Lifting Entry

2/19/04

From last time,

$$\frac{dV}{dt} = \left(\frac{V^2}{r} - g\right)\sin\gamma - \frac{D}{m}$$

$$V\frac{d\gamma}{dt} = \frac{L}{m} + \left(\frac{V^2}{r} - g\right)\cos\gamma$$

Assume (at entry velocities close to orbital) $g \cong \frac{V^2}{r}$

Equations of motion become

$$V\frac{d\gamma}{dt} = \frac{L}{m} = \frac{\rho V^2}{2}\frac{C_L A}{m} = \frac{\rho V^2}{2\beta}\left(\frac{L}{D}\right) \qquad (1)$$

$$\frac{dV}{dt} = -\frac{D}{m} = -\frac{\rho V^2}{2\beta} \qquad (2)$$

Divide (2) by (1)

$$\frac{\frac{dV}{dt}}{V\frac{d\gamma}{dt}} = \frac{-\frac{\rho V^2}{2\beta}}{\frac{\rho V^2}{2\beta}\frac{L}{D}} \Rightarrow \frac{dV}{V} = -\frac{d\gamma}{L/D} \qquad (3)$$

$$\int_{V_e}^{V}\frac{dV}{V} = -\frac{1}{L/D}\int_{\gamma_e}^{\gamma} d\gamma$$

$$\ln\frac{V}{V_e} = \frac{-(\gamma-\gamma_e)}{L/D} \Rightarrow \frac{V}{V_e} = e^{-\frac{\gamma-\gamma_e}{L/D}} \qquad (4)$$

As before, $\frac{dh}{dt} = V\sin\gamma$ (5) $\rho = \rho_0 e^{-h/h_s}$ (6)

$$\frac{d\rho}{dt} = -\frac{\rho_0}{h_s}e^{-h/h_s}\frac{dh}{dt} = \frac{-\rho}{h_s}\frac{dh}{dt} \Rightarrow \frac{d\rho}{dt} = -\frac{\rho}{h_s}V\sin\gamma \qquad (7)$$

Solve for V $\qquad V = \frac{-h_s}{\sin\gamma}\left(\frac{1}{\rho}\frac{d\rho}{dt}\right) \qquad (8)$

Substitute (8) → (1) which we rewrite as

$$\frac{d\gamma}{dt} = \frac{\rho V}{2\beta}\left(\frac{L}{D}\right) = \frac{\rho}{2\beta}\frac{L}{D}\left(-\frac{h_s}{\sin\gamma}\right)\frac{1}{\rho}\frac{d\rho}{dt}$$

$$d\gamma = -\frac{h_s}{2\beta \sin\gamma} \frac{L}{D} \, d\rho \qquad (9)$$

$$\int_{\gamma_e}^{\gamma} \sin\gamma \, d\gamma = -\frac{h_s}{2\beta} \frac{L}{D} \int_0^{\rho} d\rho \qquad (10)$$

$$\cos\gamma - \cos\gamma_e = +\frac{h_s}{2\beta} \frac{L}{D} \rho \qquad (11)$$

$$\left[\int \sin = -\cos \right]$$

$$\cos\gamma = \frac{h_s}{2\beta} \frac{L}{D} \rho + \cos\gamma_e$$

$$\gamma = -\cos^{-1}\left(\frac{h_s}{2\beta} \frac{L}{D} \rho + \cos\gamma_e \right) \qquad (12)$$

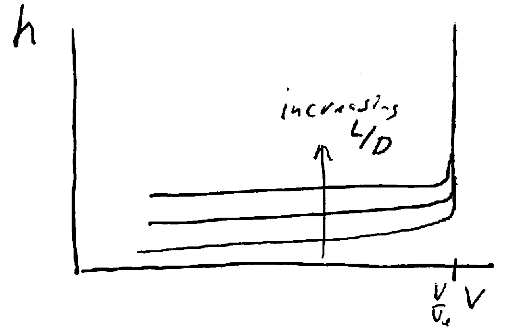
~~cos⁻¹ is ambiguous in terms of direction - We know flight path angle will be negative~~

We add a negative sign because the flight path angle is negative, and $\cos^{-1}$ is ambiguous about direction

$$\gamma = -\cos^{-1}\left(\frac{h_s}{2\beta} \frac{L}{D} \rho_0 e^{-h/h_s} + \cos\gamma_e \right) \qquad (13)$$

Rewrite (4) as $V = V_e \exp\left[-\frac{\gamma - \gamma_e}{L/D} \right] = V_e \exp\left[\frac{\gamma_e - \gamma}{L/D} \right]$
and substitute in (13)

$$\boxed{V = V_e \exp\left\{ \left(\frac{L}{D}\right)^{-1} \left[\gamma_e + \cos^{-1}\left(\frac{h_s}{2\beta} \frac{L}{D} \rho_0 e^{-h/h_s} + \cos\gamma_e \right) \right] \right\}} \qquad (14)$$



(See slides)

Find peak deceleration

- along flight path $\left| \frac{D}{m} \right| = \frac{\rho V^2}{2\beta}$

- perpendicular to flight path $\left| \frac{L}{m} \right| = \frac{\rho V^2}{2\beta} \left(\frac{L}{D} \right)$

(worried only about magnitude)

Total deceleration

$$n = \sqrt{\left(\frac{D}{m}\right)^2 + \left(\frac{L}{m}\right)^2} = \frac{1}{m}\sqrt{D^2+L^2} = \sqrt{\frac{\rho V^2}{2\beta}\, 1+\left(\frac{L}{D}\right)^2}$$

$$\boxed{n = \frac{\rho_0 V^2}{2\beta}\sqrt{1+\left(\frac{L}{D}\right)^2}\, e^{-h/h_s}} \qquad (15)$$

Plug (14) into (15), but first rewrite (15) as

$$n = \frac{\rho_0}{2\beta}\sqrt{1+\left(\frac{L}{D}\right)^2}\, e^{-h/h_s}\, V^2$$

from $\{\}^2$

$$n = \frac{\rho_0 V_e^2}{2\beta}\sqrt{1+\left(\frac{L}{D}\right)^2}\, e^{-h/h_s} \exp\left\{\frac{2}{L/D}\left[\gamma_e + \cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho_0 e^{-h/h_s} + \cos\gamma_e\right)\right]\right\}$$

Let the quantity in $[\,] \equiv X$

$$n = \frac{\rho_0 V_e^2}{2\beta}\sqrt{1+\left(\frac{L}{D}\right)^2}\, e^{-h/h_s}\, e^{\frac{2X}{L/D}} \qquad (16)$$

and

$$X = \gamma_e + \cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho_0 e^{-h/h_s} + \cos\gamma_e\right) \qquad (17)$$

set $\frac{dn}{dh} = 0$

$$0 = \frac{\rho_0 V_e^2}{2\beta}\sqrt{1+\left(\frac{L}{D}\right)^2}\left[e^{-h/h_s}\, e^{\frac{2X}{L/D}}\, \frac{2}{L/D}\frac{dX}{dh} + \left(\frac{-1}{h_s}\right)e^{-h/h_s}\, e^{\frac{2X}{L/D}}\right] \qquad (18)$$

Factoring out common terms

$$\frac{1}{h_s} = \frac{2}{L/D}\frac{dX_m}{dh} \qquad (19)$$

From (13),

$$\gamma = -\cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho_o e^{-h/h_s} + \cos\gamma_e\right)$$

$$\chi = \gamma_e - \gamma$$

$$= -\cos^{-1}\left[\cos\gamma\right] + \gamma_e \qquad \text{let } Y = \cos\gamma$$

$$\chi = \gamma_e - \cos^{-1} Y \qquad (20)$$

Trig identity: $$\frac{d(\cos^{-1}u)}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad (21)$$

$$\frac{d\chi}{dh} = -\frac{d(\cos^{-1}Y)}{dh} = \frac{+1}{\sqrt{1-Y^2}}\frac{dY}{dh}$$

$$= \frac{1}{\sqrt{1-Y^2}}\frac{d(\cos\gamma)}{dh}$$

From (13), $$\frac{d\chi}{dh} = \frac{1}{\sqrt{1-(\cos^2\gamma)}}\frac{d}{dh}\left[\frac{h_s\rho_o}{2\beta}\frac{L}{D}e^{-h/h_s} + \cos\gamma_e\right] \quad (22)$$

$$= \frac{1}{\underbrace{\sqrt{1-\cos^2\gamma}}_{\sin\gamma}}\left[\frac{L}{D}\frac{h_s\rho_o}{2\beta}\left(-\frac{1}{h_s}\right)e^{-h/h_s}\right]$$

$$\frac{d\chi}{dh} = \frac{-\rho_o}{2\beta\sin\gamma}\frac{L}{D}e^{-h/h_s} \qquad (23)$$

If we go back to n_{max} case, (19) gives

$$\frac{1}{h_s} = \frac{2}{L/D}\frac{-\rho_o}{2\beta\sin\gamma_m}\frac{L}{D}e^{-\frac{h_m}{h_s}} \qquad (24) \quad h_m, \gamma_m = \text{at } n_{max}$$

$$\sin\gamma_m = \frac{-h_s\rho_o}{\beta}e^{-h_m/h_s} \qquad (25)$$

$$\cos\gamma_m = \frac{4\cos\gamma_e \pm \sqrt{16\cos^2\gamma_e + [4+(\frac{L}{D})^2](\frac{L}{D})^2}}{4+(\frac{L}{D})^2}$$

$$\cos\gamma = \frac{h_s}{2\beta}\frac{L}{D}\rho_0 e^{-h/h_s} + \cos\gamma_e \qquad \frac{L}{D}=0 \Rightarrow \cos\gamma \equiv \cos\gamma_e$$

(25) gives $\quad \sin\gamma_m = \frac{-h_s\rho_0}{\beta} e^{-h_m/h_s} \qquad e^{-h_m/h_s} = \frac{-\beta\sin\gamma_m}{h_s\rho_0}$

$$\cos\gamma - \cos\gamma_e = \frac{h_s}{2\beta}\frac{L}{D}\rho_0 e^{-h/h_s}$$

$$\cos\gamma_m - \cos\gamma_e = -\frac{\cancel{h_s}}{2\cancel{\beta}}\frac{L}{D}\cancel{\rho_0}\frac{\cancel{\beta}\sin\gamma_m}{\cancel{h_s\rho_0}}$$

$$\cos\gamma_m - \cos\gamma_e = -\frac{1}{2}\frac{L}{D}\sin\gamma_m$$

$$(\cos\gamma_m - \cos\gamma_e)^2 = \frac{1}{4}\left(\frac{L}{D}\right)^2\sin^2\gamma_m$$

$$4(\cos^2\gamma_m - 2\cos\gamma_m\cos\gamma_e + \cos^2\gamma_e) = \left(\frac{L}{D}\right)^2(1-\cos^2\gamma_m)$$
$$= \left(\frac{L}{D}\right)^2 - \left(\frac{L}{D}\right)^2\cos^2\gamma_m$$

$$\left[4+\left(\frac{L}{D}\right)^2\right]\cos^2\gamma_m - [8\cos\gamma_e]\cos\gamma_m + \left[4\cos^2\gamma_e - \left(\frac{L}{D}\right)^2\right] = 0$$

$$\cos\gamma_m = \frac{8\cos\gamma_e \pm \sqrt{64\cos^2\gamma_e - 4[4+(\frac{L}{D})^2][4\cos^2\gamma_e - (\frac{L}{D})^2]}}{2[4+(\frac{L}{D})^2]}$$

$$= \frac{4\cos\gamma_e \pm \sqrt{16\cos^2\gamma_e - [4+(\frac{L}{D})^2][4\cos^2\gamma_e - (\frac{L}{D})^2]}}{4+(\frac{L}{D})^2}$$

$$= \frac{4}{4+(\frac{L}{D})^2}\cos\gamma_e + \frac{1}{4+(\frac{L}{D})^2}\sqrt{\cancel{16\cos^2\gamma_e} - [\cancel{16\cos^2\gamma_e} - 4(\tfrac{L}{D})^2 + 4\cos^2\gamma_e(\tfrac{L}{D})^2 - (\tfrac{L}{D})^4]}$$

$$= \frac{1}{4+(\frac{L}{D})^2}\left\{4\cos\gamma_e + \sqrt{4\left(\frac{L}{D}\right)^2 - 4\cos^2\gamma_e\left(\frac{L}{D}\right)^2 + \left(\frac{L}{D}\right)^4}\right\}$$

$$= \frac{1}{4+(\frac{L}{D})^2}\left[4\cos\gamma_e + \frac{L}{D}\sqrt{4 - 4\cos^2\gamma_e + \left(\frac{L}{D}\right)^2}\right]$$

$$\boxed{\cos \gamma_m = \frac{4\cos\gamma_e + \frac{L}{D}\sqrt{4\sin^2\gamma_e + \left(\frac{L}{D}\right)^2}}{4 + \left(\frac{L}{D}\right)^2}}$$

$\frac{L}{D} = 0 \quad \gamma_m = \gamma_e$ ✓

$$e^{-h_m/h_s} = \frac{-\beta \sin\gamma_m}{h_s \rho_o}$$

$$= -\frac{\beta}{h_s \rho_o}\left\{1 - \left[\frac{4\cos\gamma_e + \frac{L}{D}\sqrt{4\sin^2\gamma_e + \left(\frac{L}{D}\right)^2}}{4 + \left(\frac{L}{D}\right)^2}\right]^2\right\}^{1/2}$$

$\hookrightarrow V_m, \quad n_m$

$$V_m = V_e \, e^{-\frac{(\gamma_m - \gamma_e)}{L/D}}$$

$$n_m = \frac{\rho_o V_m^2}{2\beta}\sqrt{1 + \left(\frac{L}{D}\right)^2}\; e^{-h_m/h_s}$$