Robot Kinematics

- Completion of tank analysis from last time
- Description of vehicle and control axes
- Kinematic descriptions of wheel types
- Maneuverability

Tracked Vehicle Steering Example

$W = 155.7\,kN$ (75,000 lb) $\quad B = 203.2\,cm$

$l = 304.8\,cm \quad b = 76.2\,cm \quad c = 3.45\,kPa$

$\phi = 25° \quad \mu_r = 0.15 \quad \mu_t = 0.5$

1) Determine steerability – $\frac{l}{B} \leq \frac{2}{\mu_t}\left(\frac{c}{p} + \tan\phi - \mu_r\right) \leq 1.67$

Actual $\frac{l}{B} = 1.5$ so it is steerable

2) Analyze thrust during turn

$$F_o = \frac{\mu_r W}{2} + \frac{\mu_t W l}{4B} = 40.87\,kN$$

$$F_i = \frac{\mu_r W}{2} - \frac{\mu_t W l}{4B} = -17.52\,kN$$

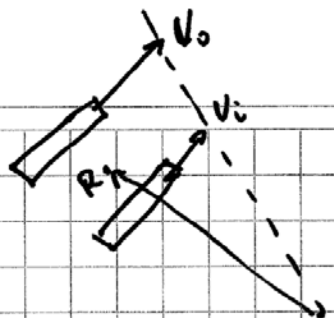
Kinematics of Tracked Skid Steering

Outside drive wheel turns at ω_o

Inside " " " " ω_i

Assume tracks do not skid or slip

Turn radius R yaw velocity Ω_z



By similar triangles $\frac{V_o}{R+\frac{B}{2}} = \frac{V_i}{R-\frac{B}{2}} \Rightarrow V_o R - V_o \frac{B}{2} = V_i R + V_i \frac{B}{2}$

$$R = \frac{B}{2} \frac{V_o + V_i}{V_o - V_i}$$

drive sprocket radius r $V = \omega r$

$$R = \frac{B}{2} \frac{\omega_o r + \omega_i r}{\omega_o r - \omega_i r} \qquad K_s = \frac{\omega_o}{\omega_i}$$

$$= \frac{B}{2} \frac{K_s + 1}{K_s - 1}$$

$$\frac{\omega_o r + \omega_i r}{2R} = \frac{\omega_o r - \omega_i r}{B}$$

$$\Omega_z = \frac{r\omega_o + r\omega_i}{2R} = \frac{r\omega_i (K_s - 1)}{B}$$

Outside track always thrusts $\Rightarrow$ always slips

$$R' = \frac{B[r\omega_o(1-i_o) + r\omega_i(1-i_i)]}{2[r\omega_o(1-i_o) - r\omega_i(1-i_i)]}$$

$$= \frac{B[K_s(1-i_o) + (1-i_i)]}{2[K_s(1-i_o) - (1-i_i)]}$$

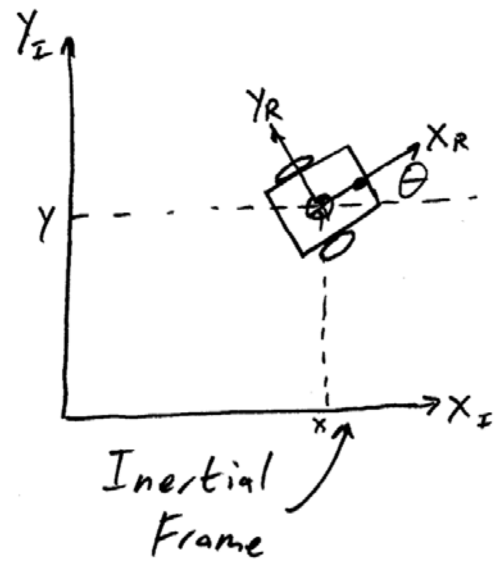
$$\Omega'_z = \frac{r\omega_o(1-i_o) + r\omega_i(1-i_i)}{2R'}$$

$$= \frac{r\omega_i\left[K_s(1-i_o) - 1-(i_i)\right]}{B}$$

$\frac{R'}{R}$ always >1 (slip causes larger turning radius)

$\frac{\Omega_z'}{\Omega_z}$ always <1 (slower turn)

Mobile Robot Kinematics



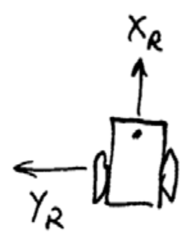
State vector in inertial frame

$$\xi_I = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

Rotational Transform

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mapping Motion $\rightarrow$ $\dot{\xi}_R = R(\theta)\,\dot{\xi}_I$



Example - $\theta = \frac{\pi}{2}$

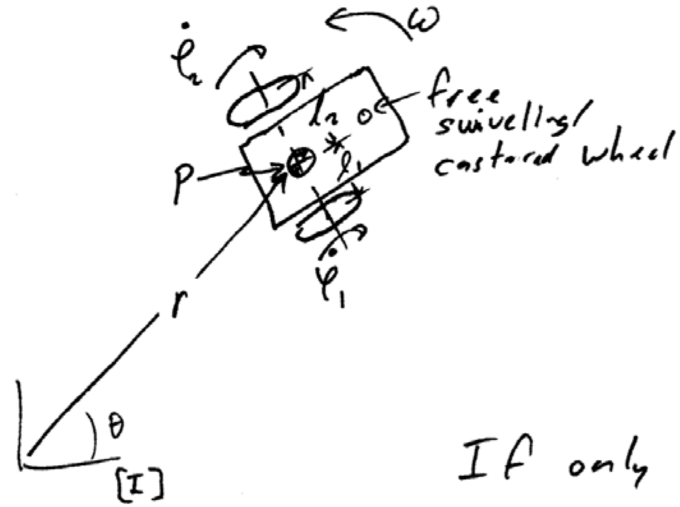
$$R\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Given a desired inertial velocity $\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$

In rover coordinates $\dot{\xi}_R = R\left(\frac{\pi}{2}\right)\dot{\xi}_I = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -\dot{x} \\ \dot{\theta} \end{bmatrix}$

Forward Kinematics (FK)

Differential drive rover - control point between two driven wheels



$$\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(l, r, \theta\, \dot{\varphi}_1, \dot{\varphi}_2)$$ (assumes $l_1 = l_2 = l$)

So in inertial frame $\dot{\xi}_I = R(\theta)^{-1} \dot{\xi}_x$

Wheels rotate with angular velocities $\dot{\varphi}_1$ and $\dot{\varphi}_2$

If only wheel 1 rotates, vehicle pivots about wheel 2

Resultant base rotation rate $\omega_1 = \dfrac{r\dot{\varphi}_1}{2l}$ ← linear velocity of wheel rim; ← rotation distance to wheel

Similarly, $\omega_2 = \dfrac{-r\dot{\varphi}_2}{2l}$

$$\dot{\xi}_I = R(\theta)^{-1} \begin{bmatrix} \dfrac{r\dot{\varphi}_1}{2} + \dfrac{r\dot{\varphi}_2}{2} \\ 0 \\ \dfrac{r\dot{\varphi}_1}{2l} + \dfrac{r\dot{\varphi}_2}{2l} \end{bmatrix}$$

$$R(\theta)^{-1} = R(\theta)^T = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Previous example: $\theta = \frac{\pi}{2}$ $\quad r=1 \quad l=1$

Assume $\dot{\varphi}_1 = 4$ and $\dot{\varphi}_2 = 2$

$$\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \leftarrow$$ moves along Y_I with speed 3 and rotation rate 1

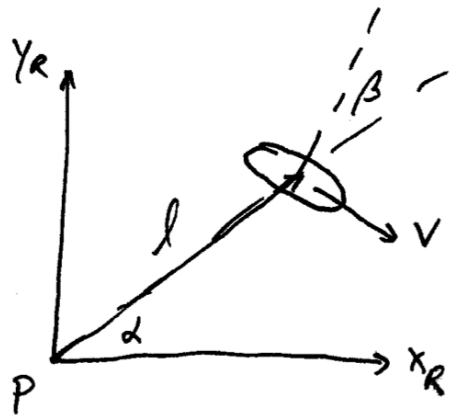
Units? $l, r = \langle m \rangle$ $\quad \dot{\varphi}_1, \dot{\varphi}_2 = \langle \text{rad/sec} \rangle$ $\quad \dot{Y}_I = 3 \text{ m/sec}$ $\quad \dot{\theta}_I = 1 \text{ rad/sec}$

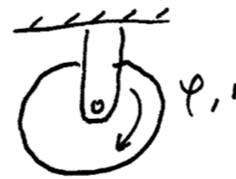
Kinematic Constraints for Various Wheel Types

Assumptions: wheels roll, but do not slip

Types of wheel mounts:
- fixed
- steered
- castored
- omni wheel

Fixed standard wheel





In rolling direction,

$$\left[\sin(\alpha+\beta) \; -\cos(\alpha+\beta) \; (-l)\cos\beta\right] R(\theta)\,\dot{\xi}_I = r\dot{\varphi}$$

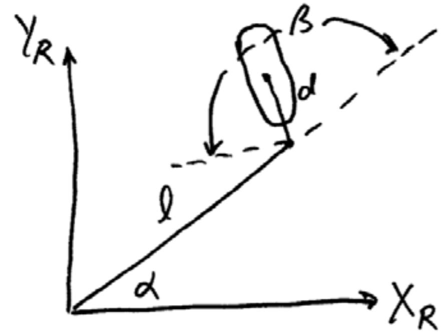
In cross direction,

$$\left[\cos(\alpha+\beta) \; \sin(\alpha+\beta) \; l\sin\beta\right] R(\theta)\,\dot{\xi}_I = 0$$

Steered Standard Wheel

→ same as fixed except $\beta = f(t) = [\beta(t)]$

Castor Wheels

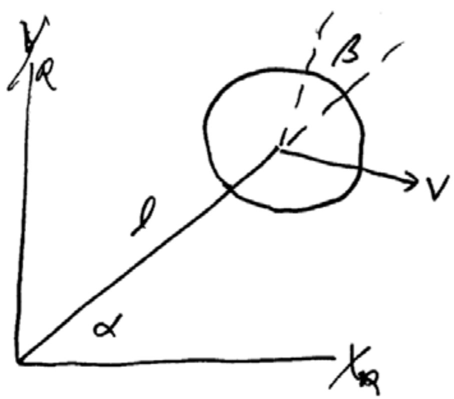


$$\left[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad (-l)\cos\beta\right] R(\theta)\,\dot{\xi}_I = r\dot{\varphi}$$

? unchanged - but β is controlled by side loads

$$\left[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad d + l\sin\beta\right] R(\theta)\,\dot{\xi}_I + d\dot{\beta} = 0$$

Spherical Wheels



$$\left[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad (-l)\cos\beta\right] R(\theta)\,\dot{\xi}_I = r\dot{\varphi}$$

$$\left[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad l\sin\beta\right] R(\theta)\,\dot{\xi}_I = 0$$

β is free variable - example - if V is in Y_R direction, $\sin(\alpha+\beta) = 0 \Rightarrow \alpha = -\beta$

Robot Kinematic Constraints

Spherical wheels and castors do not impose kinematic constraints

→ only fixed and steerable wheels control kinematics

Assume N standard wheels

↳ N_f fixed

N_s steerable

$\beta_s(t)$ are steering angles of N_s [drive angle $\varphi_s(t)$]

β_f are orientations of N_f [drive angle $\varphi_f(t)$]

$$\varphi(t) = \begin{bmatrix} \varphi_f(t) \\ \varphi_s(t) \end{bmatrix}$$

Rolling constraint of all wheels

Constraint for rolling wheels →

$$J_1(\beta_s) R(\theta) \dot{\xi}_I - J_2 \dot{\varphi} = 0$$

J_2 is $N \times N$ diagonal of radii r for each wheel

$$J_1(\beta_s) = \begin{bmatrix} J_{1f} \\ J_{1s}(\beta_s) \end{bmatrix}$$ ← not a function of β_f

J_{1f} is constant matrix of projections for each wheel

$$(N_f \times 3) \quad \left[\sin(\alpha+\beta) \; -\cos(\alpha+\beta) \; (-l)\cos\beta\right] R(\theta)\, \dot{\underset{3 \text{ vector}}{\xi_I}} - r\dot{\varphi} = 0$$

$J_{1s}(\beta_s)$ is $N_s \times 3$ column matrix

$$\left[\sin(\alpha+\beta) \; -\cos(\alpha+\beta) \; (-l)\cos\beta\right] R(\theta)\, \dot{\xi}_I - r\dot{\varphi} = 0$$

Constraint for sideslip (or lack thereof)

$$C_1(\beta_s) R(\theta) \dot{\xi}_I = 0 \qquad C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}$$

C_{1f} and C_{1s} are $(N_f \times 3)$ and $(N_s \times 3)$ matrices

$$\left[\sin(\alpha+\beta) \; -\cos(\alpha+\beta) \; (-l)\cos\beta\right] R(\theta)\, \dot{\xi}_I - r\dot{\varphi} = 0$$

$$\left[\cos(\alpha+\beta) \; \sin(\alpha+\beta) \; l\sin\beta\right] R(\theta)\, \dot{\xi}_I = 0$$

Example: Differential Drive Robot

Rolling constraints J Sliding constraints C

$$\begin{bmatrix} J_1(\beta_s) \\ C_1(\beta_s) \end{bmatrix} R(\theta)\,\dot{\xi}_I = \begin{bmatrix} J_2\varphi \\ 0 \end{bmatrix}$$

Castor is unpowered and can move in any direction $\Rightarrow$ ignore

Two remaining wheels are fixed (i.e., unsteerable)

so $J_1(\beta_s) = J_{1_f}$ and $C_1(\beta_s) = C_{1_f}$

Travel in $+X_R$ direction $\rightarrow$

$$\left.\begin{array}{lll} \text{right wheel} & \alpha = -\frac{\pi}{2} & \beta = \pi \\ \text{left wheel} & \alpha = \frac{\pi}{2} & \beta = 0 \end{array}\right\}$$

produces forward $+X$ motion for $\dot{\varphi}_\ell$ and $\dot{\varphi}_r > 0$, positive yaw for $\dot{\varphi}_r > 0$ and $\dot{\varphi}_\ell < 0$

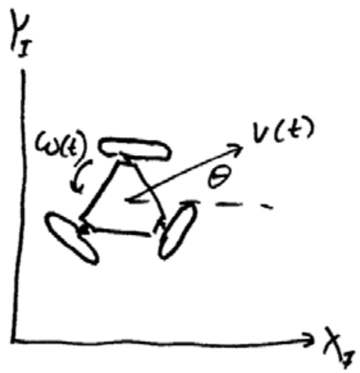
$$\begin{bmatrix} \begin{bmatrix} 1 & 0 & \ell \\ 1 & 0 & -\ell \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{bmatrix} R(\theta)\,\dot{\xi}_I = \begin{bmatrix} J_2\dot{\varphi} \\ 0 \end{bmatrix} \qquad J_2\dot{\varphi} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix}$$

Invert to get kinematic control equation

$$\dot{\xi}_I = R(\theta)^{-1} \begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} J_2 \dot{\varphi} \\ 0 \end{bmatrix} = R(\theta)^{-1} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2l} & -\frac{1}{2l} & 0 \end{bmatrix} \begin{bmatrix} J_2 \dot{\varphi} \\ 0 \end{bmatrix}$$

reminder: $J_2 \varphi = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{matrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{matrix} = \begin{matrix} r\dot{\varphi}_1 \\ r\dot{\varphi}_2 \end{matrix}$

Omniwheel example

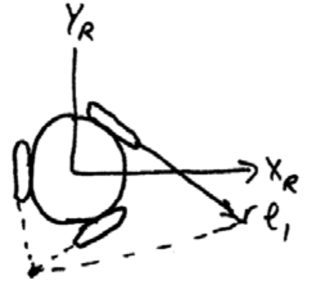


All wheels have same distance from center P of l and radius r

All wheels fixed

$$\dot{\xi}_I = R(\theta)^{-1} J_{1f}^{-1} J_2 \dot{\varphi}$$

Local reference frame



$\alpha_1 = \frac{\pi}{3}$ $\quad \alpha_2 = \pi$ $\quad \alpha_3 = -\pi/3$

$\beta = 0$ for all (all tangent to circular body)

ICR = Instantaneous Center of Rotation

$$J_{1f} = \begin{bmatrix} \sin\frac{\pi}{3} & -\cos\frac{\pi}{3} & -l \\ 0 & -\cos\pi & -l \\ \sin(-\frac{\pi}{3}) & -\cos(\frac{\pi}{3}) & -l \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & -l \\ 0 & 1 & -l \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & -l \end{bmatrix}$$

$$\dot{\zeta}_I = R(\theta)^{-1} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3l} & -\frac{1}{3l} & -\frac{1}{3l} \end{bmatrix}}_{J_{1f}^{-1}} \underbrace{J_2 \dot{\varphi}}_{\begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\varphi}_3 \end{bmatrix}}$$

Assume $l=1$ and $r=1$, $\theta=0$

if $\dot{\varphi} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ what is the robot motion?

$$\dot{\zeta}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{3}} \\ -\frac{4}{3} \\ -\frac{7}{3} \end{bmatrix} \begin{array}{l} \leftarrow \text{m/sec} \\ \leftarrow \text{m/sec} \\ \leftarrow \text{rad/sec} \end{array}$$

(if r, l in m)

Maneuverability

→ Robot must move in the environment

→ Each wheel must meet its sliding constraint

Start with $C_1(\beta_s)R(\theta)\dot{\xi}_I = 0$

$$C_{1_f} R(\theta)\dot{\xi}_I = 0$$

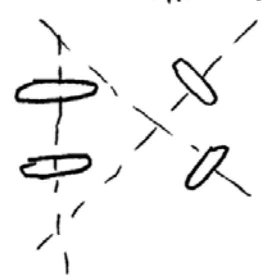
$$C_{1_s}(\beta_s) R(\theta)\dot{\xi}_I = 0$$

To be satisfied, ~~$C_1(\beta_s)$~~ $R(\theta)\dot{\xi}_I$ must lie in null space of $C_1(\beta_s)$

(space N such that for any vector $\bar{n}$ in N, $C_1(\beta_s)\bar{n} = 0$)

Or, geometrically, all wheel drive axes must pass through ICR

but



← cannot meet constraint condition

Consider a unicycle - one fixed wheel

$$C_1(\beta_s) = C_{1f} = [\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad l\sin\beta]$$

↑ rank [1]

Two parallel wheels

wheel 1: α_1, β_1, l_1 — $l_1 = l_2$

wheel 2: α_2, β_2, l_2 — $\beta_1 = \beta_2 = 0$

$\alpha_1 + \pi = \alpha_2$

$$C_1(\beta_s) = C_{1f} = \begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 & 0 \\ \cos(\alpha_1+\pi) & \sin(\alpha_1+\pi) & 0 \end{bmatrix}$$

Two constraints, but rank [1] (not independent)

If rank [2] $\Rightarrow$ can move only in straight line or along circular arc

if rank [3] $\Rightarrow$ can't move