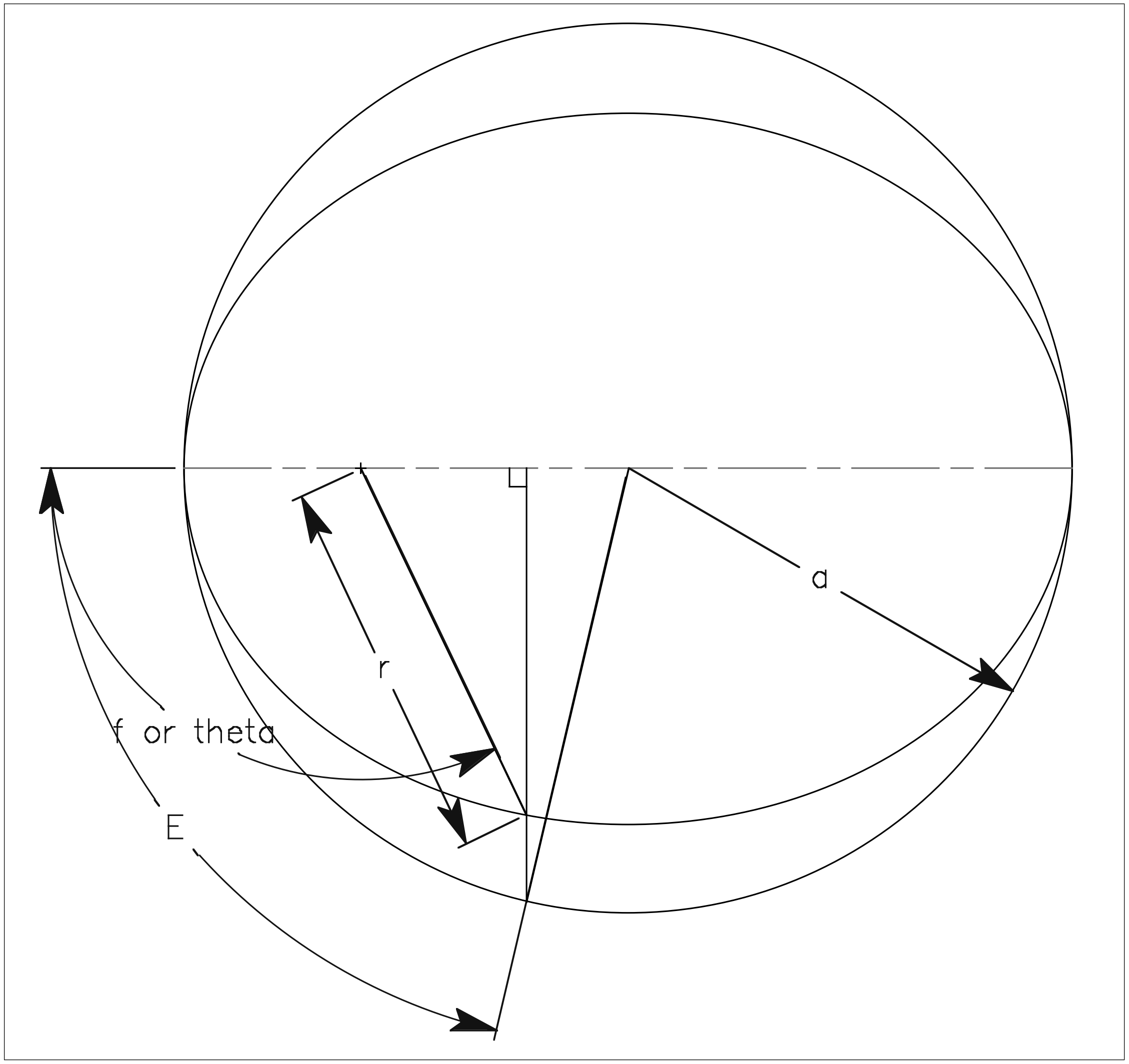
Orbital Mechanics

- Orbital Mechanics, continued
  - Time in orbits
  - Velocity components in orbit
  - Deorbit maneuvers
  - Atmospheric density models
  - Orbital decay (introduction)
- Fundamentals of Rocket Performance
  - The rocket equation
  - Mass ratio and performance
  - Structural and payload mass fractions
  - Multistaging
  - Optimal ΔV distribution between stages (introduction)

Calculating Time in Orbit



Time in Orbit

- Period of an orbit

$$P = 2\pi\sqrt{\frac{a^3}{\mu}}$$

- Mean motion (average angular velocity)

$$n = \sqrt{\frac{\mu}{a^3}}$$

- Time since pericenter passage

$$M = nt = E - e\sin E$$

➥M=mean anomaly
E=eccentric anomaly

Dealing with the Eccentric Anomaly

- Relationship to orbit

$$r = a(1 - e\cos E)$$

- Relationship to true anomaly

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

- Calculating M from time interval: iterate

$$E_{i+1} = nt + e\sin E_i$$

until it converges

Example: Time in Orbit

- Hohmann transfer from LEO to GEO
  - h_1=300 km --> r_1=6378+300=6678 km
  - r_2=42240 km
- Time of transit (1/2 orbital period)

$$a = \frac{1}{2}(r_1 + r_2) = 24,459km$$

$$t_{\text{transit}} = \frac{P}{2} = \pi\sqrt{\frac{a^3}{\mu}} = 19,034sec = 5h17m14s$$

Example: Time-based Position

Find the spacecraft position 3 hours after perigee

$$n = \sqrt{\frac{\mu}{a^3}} = 1.650 \times 10^{-4} \frac{rad}{sec}$$

$$e = 1 - \frac{r_p}{a} = 0.7270$$

$$E_{j+1} = nt + e \sin E_j = 1.783 + 0.7270 \sin e_j$$

E=0; 1.783; 2.494; 2.222; 2.361; 2.294; 2.328; 2.311; 2.320; 2.316; 2.318; 2.317; 2.317; 2.317

Example: Time-based Position (continued)

$$E = 2.317$$

$$r = a(1 + e\cos E) = 12,387km$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \Longrightarrow \theta = 160\deg$$

Have to be sure to get the position in the proper quadrant - since the time is less than 1/2 the period, the spacecraft has yet to reach apogee --> $0° < \theta < 180°$

Velocity Components in Orbit

$$r = \frac{p}{1 + e\cos\theta}$$

$$v_r = \frac{dr}{dt} = \frac{d}{dt}\left(\frac{p}{1 + e\cos\theta}\right) = \frac{-p\left(-e\sin\theta\frac{d\theta}{dt}\right)}{\left(1 + e\cos\theta\right)^2}$$

$$v_r = \frac{pe\sin\theta}{\left(1 + e\cos\theta\right)^2}\frac{d\theta}{dt}$$

$$1 + e\cos\theta = \frac{p}{r} \Longrightarrow v_r = \frac{r^2\frac{d\theta}{dt}e\sin\theta}{p}$$

$$\vec{h} = \vec{r} \times \vec{v}$$

Velocity Components in Orbit (continued)

$$\vec{h} = \vec{r} \times \vec{v} \qquad h = rv\cos\gamma = r\left(r\frac{d\theta}{dt}\right) = r^2\frac{d\theta}{dt}$$

$$v_r = \frac{r^2\frac{d\theta}{dt}e\sin\theta}{p} = \frac{he\sin\theta}{p} = \frac{\sqrt{p\mu}}{p}e\sin\theta$$

$$\boxed{v_r = \sqrt{\frac{\mu}{p}}e\sin\theta}$$

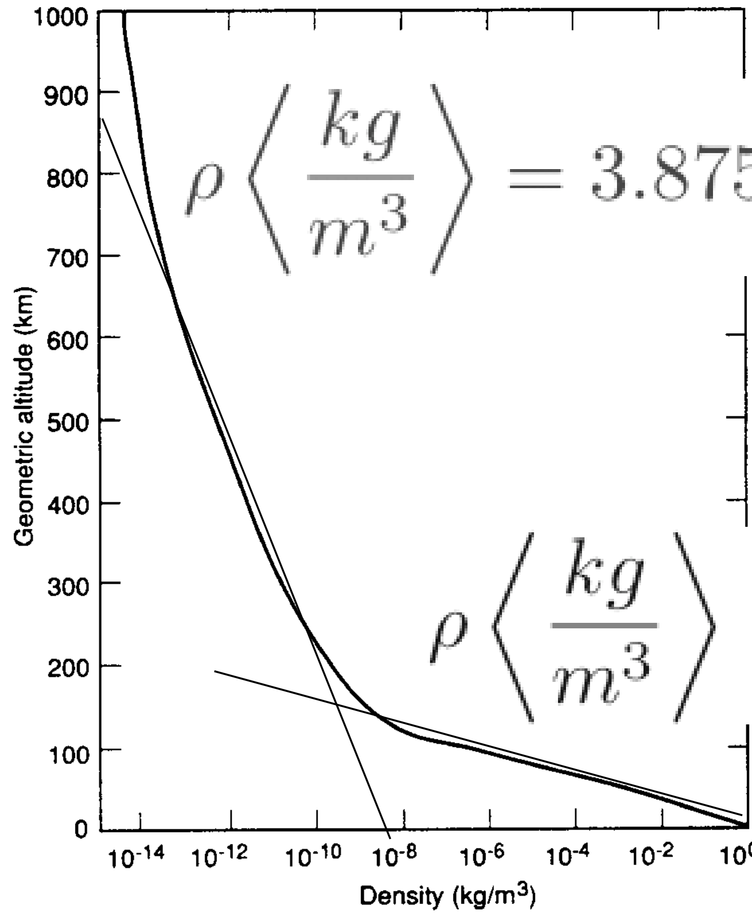
$$v_\theta = r\frac{d\theta}{dt} = r\frac{h}{r^2} = \frac{h}{r} = \frac{\sqrt{p\mu}}{r} \qquad \boxed{v_\theta = \sqrt{\frac{\mu}{p}}\left(1 + e\cos\theta\right)}$$

$$\boxed{\tan\gamma = \frac{v_r}{v_\theta} = \frac{e\sin\theta}{1 + e\cos\theta}}$$

Atmospheric Decay

- Entry Interface
  - Altitude where aerodynamic effects become significant (acc~0.05 *g*)
  - Typically 400,000 ft ~ 80 mi ~ 120 km
- Exponential Atmosphere
  - $\rho = \rho_o e^{-\frac{h}{h_{scale}}}$
  - Good for selected regions of atmosphere

Atmospheric Density with Altitude



$$\rho \left\langle \frac{kg}{m^3} \right\rangle = 3.875 \times 10^{-9} e^{-\frac{h \langle km \rangle}{59.06}}$$

$$\rho \left\langle \frac{kg}{m^3} \right\rangle = 1.226 e^{-\frac{h \langle km \rangle}{7.524}}$$

Ref: V. L. Pisacane and R. C. Moore, *Fundamentals of Space Systems* Oxford University Press, 1994

Acceleration Due to Atmospheric Drag

$$\text{drag} \equiv D = \frac{1}{2}\rho v^2 S c_D$$

$v =$ velocity; $\rho =$ density; $S =$ reference area; $c_D =$ drag coefficient

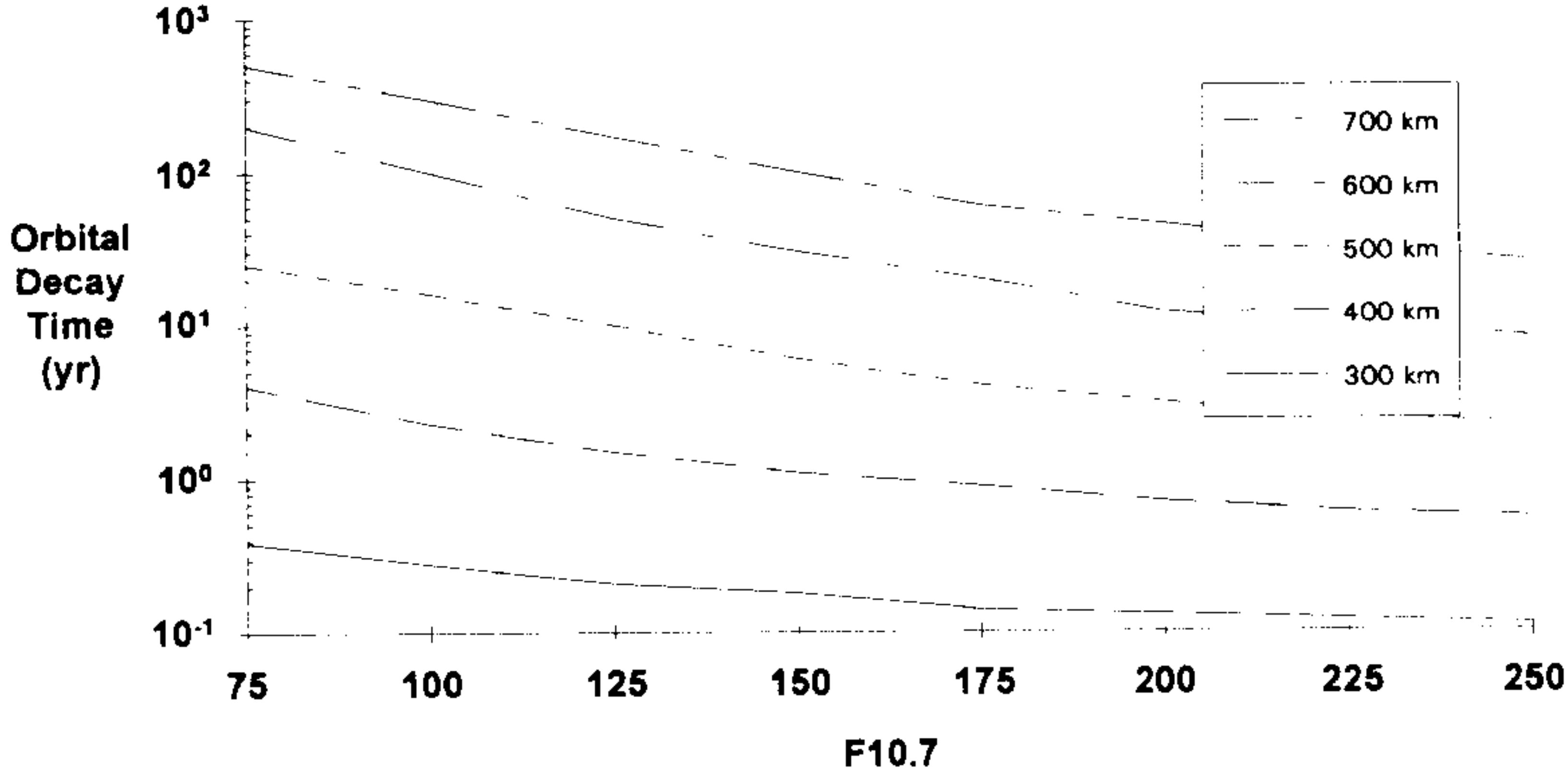
$$\text{acceleration due to drag} \equiv a_d = \frac{D}{m} = \frac{\rho v^2}{2}\frac{S c_D}{m}$$

$$\text{ballistic coefficient} \equiv \beta = \frac{m}{S c_D}$$

$$a_d = \frac{\rho v^2}{2\beta}$$

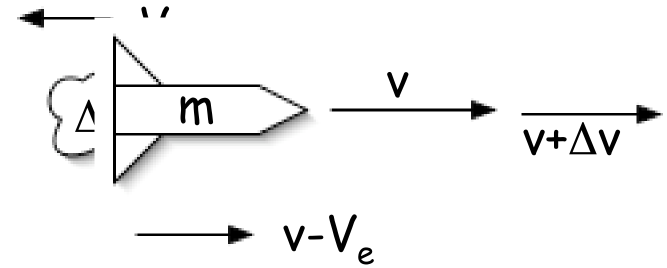
$$\text{dynamic pressure} \equiv q = \frac{\rho v^2}{2} \qquad a_d = \frac{q}{\beta}$$

Orbit Decay from Atmospheric Drag



Ref: Alan C. Tribble, *The Space Environment* Princeton University Press, 1995

Derivation of the Rocket Equation



- Momentum at time t:

$$M = mv$$

- Momentum at time t+Δt:

$$M = (m - \Delta m)(v + \Delta v) + \Delta m (v - V_e)$$

- Some algebraic manipulation gives:

$$m \Delta v = -\Delta m V_e$$

- Take to limits and integrate:

$$\int_{m_{initial}}^{m_{final}} \left(\frac{dm}{m} \right) = -\int_{V_{initial}}^{V_{final}} \left(\frac{dv}{V_e} \right)$$

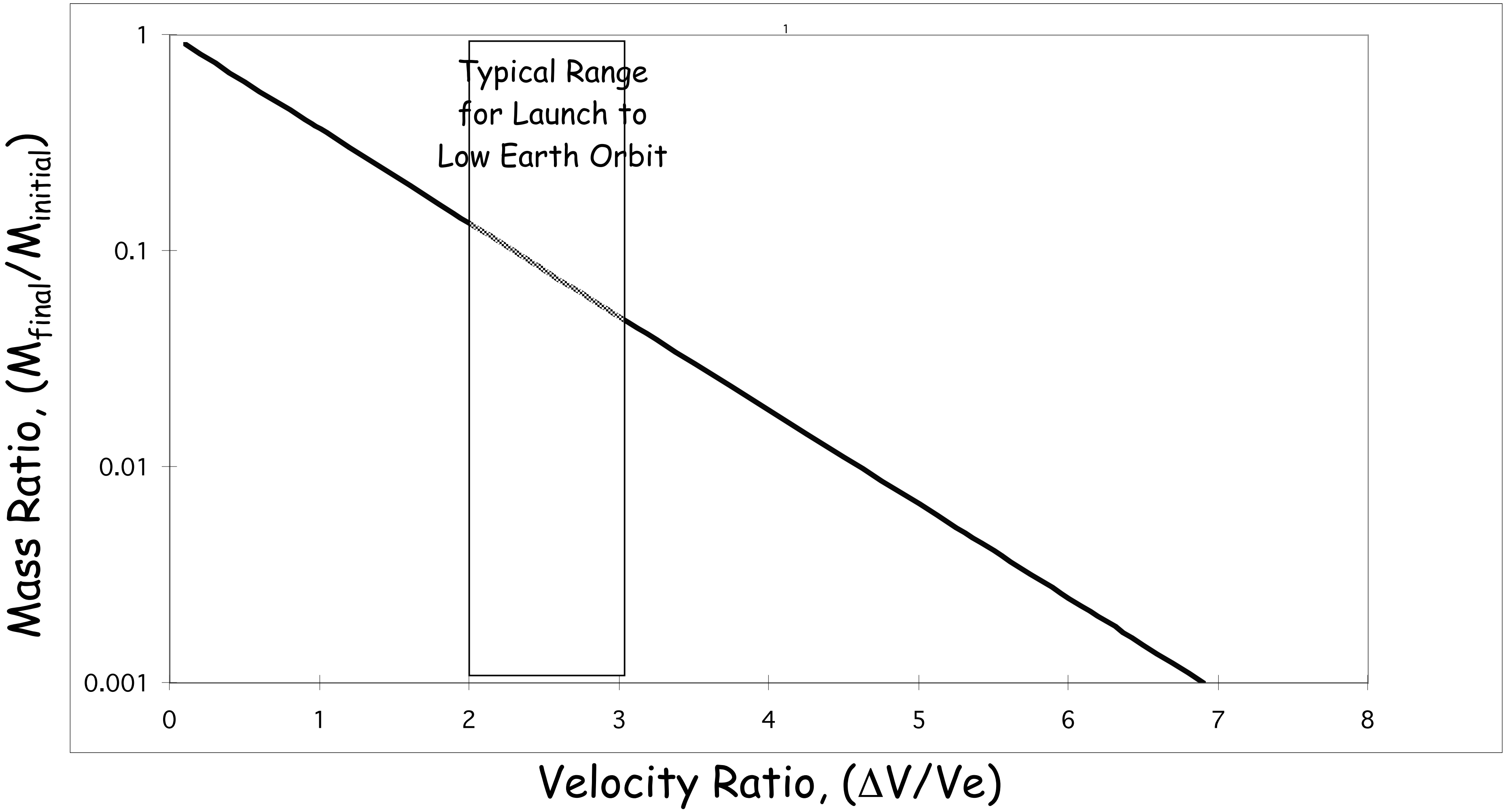
The Rocket Equation

- Alternate forms

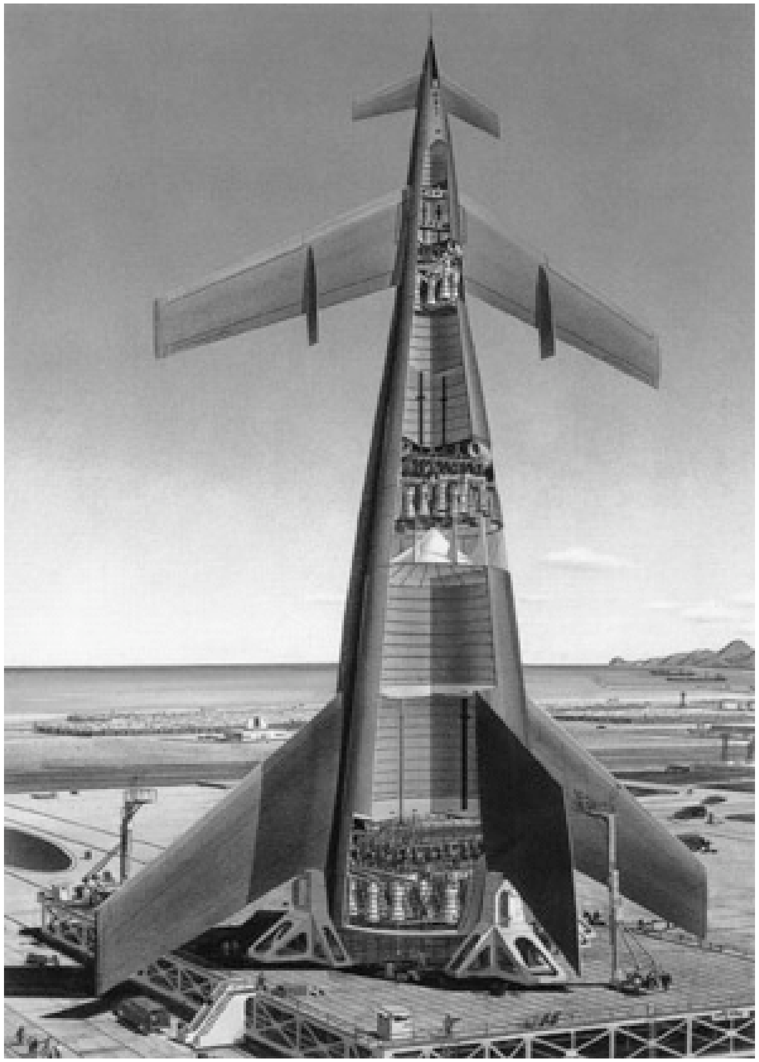
$$r \equiv \frac{m_{final}}{m_{initial}} = e^{-\frac{\Delta V}{V_e}} \quad \Delta V = -V_e \ln\left(\frac{m_{final}}{m_{initial}}\right) = -V_e \ln(r)$$

- Basic definitions/concepts
  - Mass ratio $r \equiv \frac{m_{final}}{m_{initial}}$ or $\Re = \frac{m_{initial}}{m_{final}}$
  - Nondimensional velocity change "Velocity ratio" $\frac{\Delta V}{V_e}$

Rocket Equation (First Look)



Sources and Categories of Vehicle Mass



- Payload
- Propellants
- Inert Mass
    - Structure
    - Propulsion
    - Avionics
    - Mechanisms
    - Thermal
    - Etc.

Basic Vehicle Parameters

- Basic mass summary

$$m_0 = m_L + m_p + m_i$$

$m_0 = initial\ mass$

$m_L = payload\ mass$

$m_p = propellant\ mass$

$m_i = inert\ mass$

- Inert mass fraction

$$\delta = \frac{m_i}{m_0} = \frac{m_i}{m_L + m_p + m_i}$$

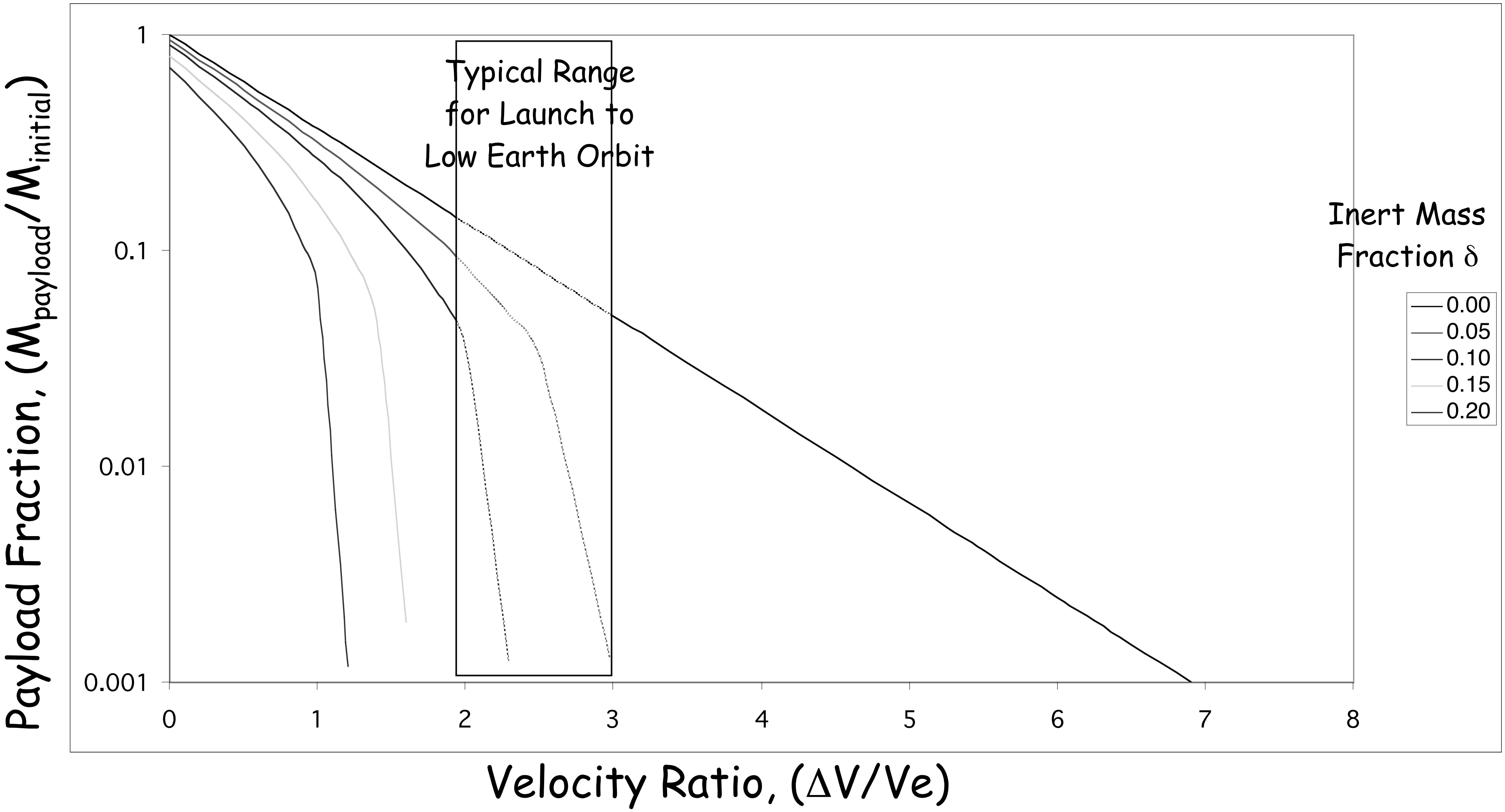
- Payload fraction

$$\lambda = \frac{m_L}{m_0} = \frac{m_L}{m_L + m_p + m_i}$$

- Parametric mass ratio

$$r = \lambda + \delta$$

Rocket Equation (including Inert Mass)



The Rocket Equation for Multiple Stages

- Assume two stages

$$\Delta V_1 = -V_{e,1} \ln\left(\frac{m_{final,1}}{m_{initial,1}}\right) = -V_{e,1} \ln(r_1)$$

$$\Delta V_2 = -V_{e,2} \ln\left(\frac{m_{final,2}}{m_{initial,2}}\right) = -V_{e,2} \ln(r_2)$$

- Assume $V_{e,1}=V_{e,2}=V_e$

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1) - V_e \ln(r_2) = -V_e \ln(r_1 r_2)$$

Continued Look at Multistaging

- Converting to masses

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1 r_2) = -V_e \ln\left(\frac{m_{final,1}}{m_{initial,1}} \frac{m_{final,2}}{m_{initial,2}}\right)$$

- Keep in mind that $m_{final,1} \sim m_{initial,2}$

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1 r_2) = -V_e \ln\left(\frac{m_{final,2}}{m_{initial,1}}\right) = -V_e \ln(r_0)$$

- r_0 has *no* physical significance!

Multistage Vehicle Parameters

- Inert mass fraction

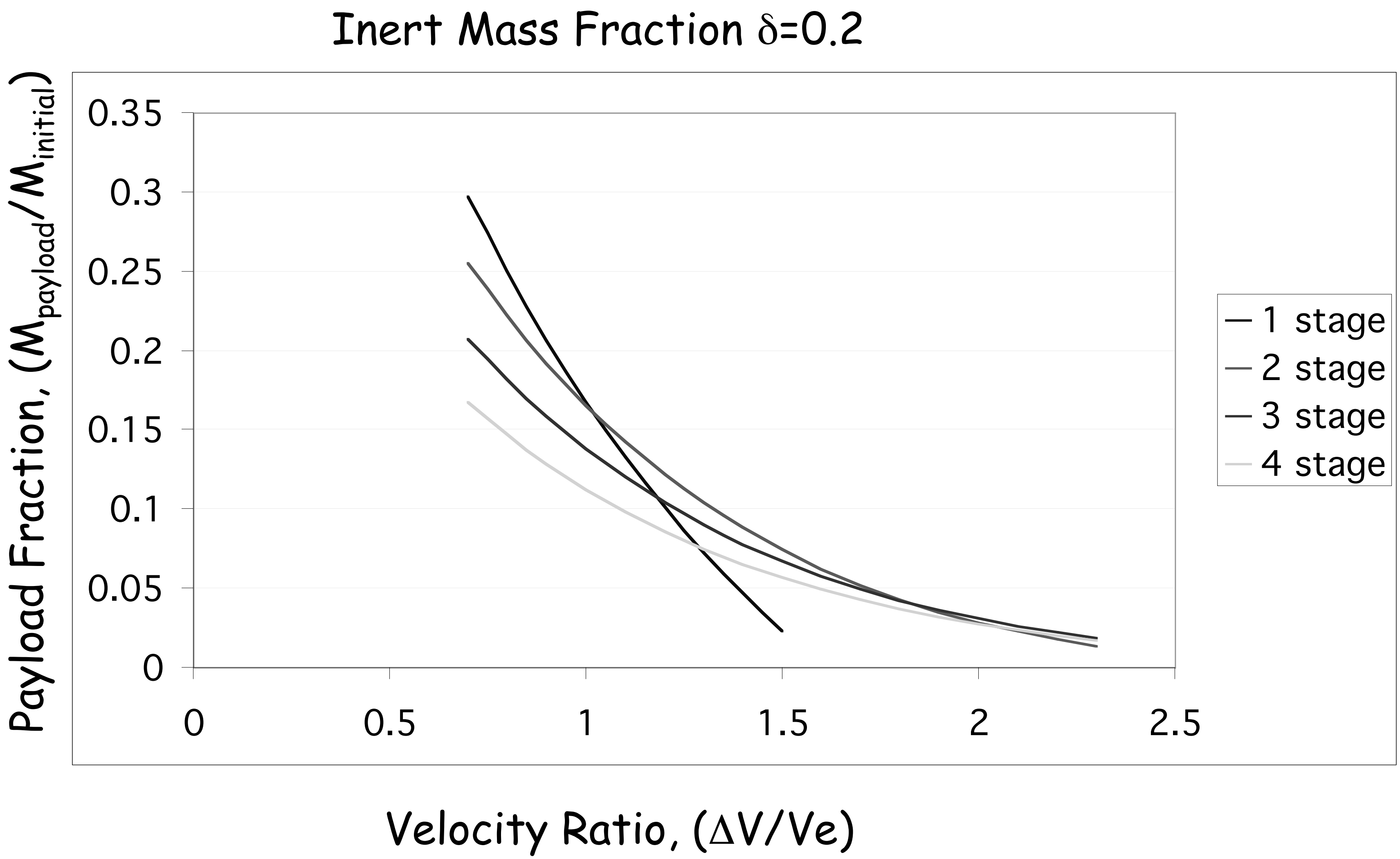
$$\delta_0 = \frac{\sum m_{i,j}}{m_0} = \sum_{j=1}^{n\ stages} \left(\delta_j \prod_{l=1}^{j-1} \lambda_l \right)$$

- Payload fraction

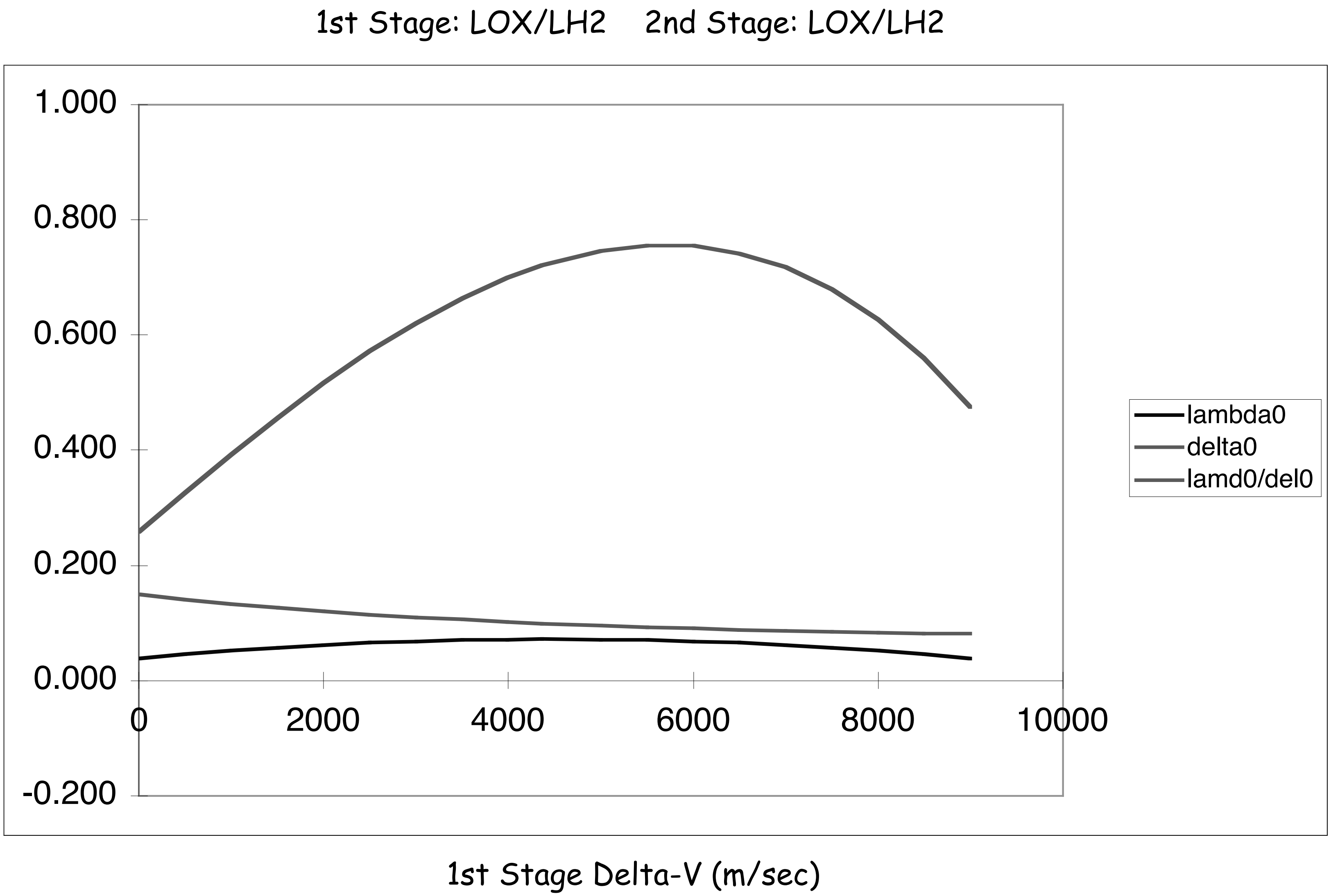
$$\lambda_0 = \frac{m_L}{m_0} = \prod_{i=1}^{n\ stages} \lambda_i$$

- Payload mass/inert mass ratio $\dfrac{\lambda_0}{\delta_0}$

Effect of Staging



Effect of ΔV Distribution



Lagrange Multipliers

- Given an objective function

$$y = f(x)$$

subject to constraint function

$$z = g(x)$$

- Create a new objective function

$$y = f(x) + \lambda\left[g(x) - z\right]$$

- Solve simultaneous equations

$$\frac{\partial y}{\partial x} = 0 \quad \frac{\partial y}{\partial \lambda} = 0$$

Optimum ΔV Distribution Between Stages

- Maximize payload fraction (2 stage case)

$$\lambda_0 = \lambda_1 \lambda_2 = (r_1 - \delta_1)(r_2 - \delta_2)$$

subject to constraint function

$$\Delta V_{total} = \Delta V_1 + \Delta V_2$$

- Create a new objective function

$$\lambda_0 = \left(e^{-\frac{\Delta V_1}{V_{e1}}} - \delta_1\right)\left(e^{-\frac{\Delta V_2}{V_{e2}}} - \delta_2\right) + K\left[\Delta V_1 + \Delta V_2 - \Delta V_{Total}\right]$$

➥Very messy for partial derivatives!

Optimum ΔV Distribution (continued)

- Use substitute objective function

$$\max(\lambda_0) \Leftrightarrow \max\left[\ln(\lambda_0)\right]$$

- Create a new constrained objective function

$$\ln(\lambda_0) = \ln(r_1 - \delta_1) + \ln(r_2 - \delta_2)$$

$$+K\left[\Delta V_{Total} + V_{e1}\ln(r_1) + V_{e2}\ln(r_2)\right]$$

- Take partials and set equal to zero

$$\frac{\partial \ln(\lambda_0)}{\partial r_1} = 0 \quad \frac{\partial \ln(\lambda_0)}{\partial r_2} = 0 \quad \frac{\partial \ln(\lambda_0)}{\partial K} = 0$$

Optimum ΔV Special Cases

- "Generic" partial of objective function

$$\frac{\partial \ln(\lambda_0)}{\partial r_i} = \frac{1}{r_i - \delta_i} + K\frac{V_{ei}}{r_i} = 0$$

- Case 1: $\delta_1 = \delta_2$ $V_{e1} = V_{e2}$

$$r_1 = r_2 \Rightarrow \Delta V_1 = \Delta V_2$$

- Case 2: $\delta_1 \neq \delta_2$ $V_{e1} = V_{e2}$

$$\frac{r_1}{\delta_1} = \frac{r_2}{\delta_2}$$

- More complex cases have to be done numerically

Sensitivity to Inert Mass

ΔV for multistaged rocket

$$\Delta V_{tot} = \sum_{k=1}^{n\ stages} \Delta V_k = \sum_{k=1}^{n} V_{e,k} \ln\left(\frac{m_{o,k}}{m_{f,k}}\right)$$

$$m_{o,k} = m_L + m_{p,k} + m_{i,k} + \sum_{j=k+1}^{n} m_{p,j} + m_{i,j}$$

$$m_{f,k} = m_L + m_{i,k} + \sum_{j=k+1}^{n} m_{p,j} + m_{i,j}$$

$$\frac{\partial \Delta V_{tot}}{\partial m_L} dm_L + \frac{\partial \Delta V_{tot}}{\partial m_{i,j}} dm_{i,j} = 0$$

Trade-off Ratio: Payload<->Inert Mass

$$\left.\frac{\partial m_L}{\partial m_{i,k}}\right|_{\partial \Delta V_{Total}=0} = \frac{-\sum_{j=1}^{k} V_{e,j}\left(\frac{1}{m_{0,j}} - \frac{1}{m_{f,j}}\right)}{\sum_{\ell=1}^{N} V_{e,\ell}\left(\frac{1}{m_{0,\ell}} - \frac{1}{m_{f,\ell}}\right)}$$

Trade-off Ratio : Payload<->Propellant

$$\left.\frac{\partial m_L}{\partial m_{p,k}}\right|_{\partial \Delta V_{Total}=0} = \frac{-\sum_{j=1}^{k} V_{e,j}\left(\frac{1}{m_{0,j}}\right)}{\sum_{\ell=1}^{N} V_{e,\ell}\left(\frac{1}{m_{0,\ell}} - \frac{1}{m_{f,\ell}}\right)}$$

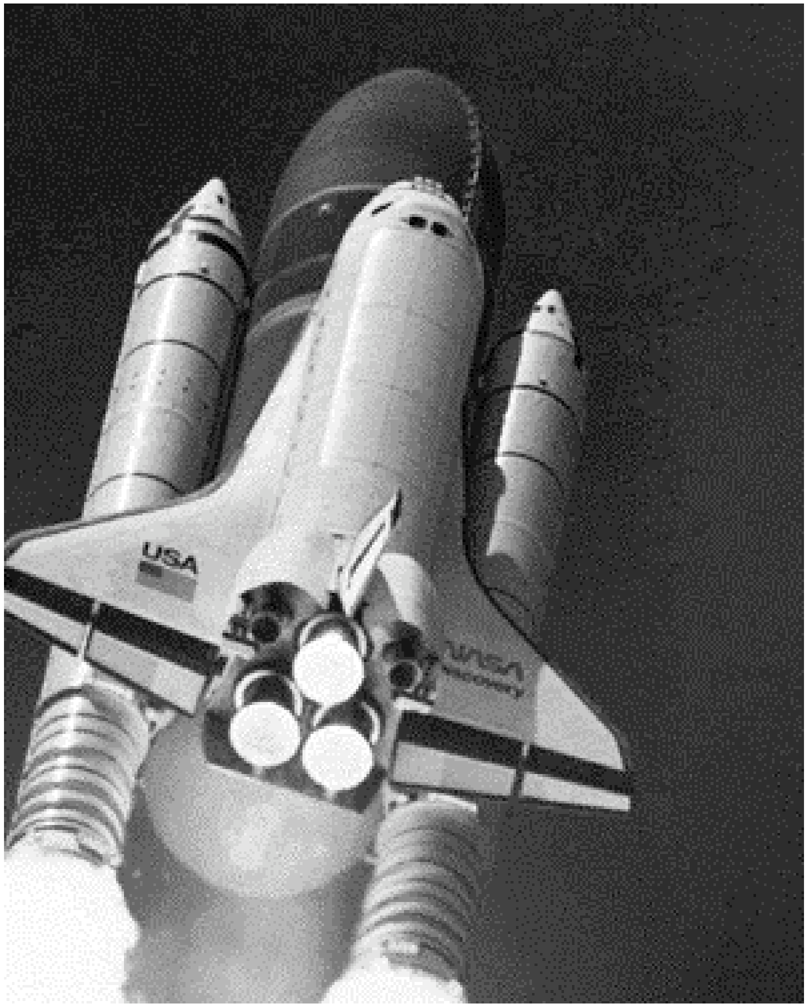
Trade-off Ratio: Payload<->Exhaust Velocity

$$\left.\frac{\partial m_L}{\partial V_{e,k}}\right|_{\partial \Delta V_{Total}=0} = \frac{\displaystyle\sum_{j=1}^{k} \ln\left(\frac{m_{0,k}}{m_{f,k}}\right)}{\displaystyle\sum_{\ell=1}^{N} V_{e,\ell}\left(\frac{1}{m_{0,\ell}} - \frac{1}{m_{f,\ell}}\right)}$$

Trade-off Ratio Example: Gemini-Titan II

	Stage 1	Stage 2
Initial Mass (kg)	150,500	32,630
Final Mass (kg)	39,370	6099
Ve (m/sec)	2900	3097
$dm_L/dm_{i,j}$	-0.1164	-1
$dm_L/dm_{p,j}$	0.04124	0.2443
$dm_L/dV_{e,j}$ (kg/m/sec)	2.870	6.459

Parallel Staging



- Multiple dissimilar engines burning simultaneously
- Frequently a result of upgrades to operational systems
- General case requires "brute force" numerical performance analysis

Parallel-Staging Rocket Equation

- Momentum at time t:

$$M = mv$$

- Momentum at time t+Δt:
(subscript "b"=boosters; "c"=core vehicle)

- Assume thrust (and mass flow rates) constant

Parallel-Staging Rocket Equation

- Rocket equation during booster burn

$$\Delta V = -\overline{V}_e \ln\left(\frac{m_{final}}{m_{initial}}\right) = -\overline{V}_e \ln\left(\frac{m_{i,b} + m_{i,c} + \chi m_{p,c} + m_{o,2}}{m_{i,b} + m_{p,b} + m_{i,c} + m_{p,c} + m_{o,2}}\right)$$

- where

$$\overline{V}_e = \frac{V_{e,b}\dot{m}_b + V_{e,c}\dot{m}_c}{\dot{m}_b + \dot{m}_c} = \frac{V_{e,b}m_{p,b} + V_{e,c}(1-\chi)m_{p,c}}{m_{p,b} + (1-\chi)m_{p,c}}$$

Analyzing Parallel-Staging Performance

Parallel stages break down into pseudo-serial stages:

- Stage "0" (boosters and core)

$$\Delta V_0 = -\overline{V}_e \ln\left(\frac{m_{i,b} + m_{i,c} + \chi m_{p,c} + m_{o,2}}{m_{i,b} + m_{p,b} + m_{i,c} + m_{p,c} + m_{o,2}}\right)$$

- Stage "1" (core alone)

$$\Delta V_1 = -V_{e,c} \ln\left(\frac{m_{i,c} + m_{o,2}}{m_{i,c} + \chi m_{p,c} + m_{o,2}}\right)$$

- Subsequent stages are as before

Modular Staging

- Use identical modules to form multiple stages
- Have to cluster modules on lower stages to make up for nonideal ΔV distributions
- Advantageous from production and development cost standpoints

Module Analysis

- All modules have the same inert mass and propellant mass
- Because δ varies with payload mass, not all modules have the same δ!
- Introduce two new parameters

$$\varepsilon = \frac{m_i}{m_i + m_p} \qquad \sigma = \frac{m_i}{m_p}$$

- Conversions $\varepsilon = \frac{\delta}{1-\lambda} \qquad \sigma = \frac{\delta}{1-\delta-\lambda}$

Rocket Equation for Modular Boosters

- Assuming n modules in stage 1,

$$r_1 = \frac{n(m_i) + m_{o,2}}{n(m_i + m_p) + m_{o,2}} = \frac{n\varepsilon + \dfrac{m_{o,2}}{m_{o,\mathrm{mod}}}}{n + \dfrac{m_{o,2}}{m_{o,\mathrm{mod}}}}$$

- If all 3 stages use same modules, n_j for stage j,

$$r_1 = \frac{n_1\varepsilon + n_2 + n_3 + \rho_L}{n_1 + n_2 + n_3 + \rho_L}$$

- where $\rho_L = \dfrac{m_L}{m_{\mathrm{mod}}}; \quad m_{\mathrm{mod}} = m_i + m_p$