Atmospheric Entry

- Orbital decay due to atmospheric drag
- Deorbit delta-V
- Ballistic entry

Energy Loss Due to Atmospheric Drag

$$\rho = \rho_o e^{-\frac{h}{h_s}} \qquad a_d = \frac{\rho v^2}{2\beta} \qquad \beta = \frac{m}{c_D A}$$

$$\text{orbital energy} \equiv E = -\frac{\mu}{2a}$$

$$\frac{dE}{dt} = \frac{\mu}{2a^2}\frac{da}{dt} = -\sqrt{\frac{\mu}{a}}\frac{\rho v^2}{2\beta}$$

Since drag is highest at perigee, the first effect of atmospheric drag is to circularize the orbit (high perigee drag lowers apogee)

$$\frac{dE}{dt} = -\sqrt{\frac{\mu}{a}}\frac{\rho}{2\beta}\frac{\mu}{a}$$

Derivation of Orbital Decay Due to Drag

Set orbital energy variation equal to energy lost by drag

$$\frac{\mu}{2a^2}\frac{da}{dt} = -\frac{\rho}{2\beta}\left(\frac{\mu}{a}\right)^{\frac{3}{2}}$$

$$\frac{da}{dt} = -\frac{\rho}{\beta}\sqrt{\mu a}$$

$$\rho = \rho_o e^{-\frac{h}{h_s}} \qquad a = h + r_E \Longrightarrow \frac{da}{dt} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{\sqrt{\mu\left(h + r_E\right)}}{\beta}\rho_o e^{-\frac{h}{h_s}}$$

Derivation of Orbital Decay (2)

This is a separable differential equation...

$$\frac{1}{\sqrt{r_E + h}} e^{\frac{h}{h_s}} dh = -\frac{\sqrt{\mu}}{\beta} \rho_o dt$$

$$\int_{h_o}^{h} \frac{1}{\sqrt{r_E + h}} e^{\frac{h}{h_s}} dh = -\frac{\sqrt{\mu}}{\beta} \rho_o \int_{t_o}^{t} dt$$

$$\text{Assume } \sqrt{r_E + h} \sim \sqrt{r_E} \text{ for } r_E \gg h$$

$$\frac{1}{\sqrt{r_E}} \int_{h_o}^{h} e^{\frac{h}{h_s}} dh = -\frac{\sqrt{\mu}}{\beta} \rho_o (t - t_o)$$

Derivation of Orbital Decay (3)

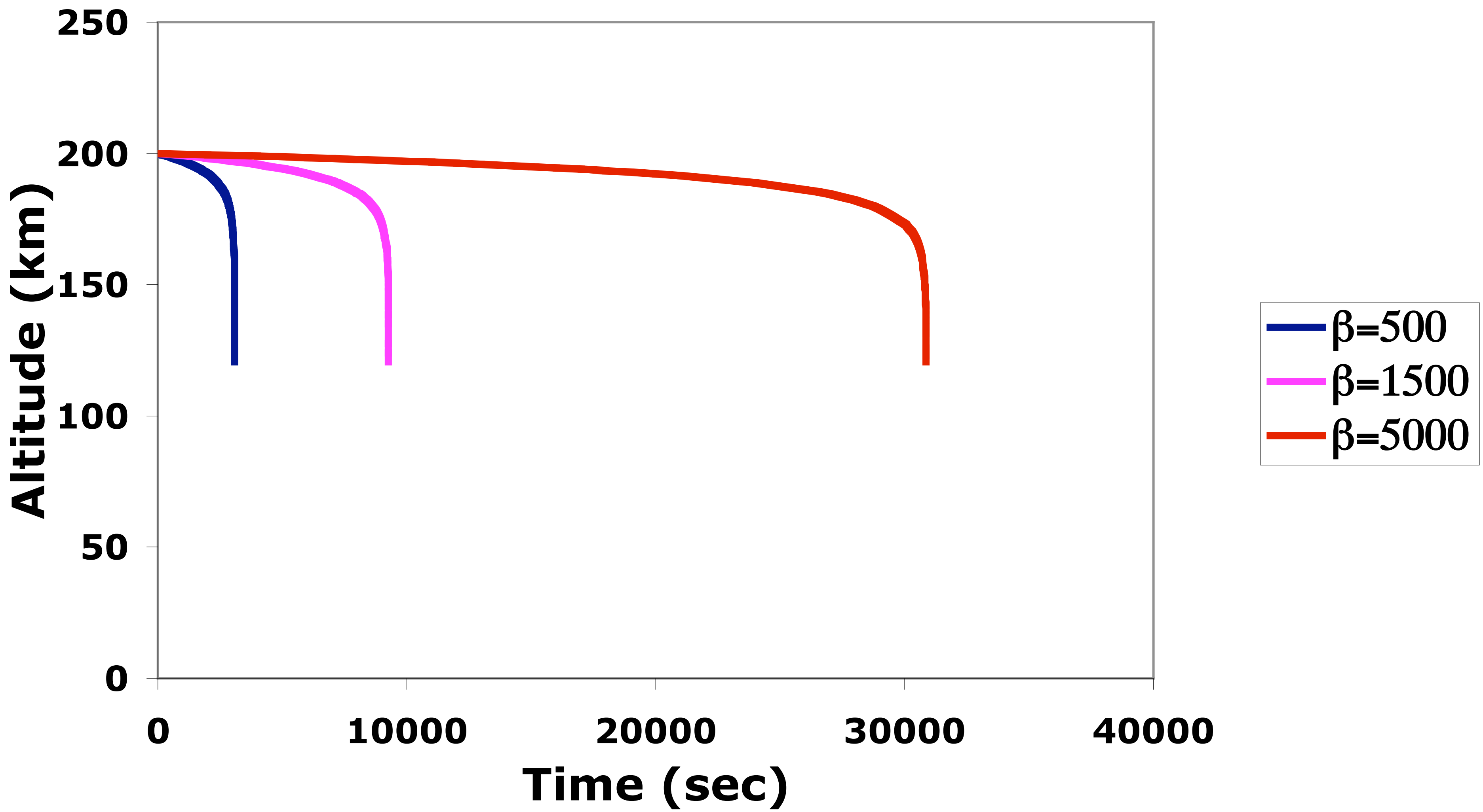
$$\frac{h_s}{\sqrt{r_E}}\left(e^{\frac{h}{h_s}} - e^{\frac{h_o}{h_s}}\right) = -\frac{\sqrt{\mu}}{\beta}\rho_o\left(t - t_o\right)$$

$$e^{\frac{h}{h_s}} - e^{\frac{h_o}{h_s}} = -\frac{\sqrt{\mu r_E}}{h_s\beta}\rho_o\left(t - t_o\right)$$

$$\boxed{h(t) = h_s \ln\left[e^{\frac{h_o}{h_s}} - \frac{\sqrt{\mu r_E}}{h_s\beta}\rho_o\left(t - t_o\right)\right]}$$

Note that some variables typically use km, and others are in meters - you have to make sure unit conversions are done properly to make this work out correctly!

Orbit Decay from Atmospheric Drag



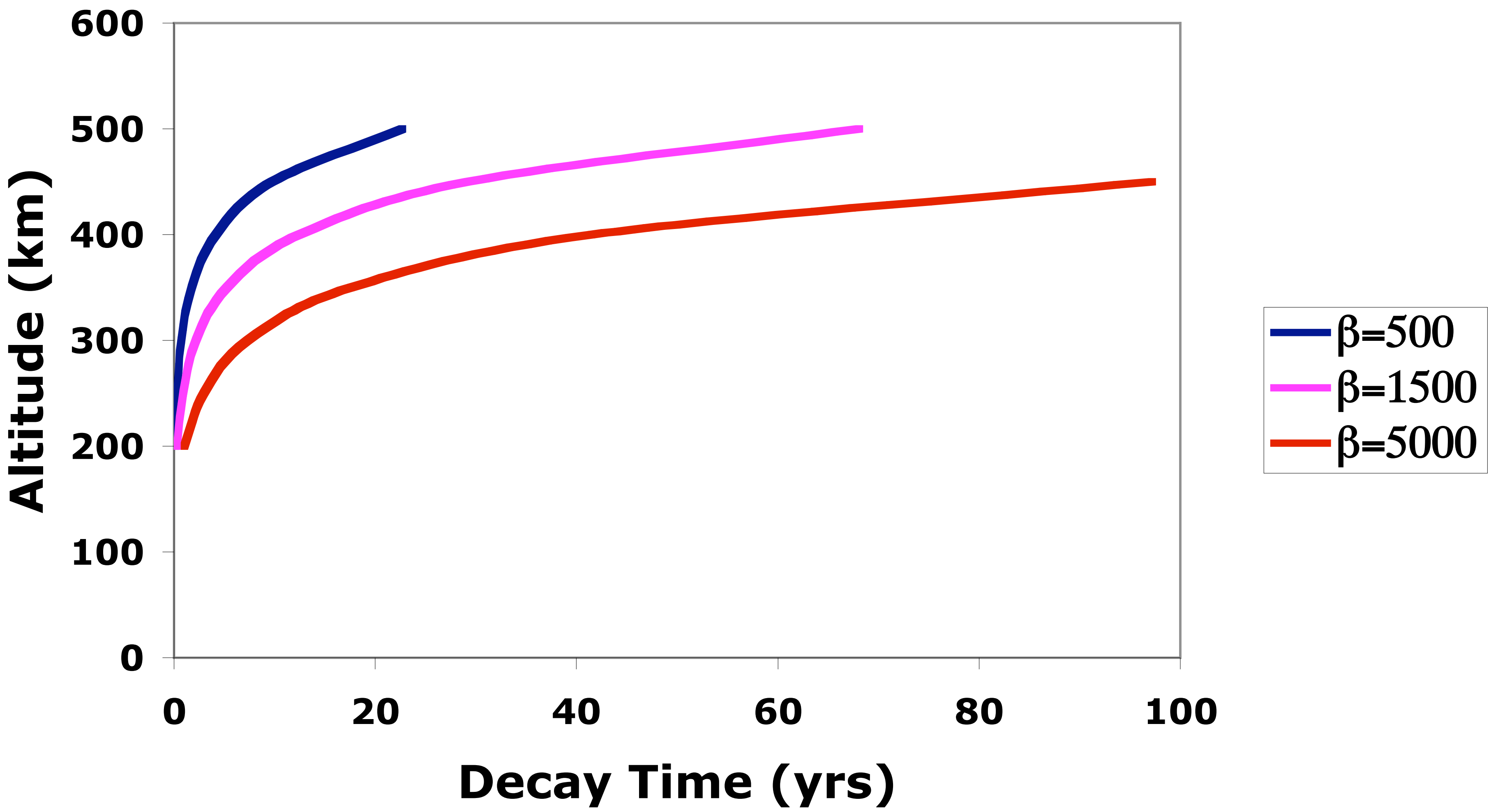
Time Until Orbital Decay

$$e^{\frac{h}{h_s}} - e^{\frac{h_o}{h_s}} = -\frac{\sqrt{\mu r_E}}{h_s \beta} \rho_o (t - t_o)$$

To find the time remaining (t_o=0) until the orbit reaches any given "critical" altitude, some algebra gives

$$t(h) = \frac{h_s \beta}{\sqrt{\mu r_E} \rho_o} \left(e^{\frac{h_o}{h_s}} - e^{\frac{h}{h_s}} \right)$$

Decay Time to r=120 km



Deorbit Delta-V

Conservation of Angular Momentum

$$|\vec{r} \times \vec{v}| = r_e v_e cos\gamma = r_a v_a$$

Conservation of Energy

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right) \qquad \frac{1}{a} = \left(\frac{2}{r} - \frac{v^2}{\mu} \right)$$

we want $v_a = f(r_a, r_e, \gamma)$

we need to eliminate a, v_e

$r_a =$ radius at apogee
$v_a =$ velocity at apogee
$r_e =$ radius at entry
$v_e =$ velocity at entry
$\gamma =$ flight path angle at entry

Derivation of Deorbit Delta-V

$$\frac{1}{a} = \left(\frac{2}{r_a} - \frac{v_a^2}{\mu}\right)$$

$$v_e^2 = \mu\left[\frac{2}{r_e} - \left(\frac{2}{r_a} - \frac{v_a^2}{\mu}\right)\right]$$

$$r_e^2 v_e^2 cos^2\gamma = r_a^2 v_a^2$$

$$v_a^2 = \frac{r_e^2}{r_a^2} cos^2(\gamma) v_e^2$$

Derivation of Deorbit Delta-V (2)

$$v_a^2 = \frac{r_e^2}{r_a^2} cos^2(\gamma)\mu \left(\frac{2}{r_e} - \frac{2}{r_a} + \frac{v_a^2}{\mu} \right)$$

$$v_a^2 \left[1 - \frac{r_e^2}{r_a^2} cos^2\gamma \right] = \frac{r_e^2}{r_a^2} cos^2\gamma \frac{2\mu}{r_a} \left(\frac{r_a}{r_e} - 1 \right)$$

$$\text{Let } \rho_e \equiv \frac{r_e}{r_a}$$

$$v_a^2 \left[1 - \rho_e^2 cos^2\gamma \right] = \rho_e^2 cos^2\gamma \frac{2\mu}{r_a} \left(\frac{1}{\rho_e} - 1 \right)$$

Derivation of Deorbit Delta-V (3)

$$v_a^2 = \frac{\rho_e^2 cos^2(\gamma)\frac{2\mu}{r_a}\left(\frac{1}{\rho_e} - 1\right)}{1 - \rho_e^2 cos^2(\gamma)}$$

$$v_a = \frac{cos\gamma\sqrt{\frac{2\mu}{r_a}}\sqrt{\rho_e - \rho_e^2}}{\sqrt{1 - \rho_e^2 cos^2(\gamma)}}$$

$$\Delta V = \sqrt{\frac{\mu}{a}} - v_a$$

$$\boxed{\Delta v_{deorbit} = \sqrt{\frac{\mu}{r_a}}\left[1 - \frac{cos\gamma\sqrt{2}\sqrt{\rho_e - \rho_e^2}}{\sqrt{1 - \rho_e^2 cos^2\gamma}}\right]}$$