Phugoid Oscillations

Dynamic response to perturbation while in shallow entry glide?

$$V\frac{d\gamma}{dt} = \frac{L}{m} + \left(\frac{V^2}{r} - g\right)\cos\gamma \qquad (\Sigma \text{ forces} \perp \text{ to } \bar{V})$$

Assume $\gamma \ll 1$ ($\sin\gamma \cong \gamma$, $\cos\gamma \cong 1$)
$\dot{\gamma}_0 = 0$ (equilibrium glide)
neglect drag

$$V\dot{\gamma} = \frac{L}{m} + \left(\frac{V^2}{r} - g\right)$$

$$\frac{dh}{dt} = \dot{h} = V\sin\gamma \cong V\gamma$$

Assume small perturbations

$$\gamma = \gamma_1 + \Delta\gamma \qquad \dot{h} = \dot{h}_1 + \Delta\dot{h}$$

$$\dot{h} = (V_1 + \Delta V)(\gamma_1 + \Delta\gamma) = \underbrace{V_1\gamma_1}_{\dot{h}_1} + \underbrace{V_1\Delta\gamma}_{\Delta\dot{h}} + \cancel{\Delta V\gamma_1}^{HOT} + \cancel{\Delta V\Delta\gamma}^{HOT}$$

$$\Delta\dot{h} = V_1\Delta\gamma$$

Neglecting drag $\Rightarrow$ $V \cong$ constant

$$V\gamma = \dot{h} \Rightarrow V\dot{\gamma} = \ddot{h}$$

$$\frac{L}{m} = \frac{L_1 + \Delta L}{m} = \frac{V^2 A C_L}{2m}\rho_0 e^{-\frac{(h_1+\Delta h)}{h_s}} = \frac{V^2 A C_L}{2m}\rho_0 e^{-h_1/h_s} e^{-\Delta h/h_s} = \frac{L_1}{m} e^{-\Delta h/h_s}$$

using Taylor's series expansion, $\frac{L}{m} \cong \frac{L_1}{m}\left(1 - \frac{\Delta h}{h_s}\right)$

Perturbed lift: $V\Delta\dot{\gamma} = \frac{\Delta L}{m} = \Delta\ddot{h}$ $\quad \frac{\Delta L}{m} = -\frac{L_1}{m}\frac{\Delta h}{h_s}$

On equilibrium glide, $\dot{\gamma} = 0 \Rightarrow \frac{L}{m} = g - \frac{V^2}{r}$

$$\frac{L_1}{mg} = 1 - \frac{V_1^2}{gr} = 1 - \frac{V_1^2}{V_c^2}$$

$$\frac{L}{mg} = \frac{L_1}{mg} + \frac{\Delta L}{mg} = \left(1 - \frac{V_1^2}{V_c^2}\right) + \frac{\Delta L}{mg}$$

$$\Delta \ddot{h} = \frac{\Delta L}{m} = \frac{-L_1}{m}\frac{\Delta h}{h_s} = -\left(1+\frac{V_1^2}{V_c^2}\right) g \frac{\Delta h}{h_s}$$

$$\boxed{\Delta \ddot{h} + \left(1-\frac{V_1^2}{V_c^2}\right)\frac{g}{h_s}\,\Delta h = 0}$$ $\Leftarrow$ simple harmonic motion (undamped!)

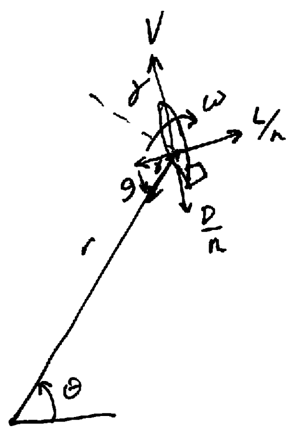
frequency: $\omega^2 = \left(1-\frac{V_1^2}{V_c^2}\right)\frac{g}{h_s}$

Period: $P = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\left(1+\frac{V_1^2}{V_c^2}\right)\frac{g}{h_s}}} \cong \frac{169\ \text{sec}}{\sqrt{1-\frac{V_1^2}{V_c^2}}}$

$V_c \sim 8000$ m/sec $\rightarrow$

V_1	P
7750 m/sec	11m 21s
6000 m/sec	4m 15s
4000 m/sec	3m 15s
2000 m/sec	2m 55s

Planar state equations



$$\omega = \dot{\gamma} - \dot{\theta}$$

$$\frac{L}{m} - g\cos\gamma = \omega V$$

$$-\frac{D}{m} - g\sin\gamma = \dot{V}$$

$$\dot{r} = \dot{h} = V\sin\gamma$$

$$r\dot{\theta} = V\cos\gamma$$

$$\omega = \dot{\gamma} - \dot{\theta} = \dot{\gamma} - \frac{V}{r}\cos\gamma$$

$$\frac{L}{m} - g\cos\gamma = \left(\dot{\gamma} - \frac{V}{r}\cos\gamma\right)V$$

$$\frac{L}{m} - \left(g - \frac{V^2}{r}\right)\cos\gamma = \dot{\gamma}V$$

$$\frac{L}{m} - \left(1 - \frac{V^2}{rg}\right)g\cos\gamma = \dot{\gamma}V$$

$$\boxed{\begin{aligned} V\dot{\gamma} &= \frac{L}{m} - \left(1 - \frac{V^2}{V_c^2}\right)g\cos\gamma \\ \dot{V} &= -\frac{D}{m} - g\sin\gamma \\ \dot{r} = \dot{h} &= V\sin\gamma \\ r\dot{\theta} &= V\cos\gamma \end{aligned}}$$

State Equations -
Coupled first order ODEs

$$\frac{L}{m} = \frac{1}{2}\frac{\rho V^2 A C_L}{m} = \frac{\rho V^2}{2}\frac{AC_D}{m}\frac{C_L}{C_D} = \frac{\rho V^2}{2\beta}\frac{L}{D}$$

$$\frac{D}{m} = \frac{1}{2}\frac{\rho V^2 A C_D}{m} = \frac{\rho V^2}{2\beta}$$

$$\rho = \rho_0 e^{-h/h_s}$$

$$h = r - r_0 \qquad g = g_0\left(\frac{r_0}{r}\right)^2$$

Numerical Integration - 4^{th} order Runge-Kutta

Given a series of equations

$$\dot{\bar{y}} = \bar{f}(t, \bar{y})$$

$$\bar{k}_1 = \Delta t \, \bar{f}(t_n, \bar{y}_n)$$

$$\bar{k}_2 = \Delta t \, \bar{f}\left(t_n + \frac{\Delta t}{2}, \bar{y}_n + \frac{\bar{k}_1}{2}\right)$$

$$\bar{k}_3 = \Delta t \, \bar{f}\left(t_n + \frac{\Delta t}{2}, \bar{y}_n + \frac{\bar{k}_2}{2}\right)$$

$$\bar{k}_4 = \Delta t \, \bar{f}(t_n + \Delta t, \bar{y}_n + \bar{k}_3)$$

$$\bar{y}_{n+1} = \bar{y}_n + \frac{\bar{k}_1}{6} + \frac{\bar{k}_2}{3} + \frac{\bar{k}_3}{3} + \frac{\bar{k}_4}{6} + O(\Delta t^5)$$