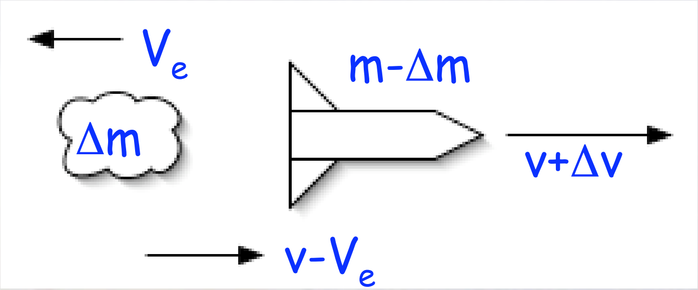
Rocket Performance

- The rocket equation
- Mass ratio and performance
- Structural and payload mass fractions
- Regression analysis
- Multistaging
- Optimal ΔV distribution between stages
- Trade-off ratios
- Parallel staging
- Modular staging

Derivation of the Rocket Equation



- Momentum at time t:

$$M = mv$$

- Momentum at time t+Δt:

$$M = (m - \Delta m)(V + \Delta v) + \Delta m(v - V_e)$$

- Some algebraic manipulation gives:

$$m\Delta v = -\Delta m V_e$$

- Take to limits and integrate:

$$\int_{m_{initial}}^{m_{final}} \frac{dm}{m} = -\int_{V_{initial}}^{V_{final}} \frac{dv}{V_e}$$

The Rocket Equation

- Alternate forms

$$r \equiv \frac{m_{final}}{m_{initial}} = e^{-\frac{\Delta V}{V_e}}$$

$$\Delta v = -V_e \ln \left(\frac{m_{final}}{m_{initial}} \right) = -V_e \ln r$$

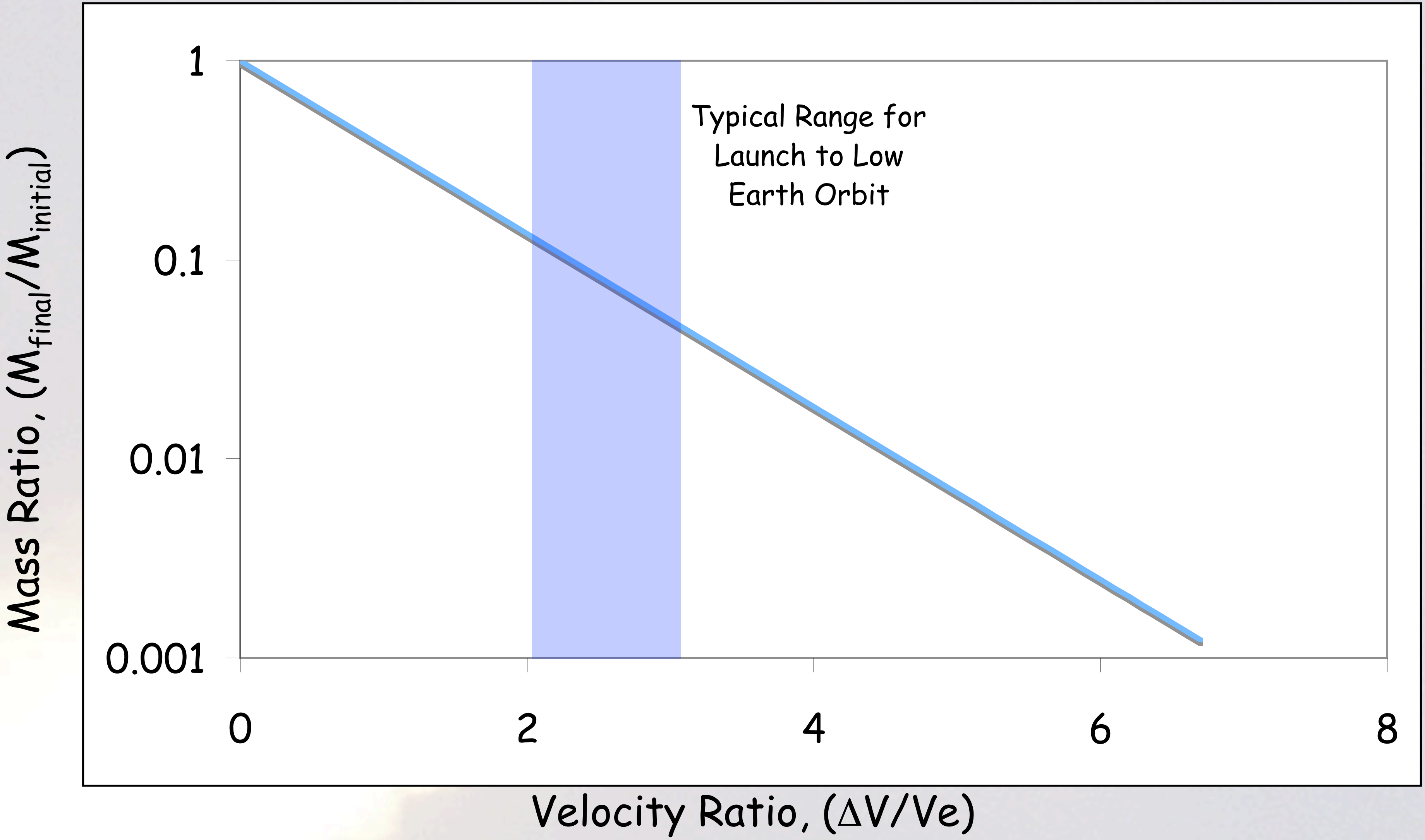
- Basic definitions/concepts
  - Mass ratio

$$r \equiv \frac{m_{final}}{m_{initial}} \text{ or } \Re \equiv \frac{m_{initial}}{m_{final}}$$

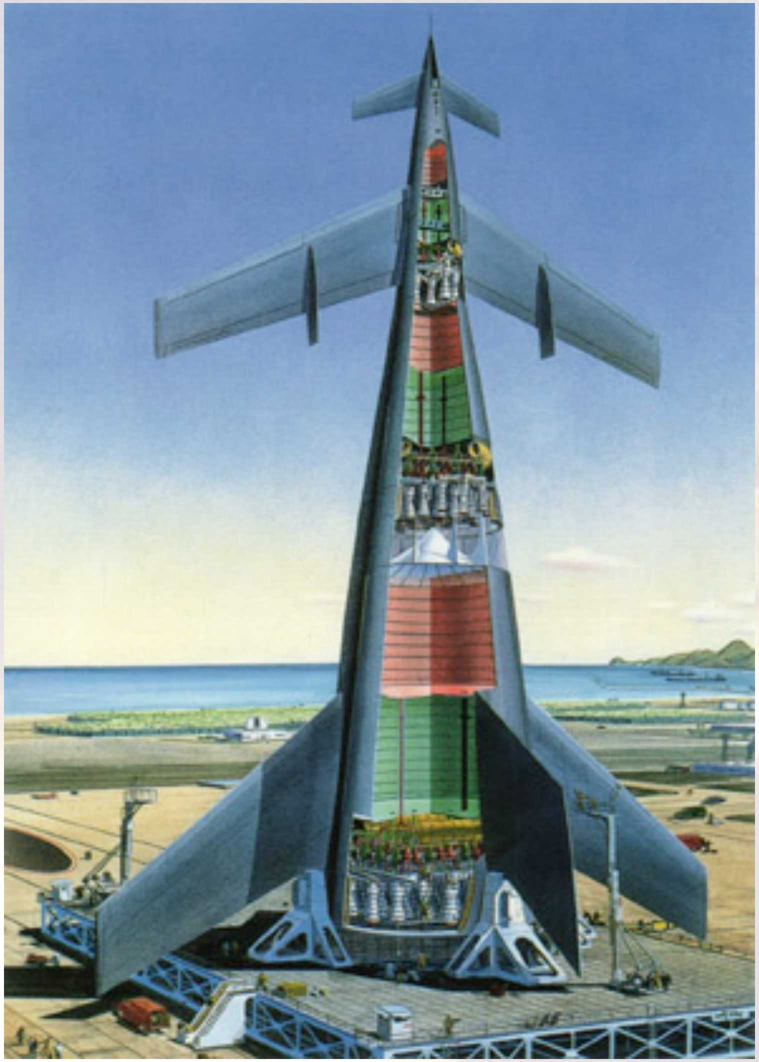
  - Nondimensional velocity change "Velocity ratio"

$$\frac{\Delta V}{V_e}$$

Rocket Equation (First Look)



Sources and Categories of Vehicle Mass



Payload
Propellants
Inert Mass
- Structure
- Propulsion
- Avionics
- Power
- Mechanisms
- Thermal
- Etc.

Basic Vehicle Parameters

- Basic mass summary

$$m_o = m_{pl} + m_{pr} + m_{in}$$

- Inert mass fraction

$$\delta \equiv \frac{m_{in}}{m_o} = \frac{m_{in}}{m_{pl} + m_{pr} + m_{in}}$$

- Payload fraction

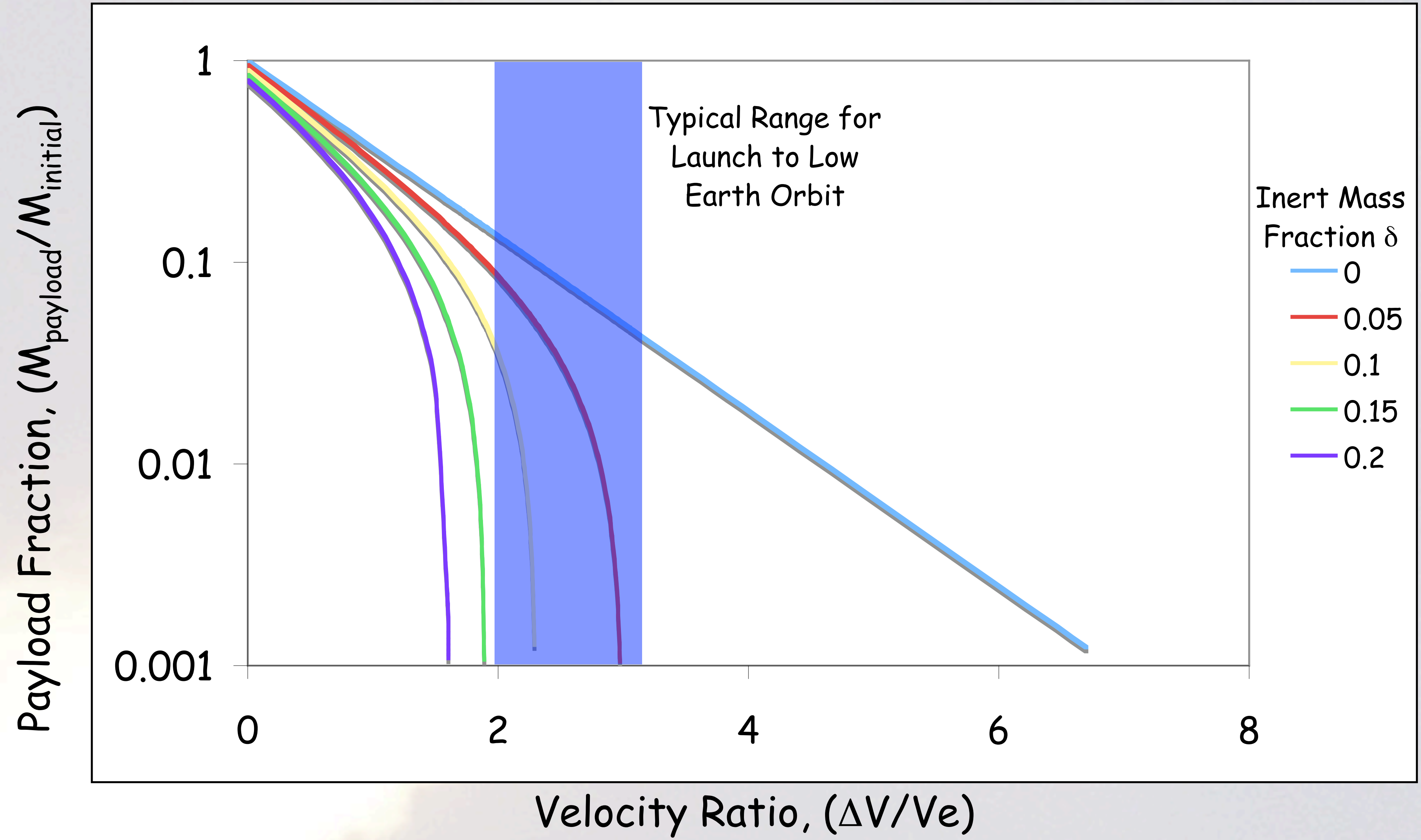
$$\lambda \equiv \frac{m_{pl}}{m_o} = \frac{m_{pl}}{m_{pl} + m_{pr} + m_{in}}$$

- Parametric mass ratio

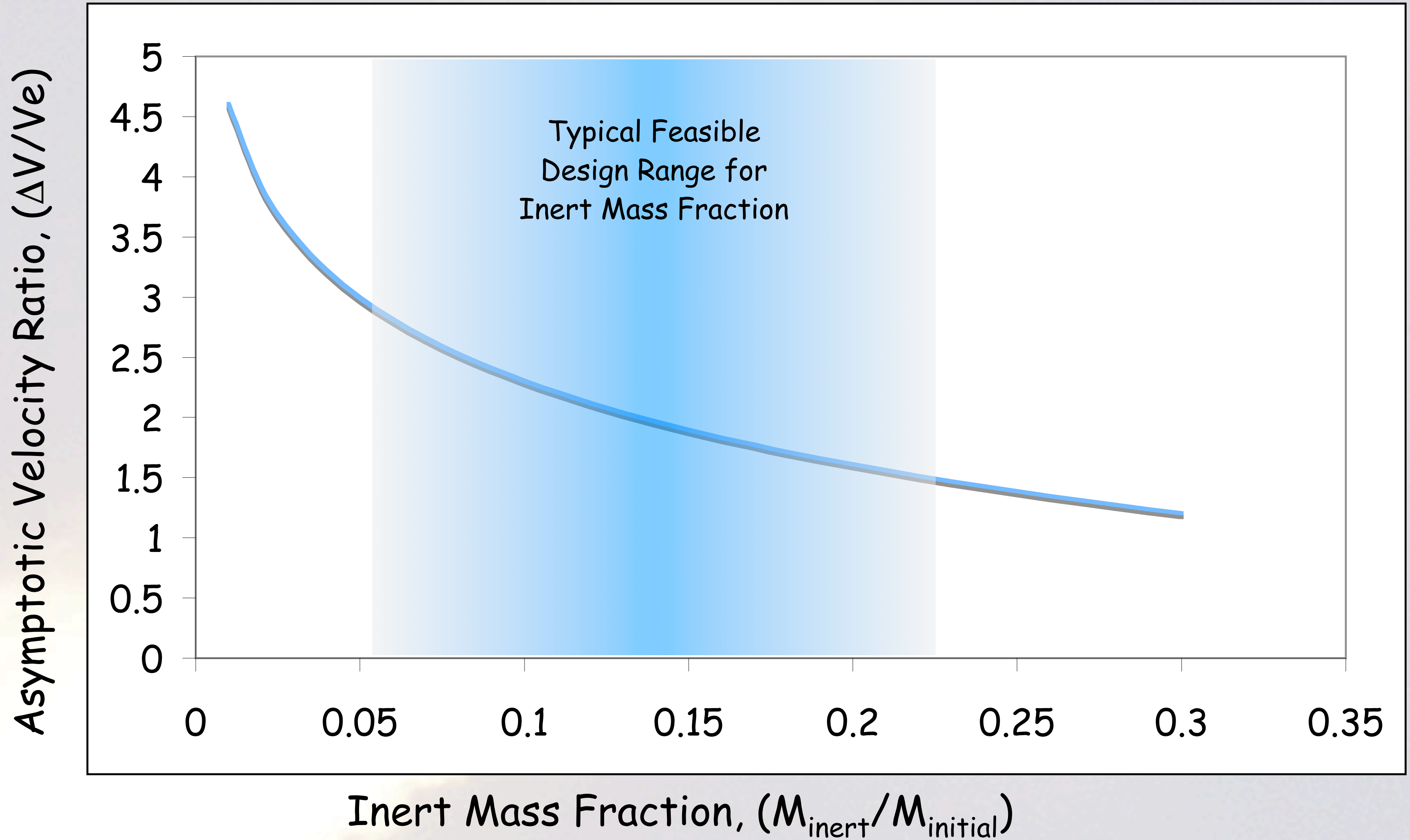
$$r = \lambda + \delta$$

m_o =initial mass
m_{pl} =payload mass
m_{pr} =propellant mass
m_{in} =inert mass

Rocket Equation (including Inert Mass)



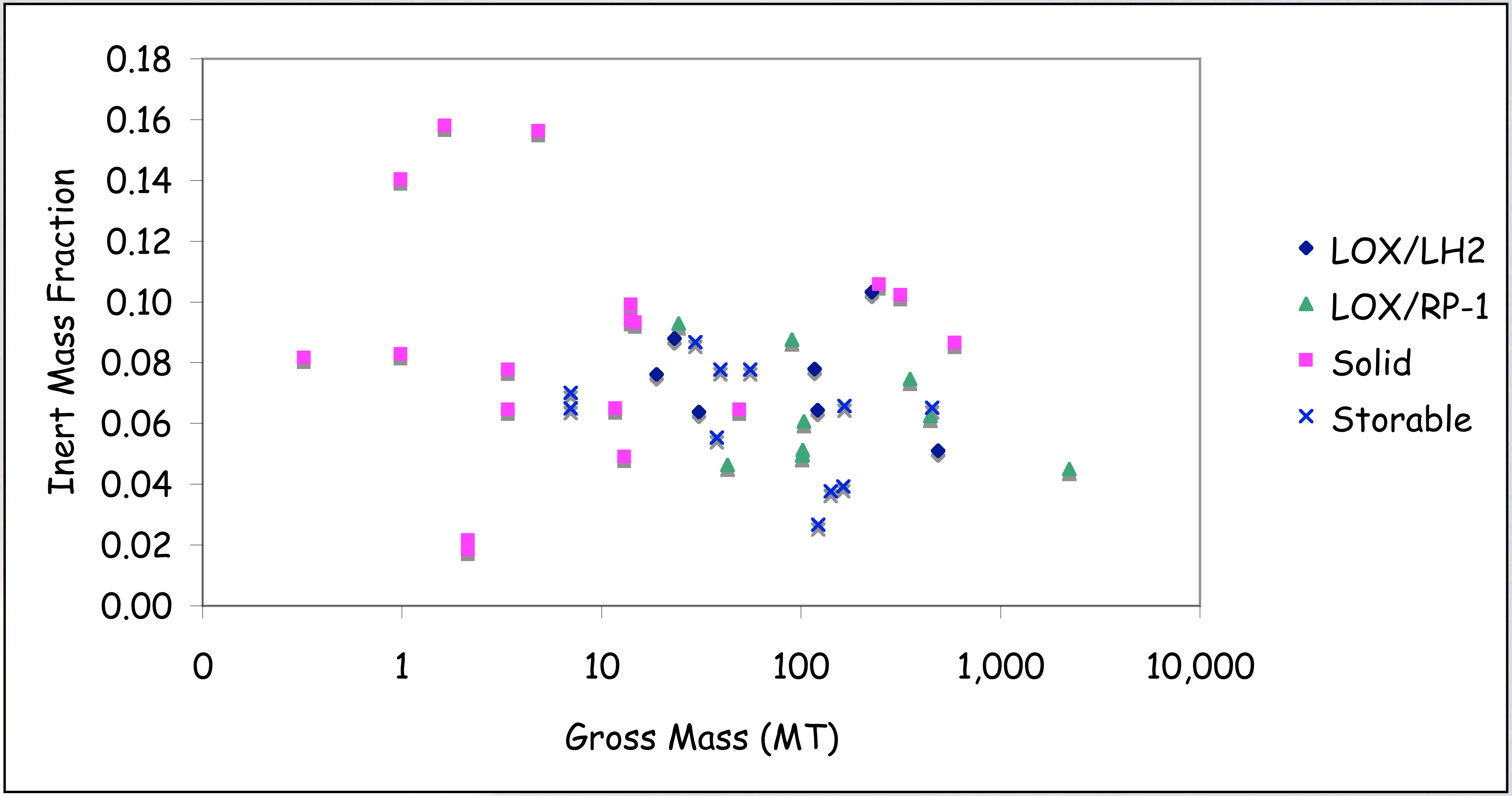
Limiting Performance Based on Inert Mass



Regression Analysis of Existing Vehicles

Veh/Stage	prop mass (lbs)	gross mass (lbs)	Type	Propellants	Isp vac (sec)	isp sl (sec)	sigma	eps	delta
Delta 6925 Stage 2	13,367	15,394	Storab	N2O4-A50	319.4		0.152	0.132	0.070
Delta 7925 Stage 2	13,367	15,394	Storab	N2O4-A50	319.4		0.152	0.132	0.065
Titan II Stage 2	59,000	65,000	Storab	N2O4-A50	316.0		0.102	0.092	0.087
Titan III Stage 2	77,200	83,600	Storab	N2O4-A50	316.0		0.083	0.077	0.055
Titan IV Stage 2	77,200	87,000	Storab	N2O4-A50	316.0		0.127	0.113	0.078
Proton Stage 3	110,000	123,000	Storab	N2O4-A50	315.0		0.118	0.106	0.078
Titan II Stage 1	260,000	269,000	Storab	N2O4-A50	296.0		0.035	0.033	0.027
Titan III Stage 1	294,000	310,000	Storab	N2O4-A50	302.0		0.054	0.052	0.038
Titan IV Stage 1	340,000	359,000	Storab	N2O4-A50	302.0		0.056	0.053	0.039
Proton Stage 2	330,000	365,000	Storab	N2O4-A50	316.0		0.106	0.096	0.066
Proton Stage 1	904,000	1,004,000	Storab	N2O4-A50	316.0	285.0	0.111	0.100	0.065
average					312.2	285.0	0.100	0.089	0.061
standard deviation					8.1		0.039	0.033	0.019

Inert Mass Fraction Data for Existing LVs



Regression Analysis

- Given a set of N data points (x_i, y_i)
- Linear curve fit: $y=Ax+B$
  - find A and B to minimize sum squared error

$$\text{error} = \sum_{i=1}^{N} (Ax_i + B - y_i)^2$$

  - Analytical solutions exist, or use Solver in Excel
- Power law fit: $y=Bx^A$

$$\text{error} = \sum_{i=1}^{N} \left[A \log(x_i) + B - \log(y_i)\right]^2$$

- Polynomial, exponential, many other fits possible

Solution of Least-Squares Linear Regression

$$\frac{\partial(\text{error})}{\partial A} = 2\sum_{i=1}^{N}(Ax_i + B - y_i)x_i = 0$$

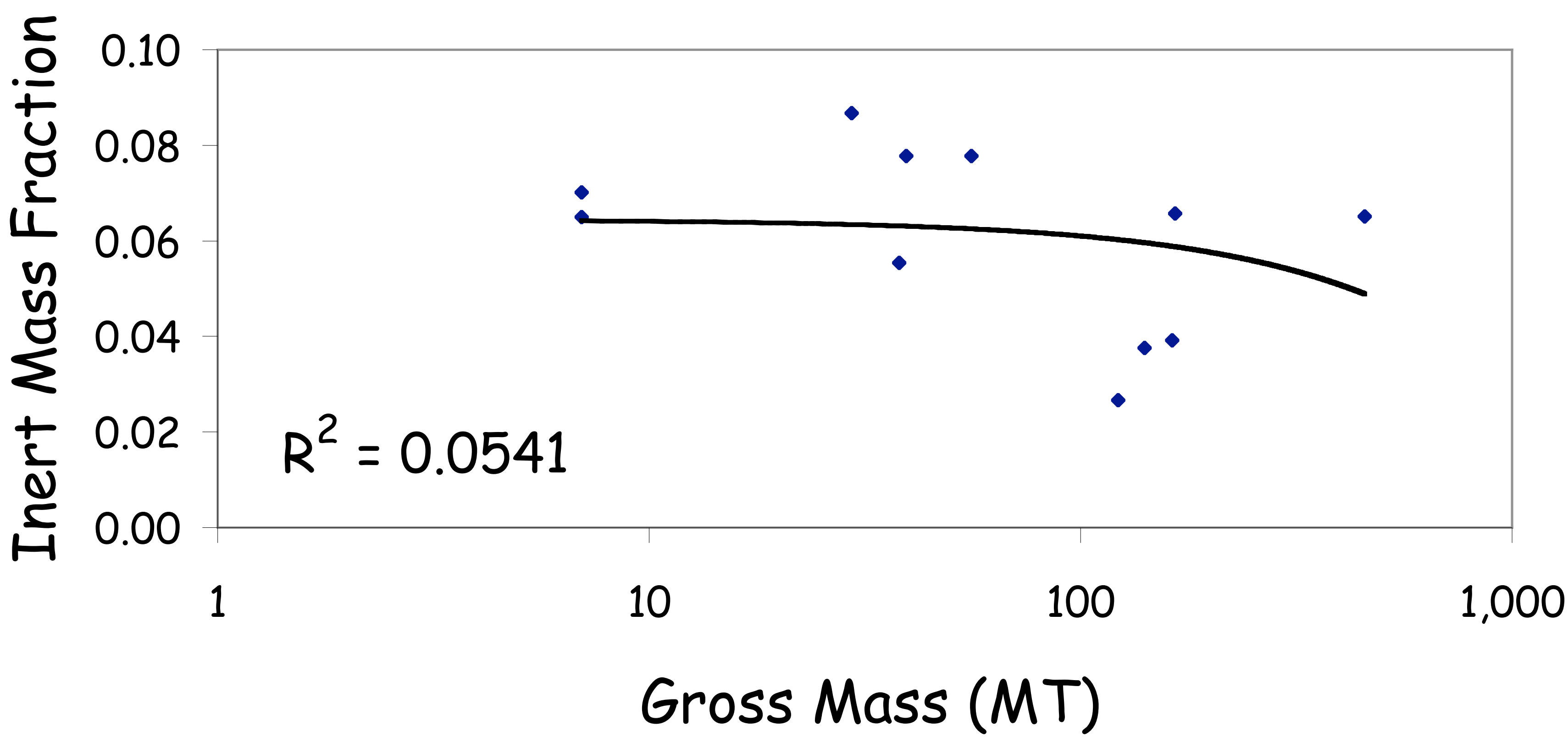
$$\frac{\partial(\text{error})}{\partial B} = 2\sum_{i=1}^{N}(Ax_i + B - y_i) = 0$$

$$A\sum_{i=1}^{N}x_i^2 + B\sum_{i=1}^{N}x_i - \sum_{i=1}^{N}x_i y_i = 0$$

$$A\sum_{i=1}^{N}x_i + NB - \sum_{i=1}^{N}y_i = 0$$

$$A = \frac{N\sum x_i y_i - \sum x_i \sum y_i}{N\sum x_i^2 - \left(\sum x_i\right)^2} \qquad B = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{N\sum x_i^2 - \left(\sum x_i\right)^2}$$

Regression Analysis - Storables



Regression Values for Design Parameters

	Vacuum Ve (m/sec)	Inert Mass Fraction	Max ΔV (m/sec)
LOX/LH2	4273	0.075	11,070
LOX/RP-1	3136	0.063	8664
Storables	3058	0.061	8575
Solids	2773	0.087	6783

The Rocket Equation for Multiple Stages

- Assume two stages

$$\Delta V_1 = -V_{e1} \ln \left(\frac{m_{final1}}{m_{initial1}} \right) = -V_{e1} \ln(r_1)$$

$$\Delta V_2 = -V_{e2} \ln \left(\frac{m_{final2}}{m_{initial2}} \right) = -V_{e2} \ln(r_2)$$

- Assume $V_{e1}=V_{e2}=V_e$

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1) - V_e \ln(r_2) = -V_e \ln(r_1 r_2)$$

Continued Look at Multistaging

- There's a historical tendency to define $r_0 = r_1 r_2$

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1 r_2) = -V_e \ln(r_0)$$

- But it's important to remember that it's really

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1 r_2) = -V_e \ln\left(\frac{m_{final1}}{m_{initial1}} \frac{m_{final2}}{m_{initial2}}\right)$$

- And that r_0 has no physical significance, since

$$m_{final1} \neq m_{initial2} \Rightarrow r_0 \neq \frac{m_{final2}}{m_{initial1}}$$

Multistage Inert Mass Fraction

- Total inert mass fraction

$$\delta_0 = \frac{m_{in,1} + m_{in,2} + m_{in,3}}{m_0} = \frac{m_{in,1}}{m_0} + \frac{m_{in,2}}{m_0} + \frac{m_{in,3}}{m_0}$$

$$\delta_0 = \frac{m_{in,1}}{m_0} + \frac{m_{in,2}}{m_{0,2}}\frac{m_{0,2}}{m_0} + \frac{m_{in,3}}{m_{0,3}}\frac{m_{0,3}}{m_{0,2}}\frac{m_{0,2}}{m_0}$$

- Convert to dimensionless parameters

$$\delta_0 = \delta_1 + \delta_2\lambda_1 + \delta_3\lambda_2\lambda_1$$

- General form of the equation

$$\delta_0 = \sum_{j=1}^{n \text{ stages}} [\delta_j \prod_{\ell=1}^{j-1} \lambda_\ell]$$

Multistage Payload Fraction

- Total payload fraction (3 stage example)

$$\lambda_0 = \frac{m_{pl}}{m_0} = \frac{m_{pl}}{m_{0,3}} \frac{m_{0,3}}{m_{0,2}} \frac{m_{0,2}}{m_0}$$

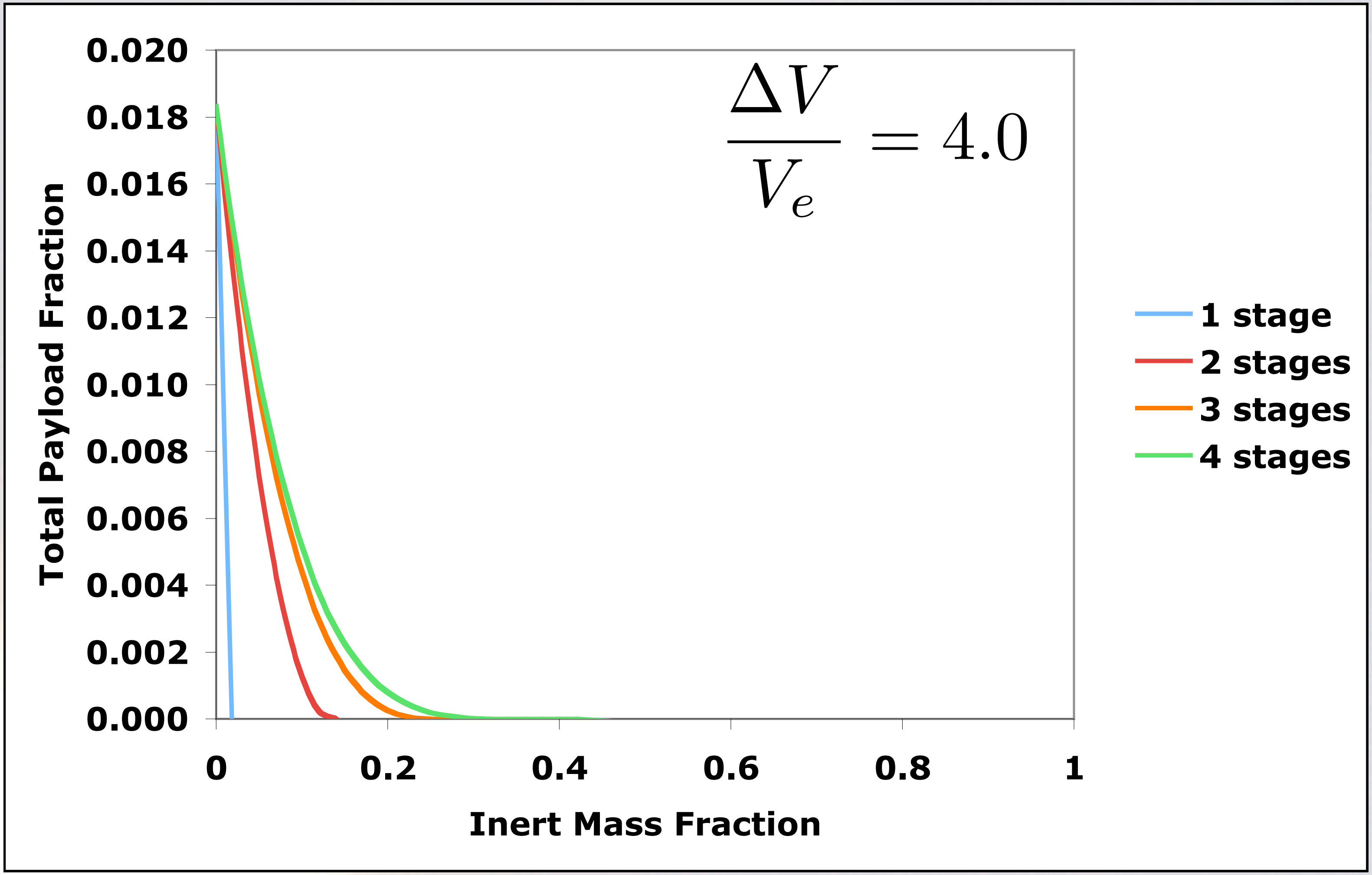
- Convert to dimensionless parameters

$$\lambda_0 = \lambda_3 \lambda_2 \lambda_1$$

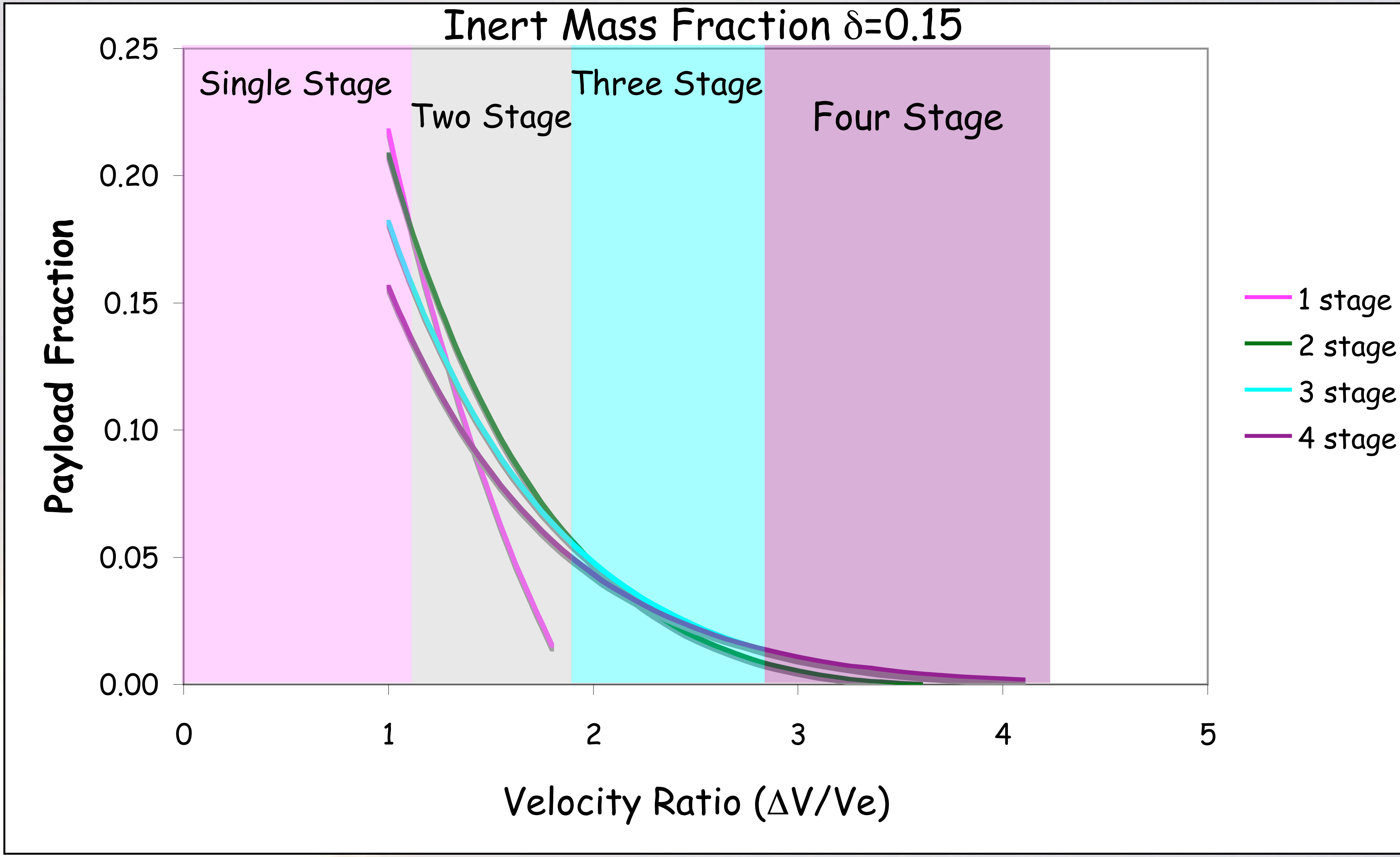
- Generic form of the equation

$$\lambda_0 = \prod_{j=1}^{n \text{ stages}} \lambda_j$$

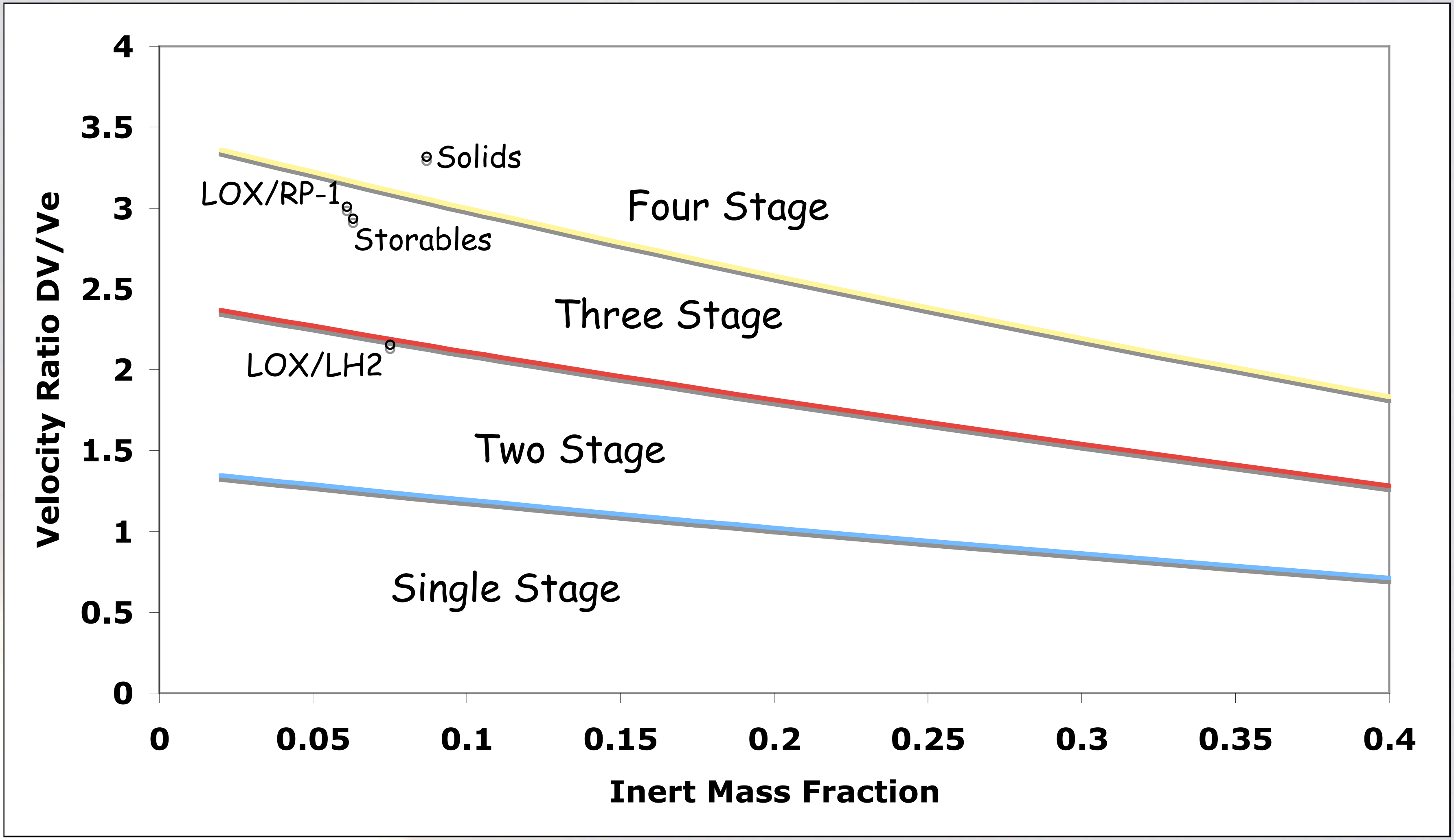
Effect of δ and ΔV/Ve on Payload



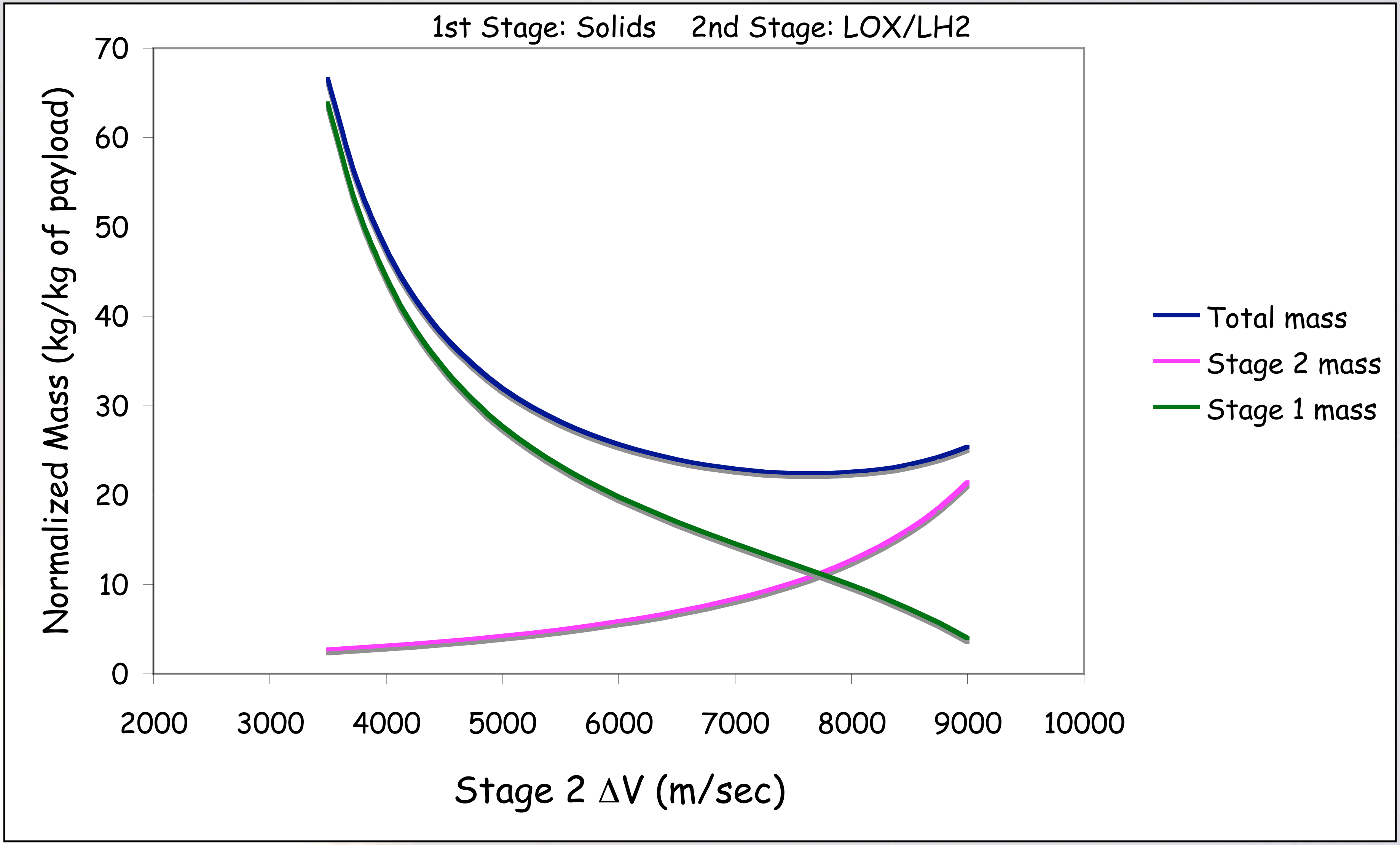
Effect of Staging



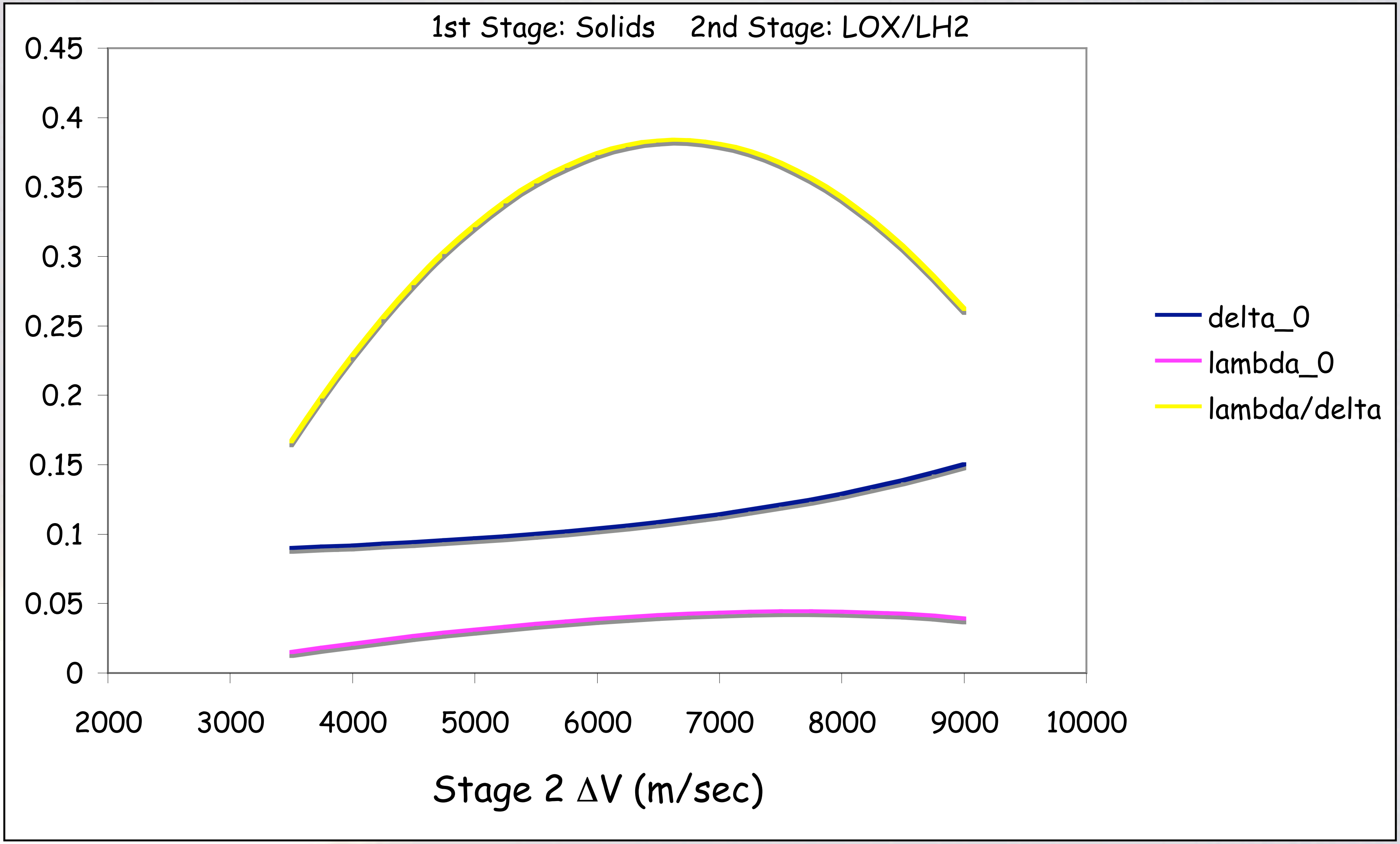
Trade Space for Number of Stages



Effect of ΔV Distribution



ΔV Distribution and Design Parameters



Lagrange Multipliers

- Given an objective function

$$y = f(x)$$

subject to constraint function

$$z = g(x)$$

- Create a new objective function

$$z = f(x) + \lambda[g(x) - z]$$

- Solve simultaneous equations

$$\frac{\partial y}{\partial x} = 0 \qquad \frac{\partial y}{\partial \lambda} = 0$$

Optimum ΔV Distribution Between Stages

- Maximize payload fraction (2 stage case)

$$\lambda_0 = \lambda_1 \lambda_2 = (r_1 - \delta_1)(r_2 - \delta_2)$$

subject to constraint function

$$\Delta V_{total} = \Delta V_1 + \Delta V_2$$

- Create a new objective function

$$\lambda_o = \left(e^{\frac{-\Delta V_1}{V_{e,1}}} - \delta_1 \right) \left(e^{\frac{-\Delta V_2}{V_{e,2}}} - \delta_2 \right) + K \left[\Delta V_1 + \Delta V_2 - \Delta V_{total} \right]$$

➡ Very messy for partial derivatives!

Optimum ΔV Distribution (continued)

- Use substitute objective function

$$max\left(\lambda_o\right) \Longleftrightarrow max\left[ln\left(\lambda_o\right)\right]$$

- Create a new constrained objective function

$$ln\left(\lambda_o\right) = ln\left(r_1 - \delta_1\right) + ln\left(r_2 - \delta_2\right) + K\left[\Delta V_1 + \Delta V_2 - \Delta V_{total}\right]$$

- Take partials and set equal to zero

$$\frac{\partial\left[ln\left(\lambda_o\right)\right]}{\partial r_1} = 0 \quad \frac{\partial\left[ln\left(\lambda_o\right)\right]}{\partial r_2} = 0 \quad \frac{\partial\left[ln\left(\lambda_o\right)\right]}{\partial K} = 0$$

Optimum ΔV Special Cases

- "Generic" partial of objective function

$$\frac{\partial\left[ln\left(\lambda_o\right)\right]}{\partial r_i}=\frac{1}{r_i-\delta_i}+K\frac{V_{e,i}}{r_i}=0$$

- Case 1: $\delta_1=\delta_2$ $V_{e,1}=V_{e,2}$

$$r_1=r_2\Longrightarrow\Delta V_1=\Delta V_2=\frac{\Delta V_{total}}{2}$$

- Same principle holds for n stages

$$r_1=r_2=\cdots=r_n\Longrightarrow$$

$$\Delta V_1=\Delta V_2=\cdots=\Delta V_n=\frac{\Delta V_{total}}{n}$$

Sensitivity to Inert Mass

ΔV for multistaged rocket

$$\Delta V_{tot} = \sum_{k=1}^{n\ stages} \Delta V_k = \sum_{k=1}^{n} V_{e,k} \ln\left(\frac{m_{o,k}}{m_{f,k}}\right)$$

where

$$m_{o,k} = m_{pl} + m_{pr,k} + m_{in.k} + \sum_{j=k+1}^{n} (m_{pr,j} + m_{in,j})$$

$$m_{f,k} = m_{pl} + m_{in.k} + \sum_{j=k+1}^{n} (m_{pr,j} + m_{in,j})$$

Finding Payload Sensitivity to Inert Mass

- Given the equation linking mass to ΔV, take

$$\frac{\partial(\Delta V_{tot})}{\partial m_{pl}} dm_{pl} + \frac{\partial(\Delta V_{tot})}{\partial m_{in,j}} dm_{in,j} = 0$$

and solve to find

$$\left.\frac{\partial m_{pl}}{\partial m_{in,k}}\right|_{\partial(\Delta V_{tot})=0} = \frac{-\sum_{j=1}^{k} V_{e,j}\left(\frac{1}{m_{o,j}} - \frac{1}{m_{f,j}}\right)}{\sum_{\ell=1}^{N} V_{e,\ell}\left(\frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}}\right)}$$

- This equation shows the "trade-off ratio" - Δpayload resulting from a change in inert mass for stage k (for a vehicle with N total stages)

Trade-off Ratio Example: Gemini-Titan II

	Stage 1	Stage 2
Initial Mass (kg)	150,500	32,630
Final Mass (kg)	39,370	6099
Ve (m/sec)	2900	3097
$dm_{pl}/dm_{in,k}$	-0.1164	-1

Payload Sensitivity to Propellant Mass

- In a similar manner, solve to find

$$\left.\frac{\partial m_{pl}}{\partial m_{pr,k}}\right|_{\partial(\Delta V_{tot})=0} = \frac{-\sum_{j=1}^{k} V_{e,j}\left(\frac{1}{m_{o,j}}\right)}{\sum_{\ell=1}^{N} V_{e,\ell}\left(\frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}}\right)}$$

- This equation gives the change in payload mass as a function of additional propellant mass (all other parameters held constant)

Trade-off Ratio Example: Gemini-Titan II

	Stage 1	Stage 2
Initial Mass (kg)	150,500	32,630
Final Mass (kg)	39,370	6099
Ve (m/sec)	2900	3097
$dm_{pl}/dm_{in,k}$	-0.1164	-1
$dm_{pl}/dm_{pr,k}$	0.04124	0.2443

Payload Sensitivity to Exhaust Velocity

- Use the same technique to find the change in payload resulting from additional exhaust velocity for stage k

$$\left.\frac{\partial m_{pl}}{\partial V_{e,k}}\right|_{\partial(\Delta V_{tot})=0} = \frac{\sum_{j=1}^{k} \ln\left(\frac{m_{o,j}}{m_{f,j}}\right)}{\sum_{\ell=1}^{N} V_{e,\ell}\left(\frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}}\right)}$$

- This trade-off ratio (unlike the ones for inert and propellant masses) has units - kg/(m/sec)

Trade-off Ratio Example: Gemini-Titan II

	Stage 1	Stage 2
Initial Mass (kg)	150,500	32,630
Final Mass (kg)	39,370	6099
Ve (m/sec)	2900	3097
$dm_{pl}/dm_{in,k}$	-0.1164	-1
$dm_{pl}/dm_{pr,k}$	0.04124	0.2443
$dm_{pl}/dV_{e,k}$ (kg/m/sec)	2.870	6.459

Parallel Staging

- Multiple dissimilar engines burning simultaneously
- Frequently a result of upgrades to operational systems
- General case requires "brute force" numerical performance analysis

Parallel-Staging Rocket Equation

- Momentum at time t:

$$M = mv$$

- Momentum at time t+Δt:
(subscript "b"=boosters; "c"=core vehicle)

$$M = (m - \Delta m_b - \Delta m_c)(v + \Delta v)$$
$$+\Delta m_b(v - V_{e,b}) + \Delta m_c(v - V_{e,c})$$

- Assume thrust (and mass flow rates) constant

Parallel-Staging Rocket Equation

- Rocket equation during booster burn

$$\Delta V = -\bar{V}_e \ln\left(\frac{m_{final}}{m_{initial}}\right) = -V_e \ln\left(\frac{m_{in,b}+m_{in,c}+\chi m_{pr,c}+m_{0,2}}{m_{in,b}+m_{pr,b}+m_{in,c}+m_{pr,c}+m_{0,2}}\right)$$

- where χ= fraction of core propellant remaining after booster burnout, and where

$$\bar{V}_e = \frac{V_{e,b}\dot{m}_b+V_{e,c}\dot{m}_c}{\dot{m}_b+\dot{m}_c} = \frac{V_{e,b}m_{pr,b}+V_{e,c}(1-\chi)m_{pr,c}}{m_{pr,b}+(1-\chi)m_{pr,c}}$$

Analyzing Parallel-Staging Performance

Parallel stages break down into pseudo-serial stages:

- Stage "0" (boosters and core)

$$\Delta V_0 = -\bar{V}_e \ln \left(\frac{m_{in,b}+m_{in,c}+\chi m_{pr,c}+m_{0,2}}{m_{in,b}+m_{pr,b}+m_{in,c}+m_{pr,c}+m_{0,2}} \right)$$

- Stage "1" (core alone)

$$\Delta V_1 = -\bar{V}_e \ln \left(\frac{m_{in,c}+m_{0,2}}{m_{in,c}+\chi m_{pr,c}+m_{0,2}} \right)$$

- Subsequent stages are as before

Parallel Staging Example: Space Shuttle

- 2 x solid rocket boosters (data below for single SRB)
  - Gross mass 589,670 kg
  - Empty mass 86,183 kg
  - Isp 269 sec
  - Burn time 124 sec
- External tank (space shuttle main engines)
  - Gross mass 750,975 kg
  - Empty mass 29,930 kg
  - Isp 455 sec
  - Burn time 480 sec
- "Payload" (orbiter + P/L) 125,000 kg

Shuttle Parallel Staging Example

$$V_{e,b} = gI_{sp,e} = (9.8)(269) = 2636\frac{m}{sec} \qquad V_{e,c} = 4459\frac{m}{sec}$$

$$\chi = \frac{480 - 124}{480} = 0.7417$$

$$\bar{V}_e = \frac{2636(1,007,000) + 4459(721,000)(1 - .7417)}{1,007,000 + 721,000(1 - .7417)} = 2921\frac{m}{sec}$$

$$\Delta V_0 = -2921 \ln\frac{862,000}{3,062,000} = 3702\frac{m}{sec}$$

$$\Delta V_1 = -4459 \ln\frac{154,900}{689,700} = 6659\frac{m}{sec}$$

$$\Delta V_{tot} = 10,360\frac{m}{sec}$$

Modular Staging

- Use identical modules to form multiple stages
- Have to cluster modules on lower stages to make up for nonideal ΔV distributions
- Advantageous from production and development cost standpoints

Module Analysis

- All modules have the same inert mass and propellant mass
- Because δ varies with payload mass, not all modules have the same δ!
- Introduce two new parameters

$$\varepsilon \equiv \frac{m_{in}}{m_{in} + m_{pr}} \qquad \sigma \equiv \frac{m_{in}}{m_{pr}}$$

- Conversions $\varepsilon = \dfrac{\delta}{1-\lambda} \qquad \sigma = \dfrac{\delta}{1-\delta-\lambda}$

Rocket Equation for Modular Boosters

- Assuming n modules in stage 1,

$$r_1 = \frac{n(m_{in}) + m_{o2}}{n(m_{in} + m_{pr}) + m_{o2}} = \frac{n\varepsilon + \frac{m_{o2}}{m_{mod}}}{n + \frac{m_{o2}}{m_{mod}}}$$

- If all 3 stages use same modules, n_j for stage j,

$$r_1 = \frac{n_1\varepsilon + n_2 + n_3 + \rho_{pl}}{n_1 + n_2 + n_3 + \rho_{pl}}$$

where $\rho_{pl} \equiv \frac{m_{pl}}{m_{mod}};\ m_{mod} = m_{in} + m_{pr}$

Modular Example

- Let's build a launch vehicle out of seven Space Shuttle Solid Rocket Boosters
  - M_{in}=86,180 kg
  - M_{pr}=503,500 kg

$$\varepsilon \equiv \frac{m_{in}}{m_{in} + m_{pr}} = 0.1461 \qquad \sigma \equiv \frac{m_{in}}{m_{pr}} = 0.1711$$

- Look at possible approaches to sequential firing

Modular Sequencing - SRB Example

- Assume no payload
- All seven firing at once - ΔV_{tot}=5138 m/sec
- 3-3-1 sequence - ΔV_{tot}=9087 m/sec
- 4-2-1 sequence - ΔV_{tot}=9175 m/sec
- 2-2-2-1 sequence - ΔV_{tot}=9250 m/sec
- 2-1-1-1-1-1 sequence - ΔV_{tot}=9408 m/sec
- 1-1-1-1-1-1-1 sequence - ΔV_{tot}=9418 m/sec
- Sequence limited by need to balance thrust laterally