ENAE 483/788D LECTURE #02
(ROCKET PERFORMANCE) PROBLEMS – FALL, 2023

In only thirteen years of operations, the Falcon 9 has become one of the world's most used (and cheapest!) launch vehicles. The mass and propellant properties of the Falcon 9 are shown in the following table. In this configuration, the rocket is capable of placing 22.8 MT into low Earth orbit (LEO).

Stage	empty mass (MT)	propellant mass (MT)	specific impulse (sec)	nominal burn time (sec)
First stage	17	407.6	312	162
Second stage	4.5	107.2	348	375

(1) Calculate

(a) Gross mass of the entire vehicle m_o

$$m_o = m_{in,1} + m_{pr,1} + m_{in,2} + m_{pr,2} + m_{pl} = 17 + 407.6 + 4.5 + 107.2 + 22.8 = \boxed{559.1 \text{ MT}}$$

(b) Inert mass fraction for the first stage δ_1

$$m_{o,2} = m_{in,2} + m_{pr,2} + m_{pl} = 4.5 + 107.2 + 22.8 = 134.5\ MT$$
$$\delta_1 = \frac{m_{in,1}}{m_{in,1} + m_{pr,1} + m_{o,2}} = \frac{17}{17 + 407.6 + 134.5} = \boxed{0.03041}$$

(c) Inert mass fraction for the second stage δ_2

$$\delta_2 = \frac{m_{in,2}}{m_{o,2}} = \frac{4.5}{134.5} = \boxed{0.03346}$$

(d) Stage inert mass fraction for the first stage ϵ_1

$$\epsilon_1 = \frac{m_{in,1}}{m_{in,1} + m_{pr,1}} = \frac{17}{17 + 407.6} = \boxed{0.04004}$$

(e) Stage inert mass fraction for the second stage ϵ_2

$$\epsilon_2 = \frac{m_{in,2}}{m_{in,2} + m_{pr,2}} = \frac{4.5}{4.5 + 107.2} = \boxed{0.04029}$$

(f) Predicted values for ϵ_1 and ϵ_2, based on the heuristic equation from lecture for storable (i.e., non-LH_2) propellants
From page 18 in the lecture slides,

$$\epsilon_{\text{storables}} = 1.6062\,(M_{\text{stage}}\langle kg\rangle)^{-0.275}$$
$$\epsilon_1 = 1.6062\,(17,000 + 407,600)^{-0.275} = \boxed{0.04551}$$
$$\epsilon_2 = 1.6062\,(4500 + 107,200)^{-0.275} = \boxed{0.06570}$$

(g) Velocity change provided by the first stage Δv_1

$$m_{f,1} = min, 1 + m_{o,2} = 17 + 134.5 = 151.5\ MT$$

$$\Delta v_1 = -g_o I_{sp} \ln \frac{m_{f,1}}{m_o} = -9.8(312) \ln \frac{151.5}{559.1} = \boxed{3992 \text{ m/sec}}$$

(h) Velocity change provided by the second stage Δv_2

$$m_{f,2} = min, 2 + m_{pl} = 4.5 + 22.8 = 27.3\ MT$$

$$\Delta v_2 = -g_o I_{sp} \ln \frac{m_{f,2}}{m_{o,2}} = -9.8(348) \ln \frac{27.3}{134.5} = \boxed{5438 \text{ m/sec}}$$

(i) Total velocity change provided by the entire launch vehicle Δv_{total}

$$\Delta v_{total} = \Delta v_1 + \Delta v_2 = 3992 + 5438 = \boxed{9431 \text{ m/sec}}$$

(j) Trade-off ratio for change in payload mass due to a change in first stage inert mass $\frac{\partial m_{pl}}{\partial m_{in,1}}$

$$\frac{\partial m_{pl}}{\partial m_{in,1}} = \frac{-gI_{sp,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right)}{gI_{sp,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + gI_{sp,2}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)}$$

$m_{o,1} = m_o$, and g appears in every term so we can divide that out (which we'll do for the next steps)

$$\frac{\partial m_{pl}}{\partial m_{in,1}} = \frac{-312\left(\frac{1}{559.1} - \frac{1}{151.5}\right)}{312\left(\frac{1}{559.1} - \frac{1}{151.5}\right) + 348\left(\frac{1}{134.5} - \frac{1}{27.3}\right)} = \boxed{\text{-0.1287}}$$

Note: this is a dimensionless number, so it is metric tons of payload lost per metric ton of inert mass, or kg of payload per kg of inert mass

(k) Trade-off ratio for change in payload mass due to a change in second stage inert mass $\frac{\partial m_{pl}}{\partial m_{in,2}}$

$$\frac{\partial m_{pl}}{\partial m_{in,2}} = \frac{-\left[I_{sp,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + I_{sp,2}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)\right]}{I_{sp,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + I_{sp,2}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)}$$

$$\frac{\partial m_{pl}}{\partial m_{in,2}} = \frac{-\left[312\left(\frac{1}{559.1} - \frac{1}{151.5}\right) + 348\left(\frac{1}{134.5} - \frac{1}{27.3}\right)\right]}{312\left(\frac{1}{559.1} - \frac{1}{151.5}\right) + 348\left(\frac{1}{134.5} - \frac{1}{27.3}\right)} = \boxed{\text{-1}}$$

(l) Trade-off ratio for change in payload mass due to a change in first stage propellant mass $\frac{\partial m_{pl}}{\partial m_{pr,1}}$

$$\frac{\partial m_{pl}}{\partial m_{pr,1}} = \frac{-I_{sp,1}\left(\frac{1}{m_{o,1}}\right)}{I_{sp,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + I_{sp,2}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)}$$

$$\frac{\partial m_{pl}}{\partial m_{pr,1}} = \frac{-312\left(\frac{1}{559.1}\right)}{312\left(\frac{1}{559.1} - \frac{1}{151.5}\right) + 348\left(\frac{1}{134.5} - \frac{1}{27.3}\right)} = \boxed{0.04785}$$

(m) Trade-off ratio for change in payload mass due to a change in second stage propellant mass $\frac{\partial m_{pl}}{\partial m_{pr,2}}$

$$\frac{\partial m_{pl}}{\partial m_{pr,2}} = \frac{-\left[I_{sp,1}\left(\frac{1}{m_{o,1}}\right) + I_{sp,2}\left(\frac{1}{m_{o,2}}\right)\right]}{I_{sp,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + I_{sp,2}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)}$$

$$\frac{\partial m_{pl}}{\partial m_{pr,2}} = \frac{-\left[312\left(\frac{1}{559.1}\right) + 348\left(\frac{1}{134.5}\right)\right]}{312\left(\frac{1}{559.1} - \frac{1}{151.5}\right) + 348\left(\frac{1}{134.5} - \frac{1}{27.3}\right)} = \boxed{0.2697}$$

(2) The Falcon Heavy is effectively three first stages strapped together to increase payload. Assuming you have to reach the same Δv_{total} you calculated in (1)(i), find the new payload to LEO assuming

(a) All three first stages burn together and are dropped together (i.e., effectively the first stage is just three times larger than it was before.)

$$m_o = 3m_{in,1} + 3m_{pr,1} + m_{in,2} + m_{pr,2} + m_{pl} = 3(17) + 3(407.6) + 4.5 + 107.2 + m_{pl} = 1385.5 + m_{pl}$$

$$m_{f,1} = 3m_{in,1} + m_{in,2} + m_{pr,2} + m_{pl} = 3(17) + 4.5 + 107.2 + m_{pl} = 162.7 + m_{pl}$$

$$\Delta v_1 = -g_o I_{sp,1} \ln \frac{m_{f,1}}{m_o} = -9.8(312) \ln \frac{162.7 + m_{pl}}{1385.5 + m_{pl}}$$

$$\Delta v_2 = -g_o I_{sp,2} \ln \frac{m_{f,2}}{m_{o,2}} = -9.8(348) \ln \frac{4.5 + m_{pl}}{111.7 + m_{pl}}$$

$$\Delta v_{tot} = \Delta v_1 + \Delta v_2 = 9431\ m/sec$$

Solving numerically, the answer is $m_{pl} = \boxed{52.21\ MT}$

(b) The two outer boosters burn together, then the center core burns by itself (effectively a three-stage rocket)

From (a), $m_o = 1385.5 + m_{pl}$

$$m_{f,1} = 3m_{in,1} + m_{pr,1} + m_{in,2} + m_{pr,2} + m_{pl} = 3(17) + 407.6 + 4.5 + 107.2 + m_{pl} = 570.3 + m_{pl}$$

$$\Delta v_1 = -g_o I_{sp,1} \ln \frac{m_{f,1}}{m_o} = -9.8(312) \ln \frac{570.3 + m_{pl}}{1385.5 + m_{pl}}$$

The "second" stage is just the Falcon 9 first stage, but we don't know the payload, so from (1)(a)

$$m_{o,2} = (559.1 - 22.8) + m_{pl} = 537 + m_{pl}$$

$$m_{f,2} = m_{o,2} - m_{pr,1(Falcon\ 9)} = 537 - 407.6 + m_{pl} = 129.4 + m_{pl}$$

$$\Delta v_2 = -g_o I_{sp,2} \ln \frac{m_{f,2}}{m_{o,2}} = -9.8(312) \ln \frac{129.4 + m_{pl}}{537 + m_{pl}}$$

The "third" stage is the same as the second stage in (a)...

$$\Delta v_3 = -g_o I_{sp,3} \ln \frac{m_{f,3}}{m_{o,3}} = -9.8(348) \ln \frac{4.5 + m_{pl}}{111.7 + m_{pl}}$$

$$\Delta v_{tot} = \Delta v_1 + \Delta v_2 + \Delta v_3 = 9431\ m/sec$$

Solving numerically, the answer is $m_{pl} = \boxed{59.07\ MT}$

(c) The center core is throttled to 75%, so the fraction of center core propellant left at booster burnout (χ) is 0.25
All three first stage modules have the same I_{sp}, so we don't have to calculate the weighted average as in the generic case. Treating the period where all three first stage modules are firing as stage "0",

$$m_{f,0} = 3m_{in,1} + 0.25m_{pr,1} + m_{in,2} + m_{pr,2} + m_{pl} = 3(17) + 0.25(407.6) + 4.5 + 107.2 + m_{pl} = 264.6 + m_{pl}$$

$$\Delta v_0 = -g_o I_{sp,1} \ln \frac{m_{f,0}}{m_o} = -9.8(312) \ln \frac{264.6 + m_{pl}}{1385.5 + m_{pl}}$$

$$m_{o,1} = m_{in,1} + 0.25m_{pr,1} + m_{in,2} + m_{pr,2} + m_{pl} = 17 + 0.25(407.6) + 4.5 + 107.2 + m_{pl} = 230.6 + m_{pl}$$

$$\Delta v_1 = -g_o I_{sp,1} \ln \frac{m_{f,1}}{m_{o,1}} = -9.8(312) \ln \frac{128.7 + m_{pl}}{230.6 + m_{pl}}$$

$$\Delta v_2 = -g_o I_{sp,2} \ln \frac{m_{f,2}}{m_{o,2}} = -9.8(348) \ln \frac{4.5 + m_{pl}}{111.7 + m_{pl}}$$

$$\Delta v_{tot} = \Delta v_0 + \Delta v_1 + \Delta v_2 = 9431 \ m/sec$$

Solving numerically, the answer is $m_{pl} = \boxed{58.89\ MT}$