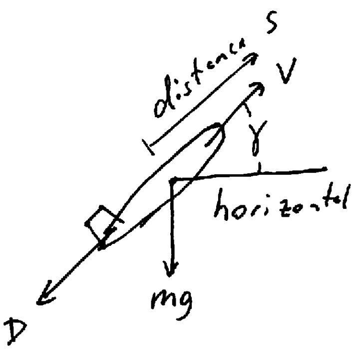
Ballistic Entry, revisited

2/12/04



System is drawn this way to represent positive definition of flight path angle γ – (really going downwards)

s = distance along the flight path

$$\frac{dV}{dt} = -g\sin\gamma - \frac{D}{m}$$

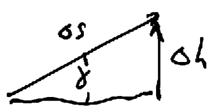
(including gravity, at least temporarily)

$$\frac{dV}{dt} = \frac{dV}{ds}\underbrace{\frac{ds}{dt}}_{\equiv V} = V\frac{dV}{ds} = \frac{1}{2}\frac{d(V^2)}{ds}$$

$$\frac{1}{2}\frac{d(V^2)}{ds} = -g\sin\gamma - \frac{D}{m}$$

Drag $D \equiv \frac{1}{2}\rho V^2 A C_D$

(Note: like everyone else I tend to use "A" and "S" interchangeably for reference area – here, I don't want to confuse it with distance "s")



$$\frac{1}{2}\frac{d(V^2)}{ds} = -g\sin\gamma - \frac{\rho V^2}{2m}AC_D$$

As before, $ds = \frac{dh}{\sin\gamma}$

$$\frac{\sin\gamma}{2}\frac{d(V^2)}{dh} = -g\sin\gamma - \frac{\rho V^2}{2m}AC_D$$

$$\rho = \rho_0 e^{-h/h_s}$$

$$\frac{d\rho}{\rho_0} = e^{-h/h_s}\left(\frac{-dh}{h_s}\right) = \frac{\rho_0 e^{-h/h_s}}{\rho_0}\left(\frac{-dh}{h_s}\right) = \frac{\rho}{\rho_0}\left(\frac{-dh}{h_s}\right)$$

$$dh = \frac{-h_s}{\rho}d\rho$$

$$\frac{\sin\gamma}{2}\frac{d(v^2)}{d\rho}\left(-\frac{\rho}{h_s}\right) = -\frac{\rho v^2}{2}\frac{AC_D}{m} - g\sin\gamma$$

Find velocity as $f(\rho)$

$$\frac{d(v^2)}{d\rho} = \frac{2gh_s}{\rho} + \underbrace{\frac{AC_D}{m}}_{\equiv\beta}\frac{h_s v^2}{\sin\gamma}$$

$$\frac{d(v^2)}{d\rho} - \frac{h_s}{\beta\sin\gamma}v^2 = \frac{2gh_s}{\rho}$$

Assume $mg \ll D$ to get homogeneous ODE

$$\frac{d(v^2)}{d\rho} - \frac{h_s}{\beta\sin\gamma}v^2 = 0 \qquad \text{separable}$$

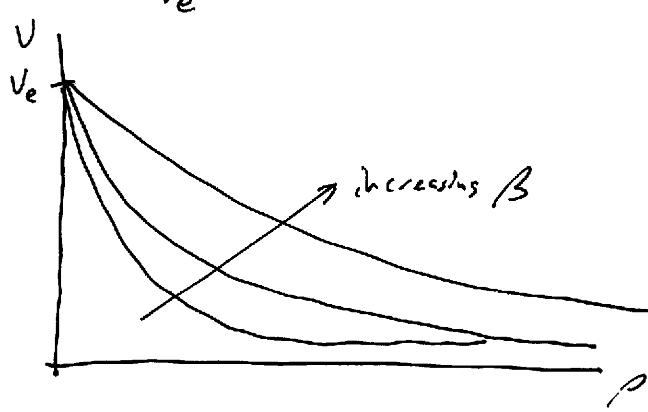
$$\frac{d(v^2)}{v^2} = \frac{h_s}{\beta\sin\gamma}\,d\rho$$

use (v^2) as integration variable

$$\int_{V_e}^{V}\frac{d(v^2)}{v^2} = \frac{h_s}{\beta\sin\gamma}\int_0^{\rho}d\rho$$

$$\ln\left(\frac{V^2}{V_e^2}\right) = 2\ln\frac{V}{V_e} = \frac{h_s\rho}{\beta\sin\gamma}$$

$$\frac{V}{V_e} = e^{\frac{h_s\rho}{2\beta\sin\gamma}}$$



What about peak deceleration?

$$n = \frac{dV}{dt} = -\frac{\rho V^2}{2\beta}$$

to find n_{max}, set $\frac{d}{dt}\left(\frac{dV}{dt}\right) = \frac{d^2V}{dt^2} = 0$

$$\frac{d^2V}{dt^2} = -\frac{1}{2\beta}\left(2\rho V\frac{dV}{dt} + V^2\frac{d\rho}{dt}\right)$$

$$= -\frac{1}{2\beta}\left(-\frac{2\rho^2V^3}{2\beta} + V^2\frac{d\rho}{dt}\right) = 0$$

$$V^2\frac{d\rho}{dt} = \frac{2\rho^2V^3}{2\beta} \Rightarrow \rho^2 V = \beta\frac{d\rho}{dt}$$

From exponential atmosphere,

$$\frac{d\rho}{dt} = -\frac{\rho_0}{h_s}e^{-h/h_s}\frac{dh}{dt} = -\frac{\rho}{h_s}\frac{dh}{dt}$$

and $\frac{dh}{dt} = V\sin\gamma$

$$\frac{d\rho}{dt} = \frac{-\rho V}{h_s}\sin\gamma \quad \text{so}$$

$$\cancel{\rho}^{2}\cancel{V} = \beta\left(-\frac{\cancel{\rho V}}{h_s}\sin\gamma\right)$$

$$\rho_{n_{max}} = -\frac{\beta}{h_s}\sin\gamma$$

At surface, $\rho_{\cancel{max}} = \rho_0$ $\qquad \rho_0 = -\frac{\beta_{crit}}{h_s}\sin\gamma$

$$\beta_{crit} = -\frac{\rho_0 h_s}{\sin\gamma}$$

< 0

⇐ value of β at which vehicle hits ground before max deceleration!

$$\frac{dV}{dt} = \frac{\rho V^2}{2\beta} \qquad \left|\frac{dV}{dt}\right|_{max} = \frac{\rho_{max} V^2}{2\beta}$$

$$= \frac{V^2}{2\beta}\left(-\frac{\beta}{h_s}\sin\gamma\right) = -\frac{1}{2}\frac{V^2}{h_s}\sin\gamma$$

$$\frac{V}{V_e} = e^{\frac{h_s \rho}{2\beta\sin\gamma}} \Rightarrow \frac{V_{\rho_{max}}}{V_e} = e^{\frac{h_s}{2\beta\sin\gamma}\left(-\frac{\beta}{h_s}\sin\gamma\right)} = e^{-1/2}$$

$$\left|\frac{dV}{dt}\right|_{max} = -\frac{1}{2}\frac{\left(V_e e^{-1/2}\right)^2}{h_s}\sin\gamma$$

$$= -\frac{V_e^2 \sin\gamma}{2 h_s e}$$

Example: $V = 6000$ m/sec
$h_s = 7100$ m/sec
$\rho_0 = 1.22$ kg/m^3

$$\beta_{crit} = \frac{-(1.22\ \mathrm{kg/m^3})(7100\ \mathrm{m})}{\sin(-90°)} = 8660\ \mathrm{kg/m^2}\ (\approx 1775\ \mathrm{lb/ft^2})$$

Peak deceleration

$$\left|\frac{dV}{dt}\right|_{max} = -\frac{V_e^2 \sin\gamma}{2 h_s e} = 933\ \mathrm{m/sec^2} \cong 95\ g\text{'s}$$

Terminal velocity

Full form of ODE: $\frac{d(V^2)}{d\rho} - \frac{h_s}{\beta\sin\gamma}V^2 = \frac{2 g h_s}{\rho}$

At terminal velocity, V = constant (typically at surface $\rho = \rho_0$)

$$V_T^2 = \frac{-2\beta g h_s \sin\gamma}{h_s \rho_0} \Rightarrow \boxed{V_T = \sqrt{-\frac{2\beta g \sin\gamma}{\rho_0}}}$$

Assume a sphere, 6.75" in diameter (=.1714m ϕ)

$C_D = 0.2$

	Iron	Aluminum	Balsa wood
Weight	40 lb	15.6 lbs	14.5 oz
β (kg/m²)	3938	1532	89
for max deceleration: ρ (kg/m³)	0.555	0.216	0.0125
for max deceleration: h (m)	5600	12,300	32,500
V_{impact} (m/sec)	1998	355	4.4×10^{-18}*
$V_{terminal}$ (m/sec)	251	156	38

*assumes no gravity (bad assumption!)