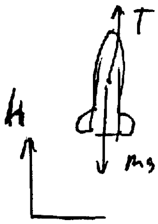
Sounding Rocket
No atmosphere
Vertical Trajectory




$$\frac{dh}{dt} = V$$

$$m\frac{dV}{dt} = T - mg \qquad T = \dot{m} V_e, \qquad \dot{m} = -\frac{dm}{dt}$$

$$m\frac{dV}{dt} = -\frac{dm}{dt}V_e - mg$$

$$dV = -\frac{dm}{m}V_e - g$$

$$\int_0^V dV = -g\int_0^t dt - V_e\int_{m_0}^{m}\frac{dm}{m}$$

Assume Thrust const $\Rightarrow \dot{m}$ const

$$m = m_0 - \dot{m}t \quad \Rightarrow \quad t = \frac{m_0 - m}{\dot{m}}$$

$$V = \underbrace{-gt}_{\text{gravity loss}} - V_e \ln\frac{m}{m_0} + V_0 \quad = \frac{dh}{dt}$$

$$\frac{dh}{dt} = -gt - V_e \ln\frac{m_0 - \dot{m}t}{m_0}$$

$$\int_0^h dh = -g\int_0^t t\,dt - V_e\int_0^t \ln\frac{m_0 - \dot{m}t}{m_0}\,dt$$

$u = \dfrac{m_0 - \dot{m}t}{m_0}$

$du = -\dfrac{\dot{m}}{m_0}dt$

$$h = -\frac{1}{2}gt^2 - V_e\int \ln u\left(-\frac{m_0}{\dot{m}}\right)du$$

$$+\frac{V_e m_0}{\dot{m}}\left[u\ln u - u\right]$$

$$= \frac{V_e m_0}{\dot{m}}\left[\frac{m_0 - \dot{m}t}{m_0}\ln\frac{m_0 - \dot{m}t}{m_0} - \frac{m_0 - \dot{m}t}{m_0} + 1\right]$$

$t = \dfrac{m_0 - m}{\dot{m}}$

$$h = -\frac{1}{2} g \left(\frac{m_0 - m}{\dot m}\right)^2 + \frac{V_e m_0}{\dot m}\left(\frac{m}{m_0} \ln \frac{m}{m_0} - \frac{m}{m_0} + 1\right)$$

Burnout conditions $\Rightarrow$ $V = V_{bo}$, $m = m_f$

$$\frac{\cancel{m_0} - \cancel{\dot m}\left(\frac{\cancel{m_0} - m}{\cancel{\dot m}}\right)}{m_0} = \frac{m}{m_0}$$

$$V_{bo} = -\frac{g}{\dot m}(m_0 - m_f) - V_e \ln \frac{m_f}{m_0} \quad \text{(assuming } V_0 = 0\text{)}$$

$$h_{bo} = -\frac{g}{2}\left(\frac{m_0 - m_f}{\dot m}\right)^2 + \frac{V_e}{\dot m}\left(m_f \ln \frac{m_f}{m_0} - m_f + m_0\right)$$

Coasting after burnout ($T = 0$, drag $= 0$)

Energy $E = \frac{1}{2} m_f V_{bo}^2 + m_f g h_{bo} = \text{const} = \frac{1}{2} m_f V^2 + m_f g h$

$$E = \frac{1}{2} m_f \left[-\frac{g}{\dot m}(m_0 - m_f) - V_e \ln \frac{m_f}{m_0}\right]^2 + m_f g \left[-\frac{g}{2}\left(\frac{m_0 - m_f}{\dot m}\right)^2 + \frac{V_e}{\dot m}\left(m_f \ln \frac{m_f}{m_0} - m_f + m_0\right)\right]$$

at max $h$ $\quad E = m_f g h_{max}$ $\quad$ ($V = 0$)

$$\cancel{m_f} g h_{max} = \frac{1}{2}\cancel{m_f}\left[-\frac{g}{\dot m}(m_0 - m_f) - V_e \ln \frac{m_f}{m_0}\right]^2 + \cancel{m_f} g \left[-\frac{g}{2}\left(\frac{m_0 - m_f}{\dot m}\right)^2 + \frac{V_e}{\dot m}\left(m_f \ln \frac{m_f}{m_0} - m_f + m_0\right)\right]$$

$$g h_{max} = \cancel{\frac{1}{2}\frac{g^2}{\dot m^2}(m_0 - m_f)^2} + \cancel{\frac{2}{2}}\frac{g V_e}{\dot m}(m_0 - m_f) \ln \frac{m_f}{m_0} + \frac{V_e^2}{2}\left(\ln \frac{m_f}{m_0}\right)^2$$

$$+ g\left[\cancel{-\frac{g}{2}\left(\frac{m_0 - m_f}{\dot m}\right)^2} + \frac{V_e}{\dot m}\left(m_f \ln \frac{m_f}{m_0} - m_f + m_0\right)\right]$$

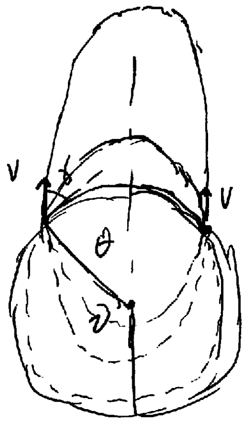
$$h_{max} = \frac{v_e}{\dot{m}}(m_0 - \cancel{m_f}) \ln \frac{m_f}{m_0} + \frac{v_e^2}{2g}\left(\ln \frac{m_f}{m_0}\right)^2$$

$$+ \cancel{\frac{v_e}{\dot{m}} m_f \ln \frac{m_f}{m_0}} + \frac{v_e}{\dot{m}}(m_0 - m_f)$$

$$= \frac{1}{2}\frac{v_e^2}{g}\left(\ln \frac{m_f}{m_0}\right)^2 + \frac{v_e}{\dot{m}} m_0 \ln \frac{m_f}{m_0} + \frac{v_e}{\dot{m}}(m_0 - m_f)$$

want high $v_e$, low $r \equiv \frac{m_f}{m_0}$, high $\dot{m}$ (short $t_b$)

Ballistic Travel




Travel along great circle distance of $2\theta$

Velocity on suborbital arc $= V = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$

Total $\Delta V = 2V$

Find orbit with minimum $a$ that connects start + finish

$$r = \frac{p}{1 + e\cos\nu} \qquad \nu = \pi - \theta \qquad \cos\nu = -\cos\theta$$

$$= \frac{a(1-e^2)}{1 - e\cos\theta} \qquad h = rv\cos\gamma$$

$$a = \frac{r(1 - e\cos\theta)}{1-e^2} \qquad = r\sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}\cos\gamma$$

$$V = \mu\sqrt{\frac{2}{r} - \frac{1-e^2}{r(1-e\cos\theta)}}$$

$$-2e\cancel{r} + \cancel{2r}e^2\cos\theta + \cancel{r}\cos\theta \cancel{-re^2\cos\theta}$$

$$\frac{\partial V}{\partial e} = 0 \Rightarrow 0 = \frac{\mu}{2}\left[\frac{2}{r} - \frac{1-e^2}{r(1-e\cos\theta)}\right]^{-1/2}\left(-\frac{r(1-e\cos)(-2e) - (1-e^2)r(-\cos\theta)}{r^2(1-e\cos\theta)^2}\right)$$

$$\int \ln x = x\ln x - x + C$$

$$e^2\cos\theta - 2e + \cos\theta = 0$$

$$\frac{2}{r} - \frac{1-e^2}{r(1-e\cos\theta)} = 0$$

$\theta \to 0 \quad e \to 1$

$\theta \to \frac{\pi}{2} \quad e \to \frac{0}{0}$

$$\frac{1-e^2}{\cancel{r}(1-e\cos\theta)} = \frac{2}{\cancel{r}} \qquad 1 - e^2 = 2 - e\cos\theta$$

$$e^2 - e\cos\theta + 1 = 0$$

L'Hospital's rule

$$\frac{1-\sin\theta}{\cos\theta} = e \qquad \frac{-\cos\theta}{-\sin\theta} = 0$$

$$\frac{\cos\theta \pm \sqrt{\cos^2\theta - 4}}{2} \qquad \frac{1 \pm \sin\theta}{\cos\theta} = e$$

$$\frac{2 \pm \sqrt{4 - 4\cos^2\theta}}{2\cos\theta} \qquad \frac{\cancel{2} \pm \cancel{2}\sqrt{1-\cos^2\theta}}{\cancel{2}\cos\theta} = e$$

$$T = m\left(g - \frac{v^2}{r}\right) = \dot{m} c \qquad \dot{m} = \frac{m}{c}\left(g - \frac{v^2}{r}\right)$$

$$\frac{dm}{dt} = -\dot{m} = -\frac{m}{c}\left(g - \frac{v^2}{r}\right)$$

$$\frac{dm}{m} = -\frac{g - v^2/r}{c}\, dt$$

$$\ln \frac{m_f}{m_0} = -\frac{g - \frac{v^2}{r}}{c}\, t \qquad t = \frac{d}{V} \qquad gt$$

$$\frac{gd}{V} \qquad \left(g - \frac{v^2}{r}\right)\frac{d}{V}$$

$$= -\frac{gd}{cV} + \frac{Vd}{rc}$$

$$\ln \frac{m_{f_1}}{m_{0_1}} = -\frac{V}{c}$$

$$\frac{dg}{V}$$

$$\ln \frac{m_{f_3}}{m_{0_1}} = -\frac{2V}{c} - \frac{gd}{cV} + \frac{Vd}{rc}$$

$$\frac{\partial}{\partial V} = 0 \Rightarrow -\frac{2}{c} + \frac{gd}{cV^2} + \frac{d}{rc} = 0$$

$$\frac{gd}{\cancel{c}V^2} = \frac{2}{\cancel{c}} - \frac{d}{r\cancel{c}}$$

$$\frac{gd}{V^2} = 2 - \frac{d}{r} \qquad V^2 = \frac{gd}{2 - \frac{d}{r}}$$

$$V_{opt} = \sqrt{\frac{gd}{2 - \frac{d}{r}}} \quad \text{no centripetal} \quad \sqrt{\frac{gd}{2}}$$

$$F = mg = \frac{GMm}{r^2}$$

$$g = \frac{\mu}{r^2}$$