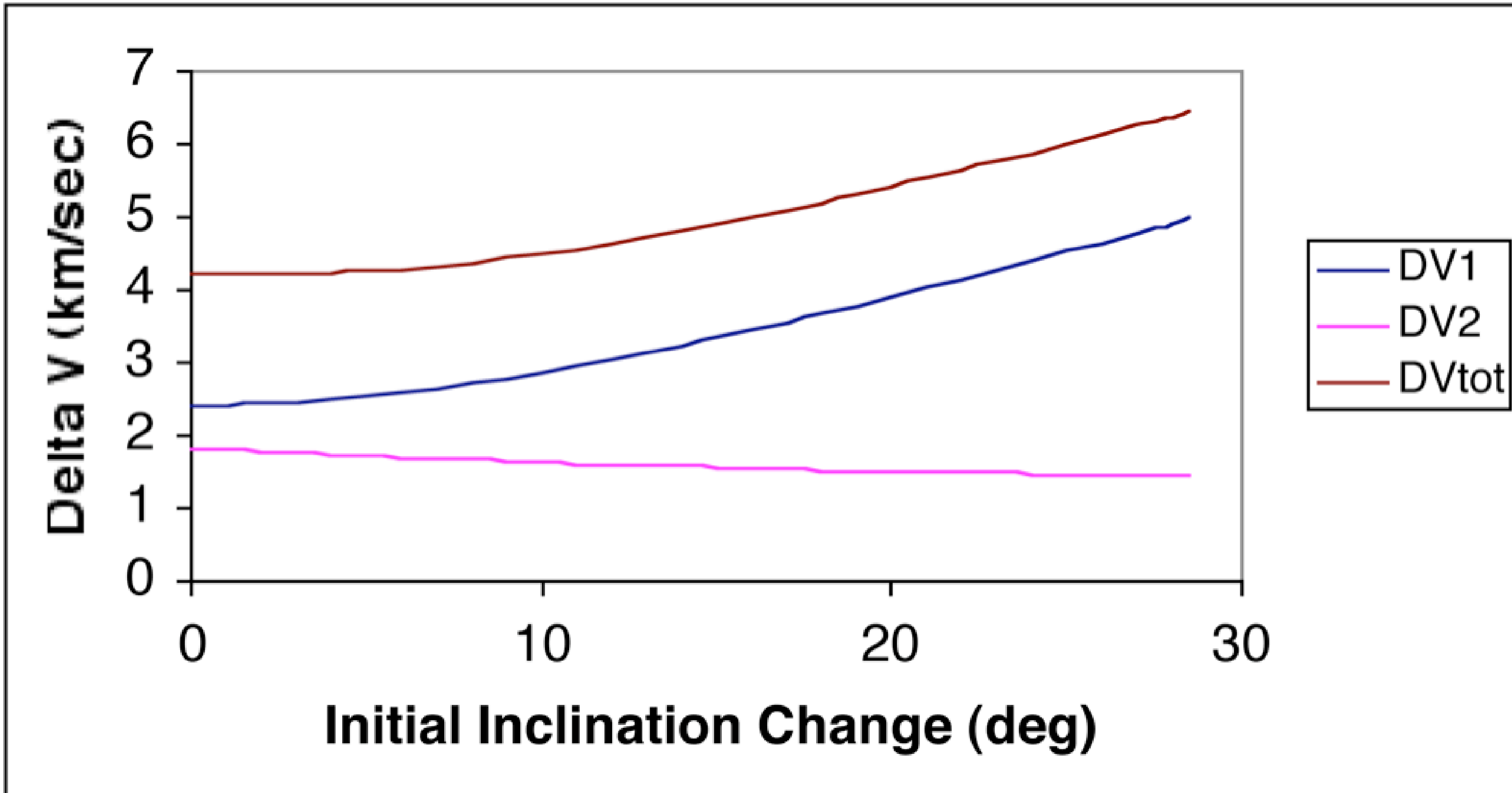
- Initial vehicle velocity $$v_1 = \sqrt{\frac{\mu}{r_1}}$$
- Needed final velocity $$v_p = \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{2r_2}{r_1 + r_2}}$$
- Delta-V $$\Delta v_1 = \sqrt{v_1^2 + v_p^2 - 2v_1 v_p \cos(\Delta i_1)}$$

Second Maneuver with Plane Change Δi_2



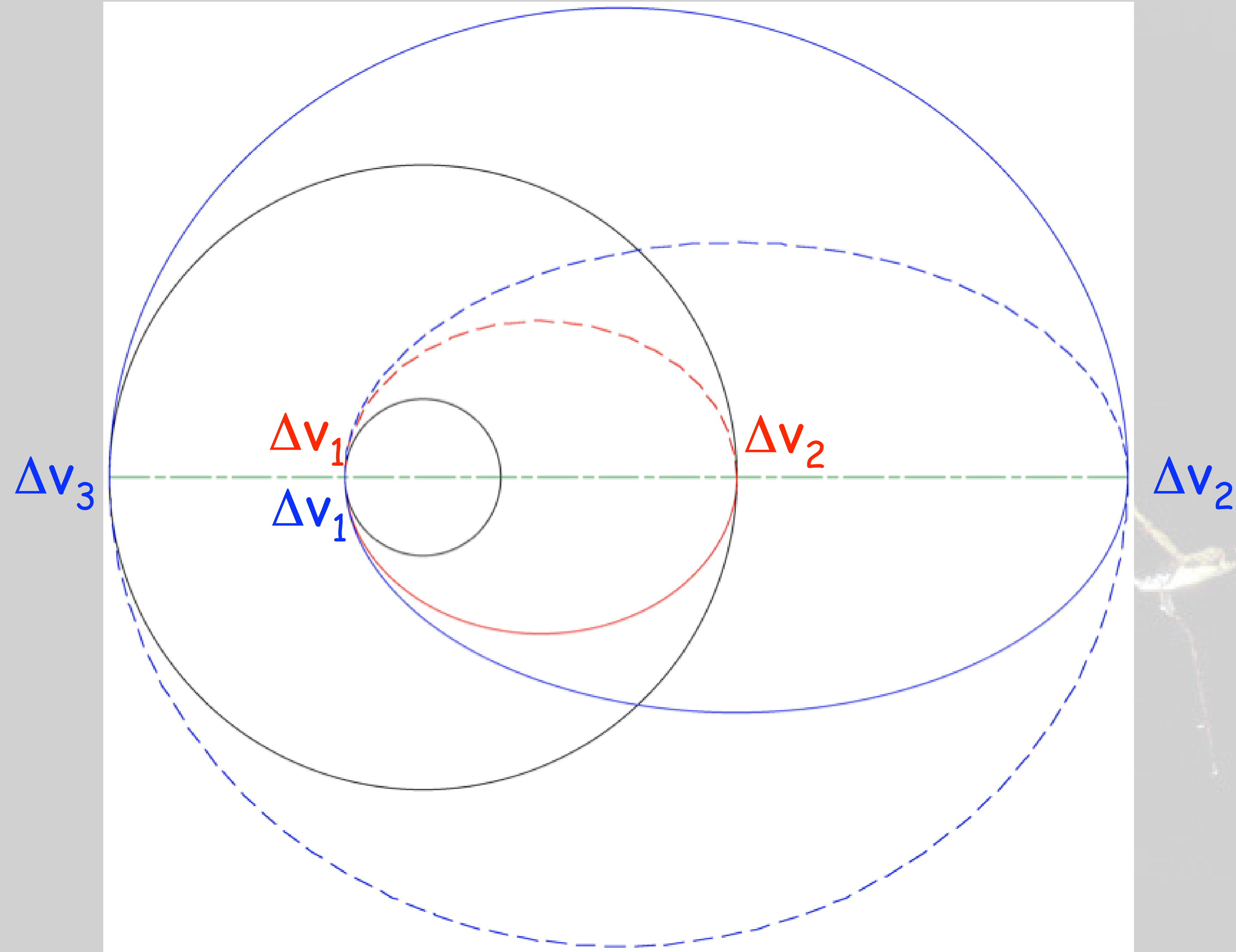
- Initial vehicle velocity $$v_a = \sqrt{\frac{\mu}{r_2}}\sqrt{\frac{2r_1}{r_1 + r_2}}$$
- Needed final velocity $$v_2 = \sqrt{\frac{\mu}{r_2}}$$
- Delta-V $$\Delta v_2 = \sqrt{v_2^2 + v_a^2 - 2v_2 v_a \cos(\Delta i_2)}$$

Sample Plane Change Maneuver

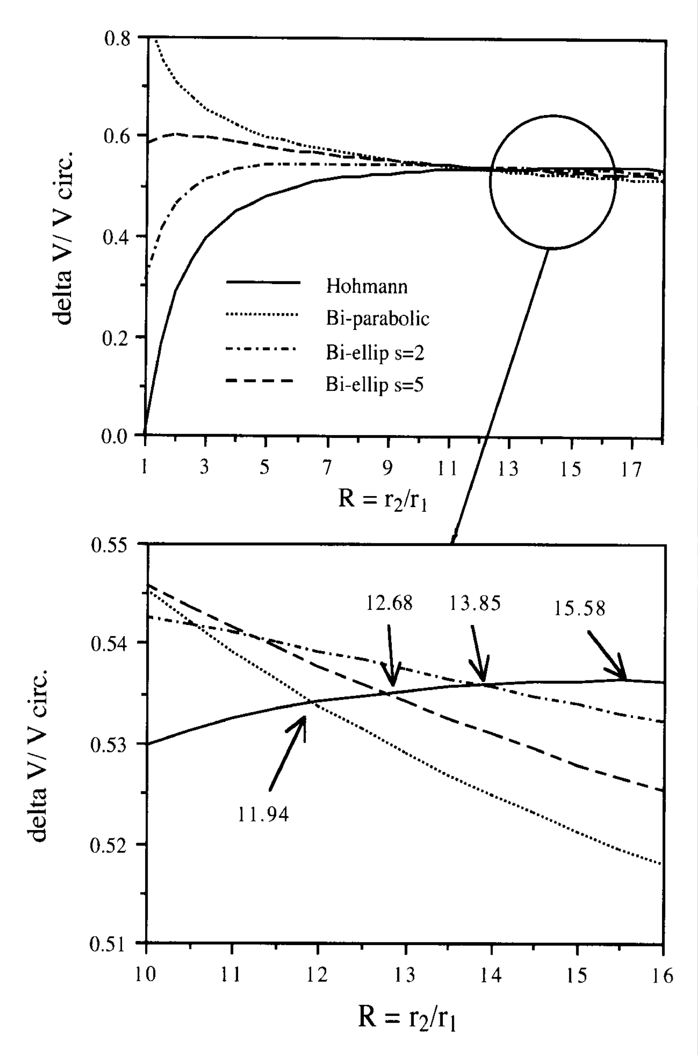


Optimum initial plane change = 2.20°

Bielliptic Transfer

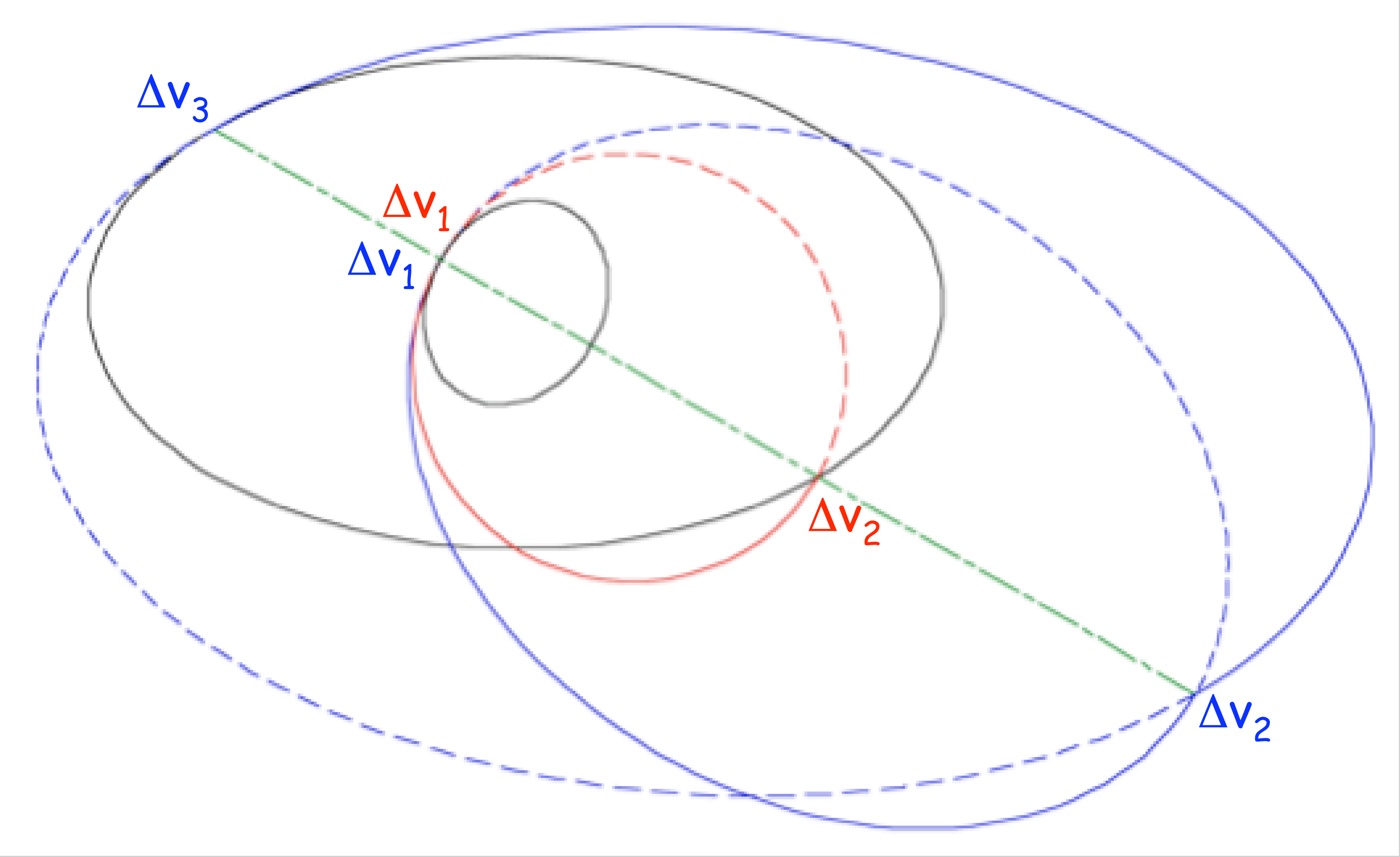


Coplanar Transfer Velocity Requirements

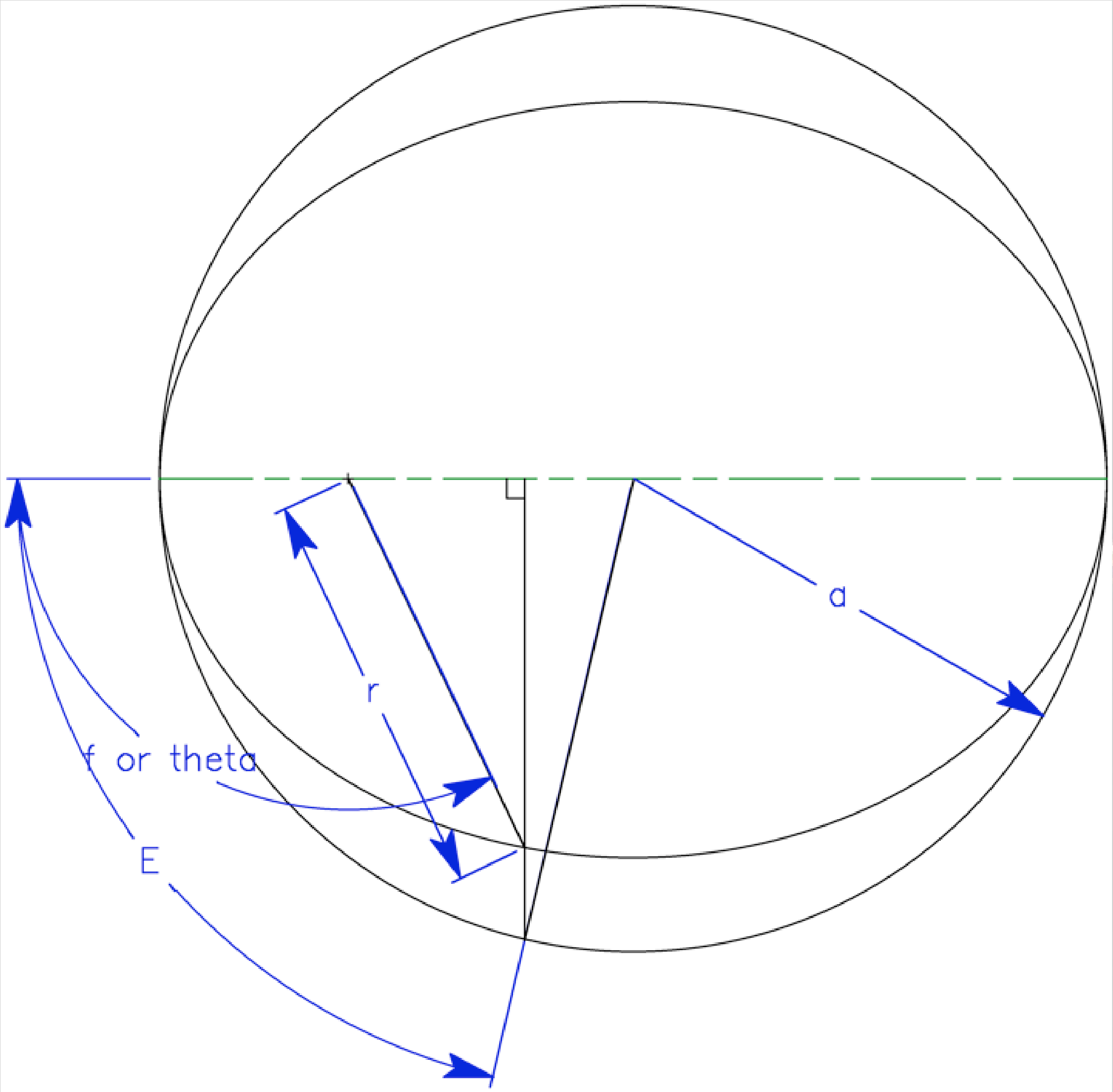


Ref: J. E. Prussing and B. A. Conway, Orbital Mechanics Oxford University Press, 1993

Noncoplanar Bielliptic Transfers



Calculating Time in Orbit



Time in Orbit

- Period of an orbit

$$P = 2\pi\sqrt{\frac{a^3}{\mu}}$$

- Mean motion (average angular velocity)

$$n = \sqrt{\frac{\mu}{a^3}}$$

- Time since pericenter passage

$$M = nt = E - e\sin E$$

➥M=mean anomaly

Dealing with the Eccentric Anomaly

- Relationship to orbit

$$r = a(1 - e\cos E)$$

- Relationship to true anomaly

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

- Calculating M from time interval: iterate

$$E_{i+1} = nt + e\sin E_i$$

until it converges

Example: Time in Orbit

- Hohmann transfer from LEO to GEO
  - h_1=300 km --> r_1=6378+300=6678 km
  - r_2=42240 km
- Time of transit (1/2 orbital period)

$$a = \frac{1}{2}(r_1 + r_2) = 24,459km$$

$$t_{\text{transit}} = \frac{P}{2} = \pi\sqrt{\frac{a^3}{\mu}} = 19,034sec = 5h17m14s$$

Example: Time-based Position

Find the spacecraft position 3 hours after perigee

$$n = \sqrt{\frac{\mu}{a^3}} = 1.650 \times 10^{-4} \frac{rad}{sec}$$

$$e = 1 - \frac{r_p}{a} = 0.7270$$

$$E_{j+1} = nt + e \sin E_j = 1.783 + 0.7270 \sin E_j$$

E=0; 1.783; 2.494; 2.222; 2.361; 2.294; 2.328; 2.311; 2.320; 2.316; 2.318; 2.317; 2.317; 2.317

Example: Time-based Position (continued)

$$E = 2.317$$

$$r = a(1 + e\cos E) = 12,387km$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \Longrightarrow \theta = 160\deg$$

Have to be sure to get the position in the proper quadrant - since the time is less than 1/2 the period, the spacecraft has yet to reach apogee --> $0° < \theta < 180°$

Velocity Components in Orbit

$$r = \frac{p}{1 + e\cos\theta}$$

$$v_r = \frac{dr}{dt} = \frac{d}{dt}\left(\frac{p}{1 + e\cos\theta}\right) = \frac{-p\left(-e\sin\theta\,\frac{d\theta}{dt}\right)}{\left(1 + e\cos\theta\right)^2}$$

$$v_r = \frac{pe\sin\theta}{\left(1 + e\cos\theta\right)^2}\frac{d\theta}{dt}$$

$$1 + e\cos\theta = \frac{p}{r} \implies v_r = \frac{r^2\frac{d\theta}{dt}e\sin\theta}{p}$$

$$\vec{h} = \vec{r} \times \vec{v}$$

Velocity Components in Orbit (continued)

$$\vec{h} = \vec{r} \times \vec{v}$$

$$h = rv\cos\gamma = r\left(r\frac{d\theta}{dt}\right) = r^2\frac{d\theta}{dt}$$

$$v_r = \frac{r^2\frac{d\theta}{dt}e\sin\theta}{p} = \frac{he\sin\theta}{p} = \frac{\sqrt{p\mu}}{p}e\sin\theta$$

$$v_r = \sqrt{\frac{\mu}{p}}e\sin\theta$$

$$v_\theta = r\frac{d\theta}{dt} = r\frac{h}{r^2} = \frac{h}{r} = \frac{\sqrt{p\mu}}{r}$$

$$v_\theta = \sqrt{\frac{\mu}{p}}(1 + e\cos\theta)$$

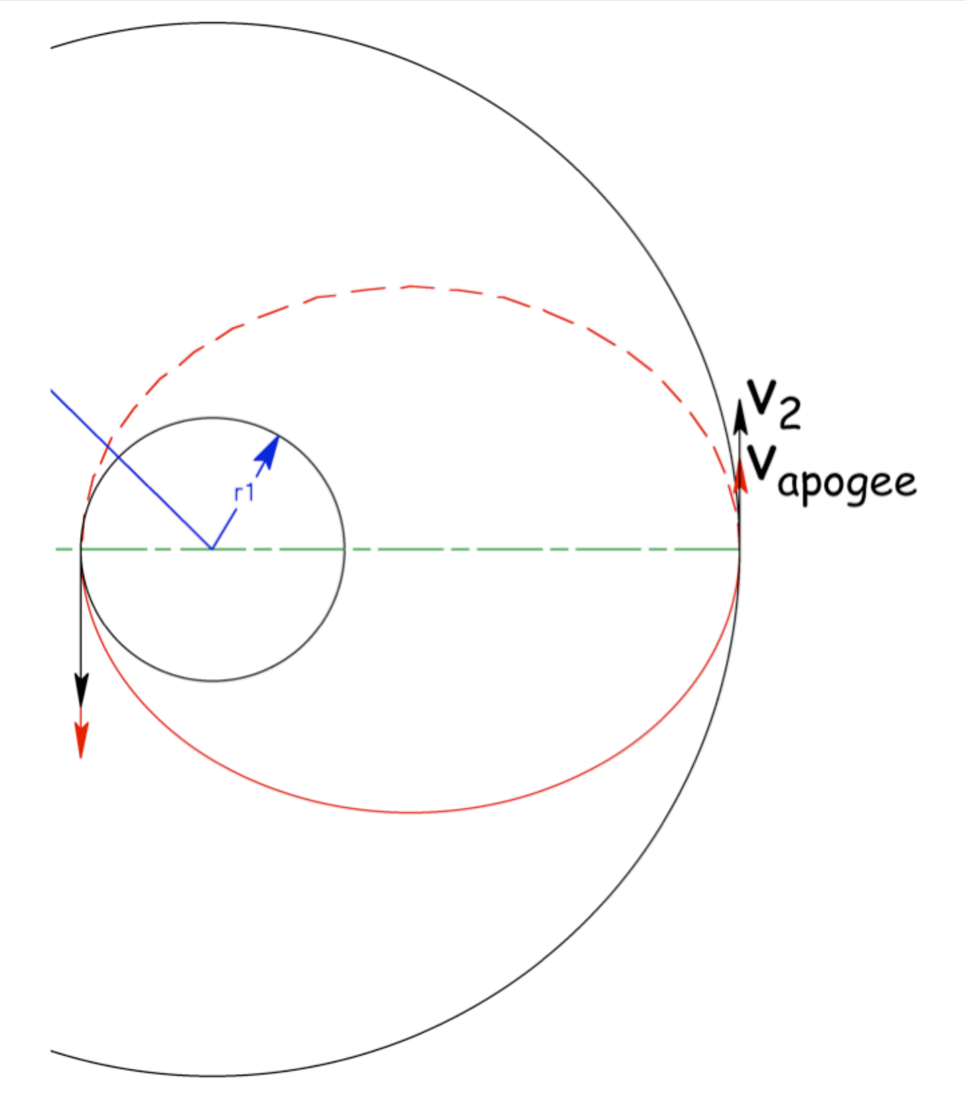
$$\tan\gamma = \frac{v_r}{v_\theta} = \frac{e\sin\theta}{1 + e\cos\theta}$$

Patched Conics

- Simple approximation to multi-body motion (e.g., traveling from Earth orbit through solar orbit into Martian orbit)
- Treats multibody problem as "hand-offs" between gravitating bodies --> reduces analysis to sequential two-body problems
- Caveat Emptor: There are a number of formal methods to perform patched conic analysis. The approach presented here is a very simple, convenient, and not altogether accurate method for performing this calculation. Results will be accurate to a few percent, which is adequate at this level of design analysis.

Example: Lunar Orbit Insertion

- v_2 is velocity of moon around Earth
- Moon overtakes spacecraft with velocity of $(v_2 - v_{apogee})$
- This is the velocity of the spacecraft relative to the moon while it is effectively “infinitely” far away (before lunar gravity accelerates it) = “hyperbolic excess velocity”



Planetary Approach Analysis

- Spacecraft has v_h hyperbolic excess velocity, which fixes total energy of approach orbit

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} = \frac{v_h^2}{2}$$

- Vis-viva provides velocity of approach

$$v = \sqrt{v_h^2 + \frac{2\mu}{r}}$$

- Choose transfer orbit such that approach is tangent to desired final orbit at periapse

$$\Delta v = \sqrt{v_h^2 + \frac{2\mu}{r}} - \sqrt{\frac{\mu}{r}}$$

Patched Conic Example - Lunar Approach

- Lunar orbital velocity around the Earth

$$v_m = \sqrt{\frac{\mu}{r_m}} = \sqrt{\frac{398,604}{384,400}} = 1.018\ \frac{km}{sec}$$

- Apogee velocity of Earth transfer orbit from initial 400 km low Earth orbit

$$v_a = v_m\sqrt{\frac{2r_1}{r_1 + r_m}} = 1.018\sqrt{\frac{6778}{6778 + 384,400}} = 0.134\ \frac{km}{sec}$$

- Velocity difference between spacecraft "infinitely" far away and moon (hyperbolic excess velocity)

$$v_h = v_m - v_a = v_m = 1.018 - 0.134 = 0.884\ \frac{km}{sec}$$

Patched Conic - Lunar Orbit Insertion

- The spacecraft is now in a hyperbolic orbit of the moon. The velocity it will have at the perilune point tangent to the desired 100 km low lunar orbit is

$$v_{pm} = \sqrt{v_h^2 + \frac{2\mu_m}{r_{LLO}}} = \sqrt{1.018^2 + \frac{2(4667.9)}{1878}} = 2.451 \ \frac{km}{sec}$$

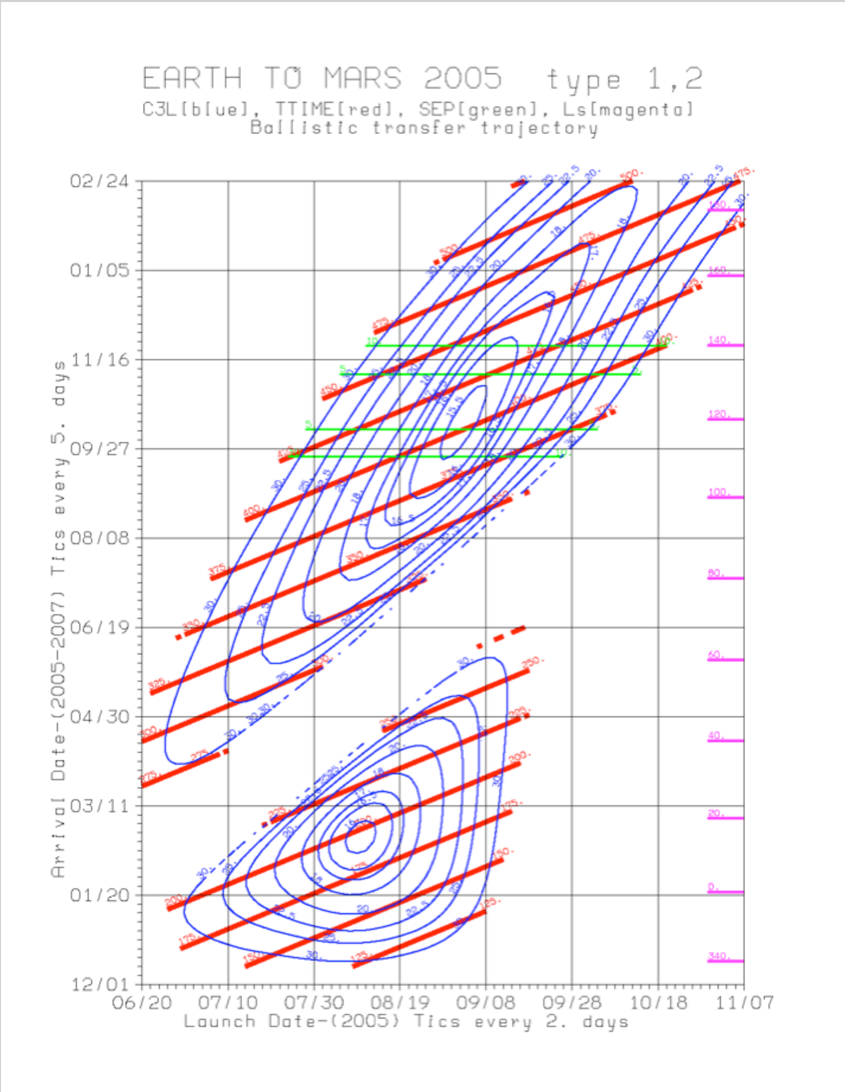
- The required delta-V to slow down into low lunar orbit is

$$\Delta v = v_{pm} - v_{cm} = 2.451 - \sqrt{\frac{4667.9}{1878}} = 0.874 \ \frac{km}{sec}$$

ΔV Requirements for Lunar Missions

To: / From:	Low Earth Orbit	Lunar Transfer Orbit	Low Lunar Orbit	Lunar Descent Orbit	Lunar Landing
Low Earth Orbit		3.107 km/sec			
Lunar Transfer Orbit	3.107 km/sec		0.837 km/sec		3.140 km/sec
Low Lunar Orbit		0.837 km/sec		0.022 km/sec	
Lunar Descent Orbit			0.022 km/sec		2.684 km/sec
Lunar Landing		2.890 km/sec		2.312 km/sec	

Interplanetary "Pork Chop" Plots



- Summarize a number of critical parameters
  - Date of departure
  - Date of arrival
  - Hyperbolic energy ("C3")
  - Transfer geometry
- Launch vehicle determines available C3 based on window, payload mass

Space Launch - The Physics

- Minimum orbital altitude is ~200 km

$$\frac{\text{Potential Energy}}{\text{kg in orbit}} = gh = 1.96x10^6 \frac{J}{kg}$$

- Circular orbital velocity there is 7784 m/sec

$$\frac{\text{Kinetic Energy}}{\text{kg in orbit}} = \frac{1}{2}v^2 = 30x10^6 \frac{J}{kg}$$

- Total energy per kg in orbit

$$\frac{\text{Total Energy}}{\text{kg in orbit}} = PE + KE = 32x10^6 \frac{J}{kg}$$

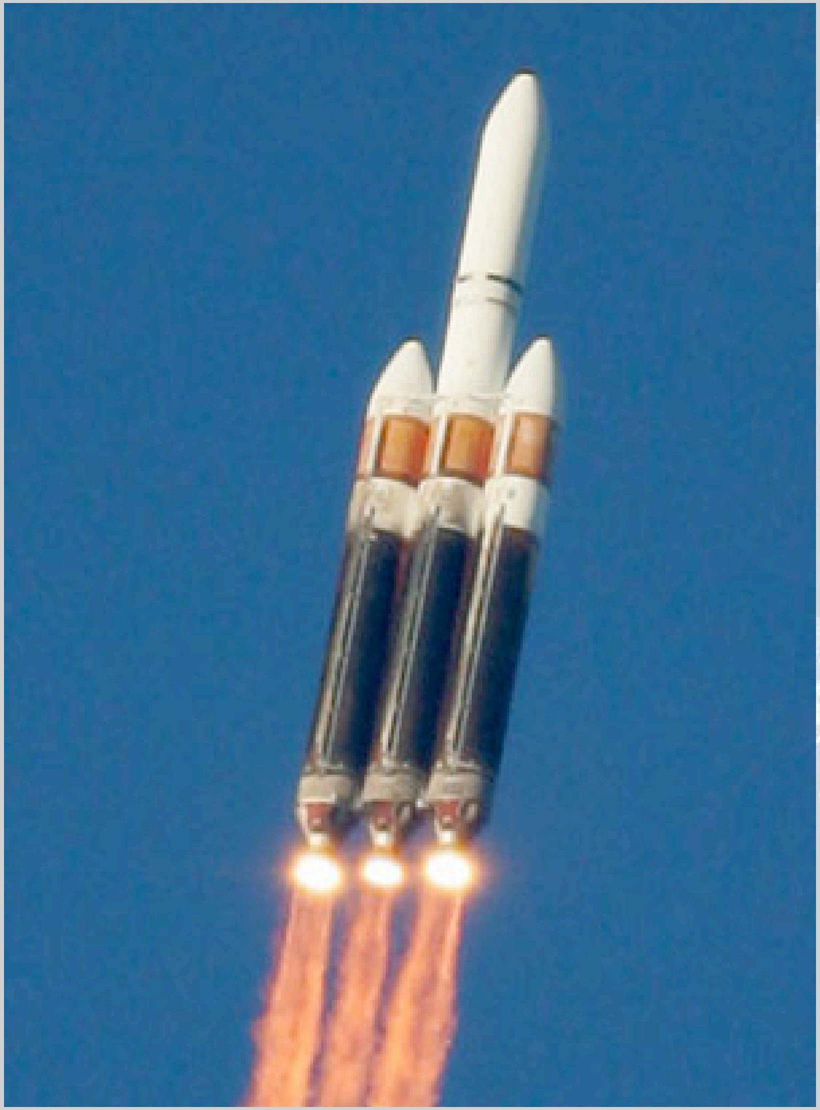
Theoretical Cost to Orbit

- Convert to usual energy units

$$\frac{\text{Total Energy}}{\text{kg in orbit}} = 32x10^6 \frac{J}{kg} = 8.888 \frac{kWhrs}{kg}$$

- Domestic energy costs are ~$0.05/kWhr
  - Theoretical cost to orbit <u>$0.44/kg</u>

Actual Cost to Orbit



Delta IV Heavy
- 23,000 kg to LEO
- $250 M per flight

$10,900/kg of payload

Factor of 25,000x higher than theoretical energy costs!

What About Airplanes?

- For an aircraft in level flight,

$$\frac{\text{Weight}}{\text{Thrust}} = \frac{\text{Lift}}{\text{Drag}}, \text{ or } \frac{mg}{T} = \frac{L}{D}$$

- Energy = force x distance, so

$$\frac{\text{Total Energy}}{\text{kg}} = \frac{\text{Thrust x Distance}}{\text{mass}} = \frac{Td}{m} = \frac{gd}{L/D}$$

- For an airliner (L/D=25) to equal orbital energy, d=81,000 km (2 roundtrips NY-Sydney)

Equivalent Airline Costs?

- Average economy ticket NY-Sydney round-round-trip (Travelocity 1/28/04) ~$1300
- Average passenger (+ luggage) ~100 kg
- Two round trips = $26/kg
  - Factor of 60x more than electrical energy costs
  - Factor of 420x less than current launch costs
- But…

  you get to refuel at each stop!

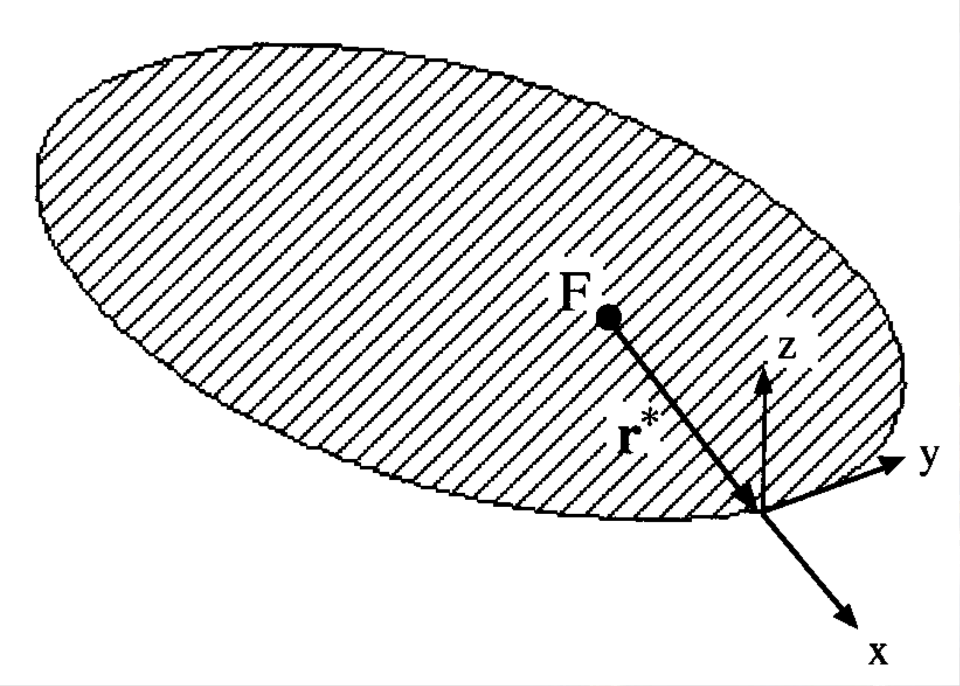
Equivalence to Air Transport

- 81,000 km ~ twice around the world
- Voyager - one of only two aircraft to ever circle the world non-stop, non-refueled - once!

Orbital Entry - The Physics

- 32 MJ/kg dissipated by friction with atmosphere over ~8 min = 66kW/kg
- Pure graphite (carbon) high-temperature material: c_p=709 J/kg°K
- Orbital energy would cause temperature gain of 45,000°K!
- (We'll cover this in much greater detail in an upcoming lecture)

Hill's Equations (Proximity Operations)



Ref: J. E. Prussing and B. A. Conway, Orbital Mechanics
Oxford University Press, 1993

$$\ddot{x} = 3n^2x + 2n\dot{y} + a_{dx}$$

$$\ddot{y} = -2n\dot{x} + a_{dy}$$

$$\ddot{z} = -n^2z + a_{dz}$$

Clohessy-Wiltshire ("CW") Equations

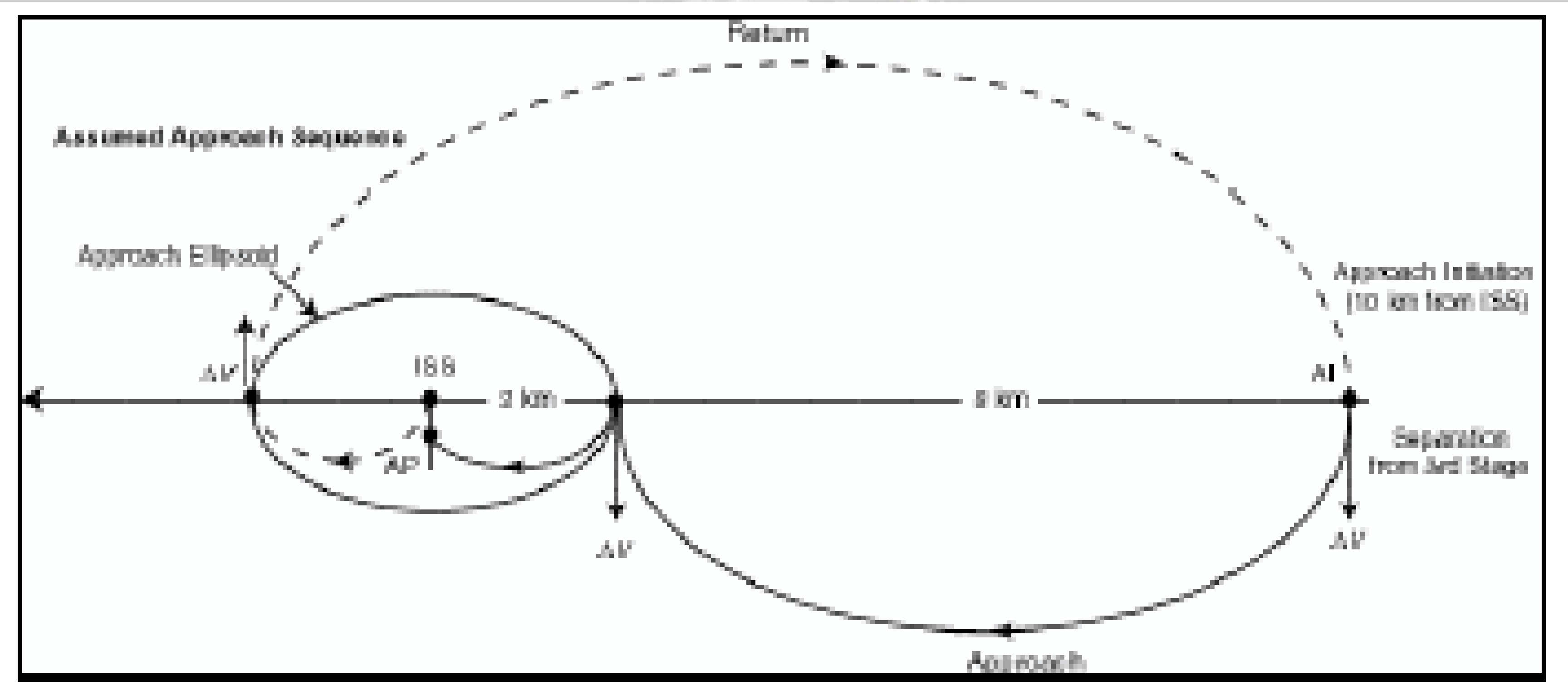
$$x(t) = \left[4 - 3\cos(nt)\right]x_o + \frac{\sin(nt)}{n}\dot{x}_o + \frac{2}{n}\left[1 - \cos(nt)\right]\dot{y}_o$$

$$y(t) = 6\left[\sin(nt) - nt\right]x_o + y_o - \frac{2}{n}\left[1 - \cos(nt)\right]\dot{x}_o + \frac{4\sin(nt) - 3nt}{n}\dot{y}_o$$

$$z(t) = z_o\cos(nt) + \frac{\dot{z}_o}{n}\sin(nt)$$

$$\dot{z}(t) = -z_o n\sin(nt) + \dot{z}_o\sin(nt)$$

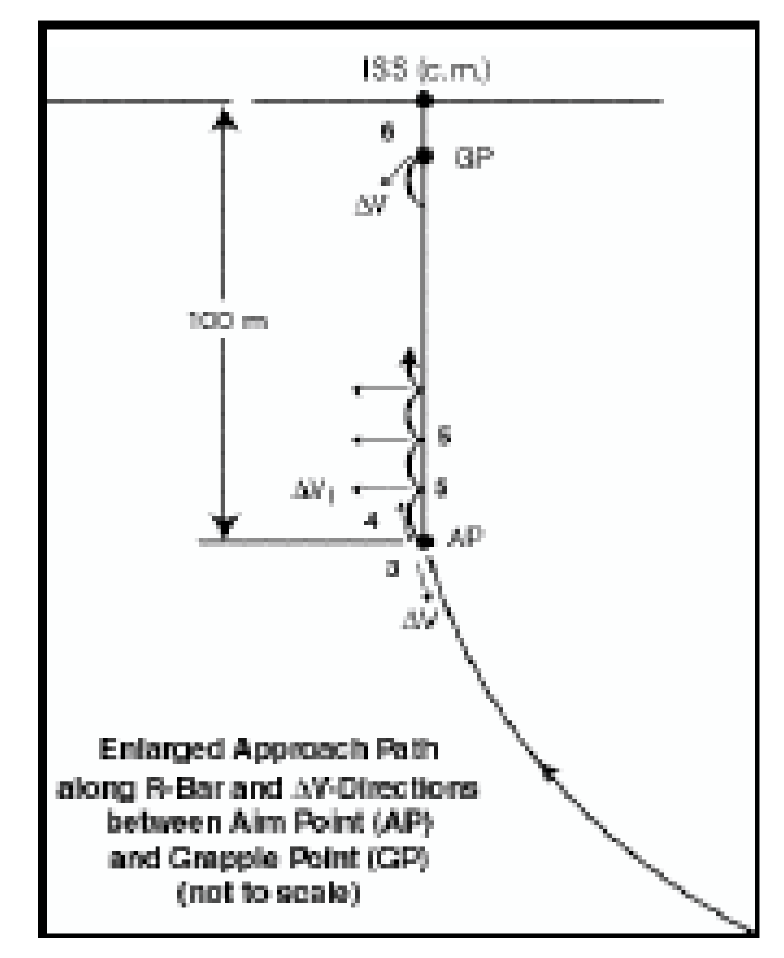
"V-Bar" Approach



Ref: Collins, Meissinger, and Bell, Small Orbit Transfer Vehicle (OTV) for On-Orbit Satellite Servicing and Resupply, 15th USU Small Satellite Conference, 2001

"R-Bar" Approach

- Approach from along the radius vector ("R-bar")
- Gravity gradients decelerate spacecraft approach velocity - low contamination approach
- Used for Mir, ISS docking approaches



Ref: Collins, Meissinger, and Bell, Small Orbit Transfer Vehicle (OTV) for On-Orbit Satellite Servicing and Resupply, 15th USU Small Satellite Conference, 2001

References for Lecture 3

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