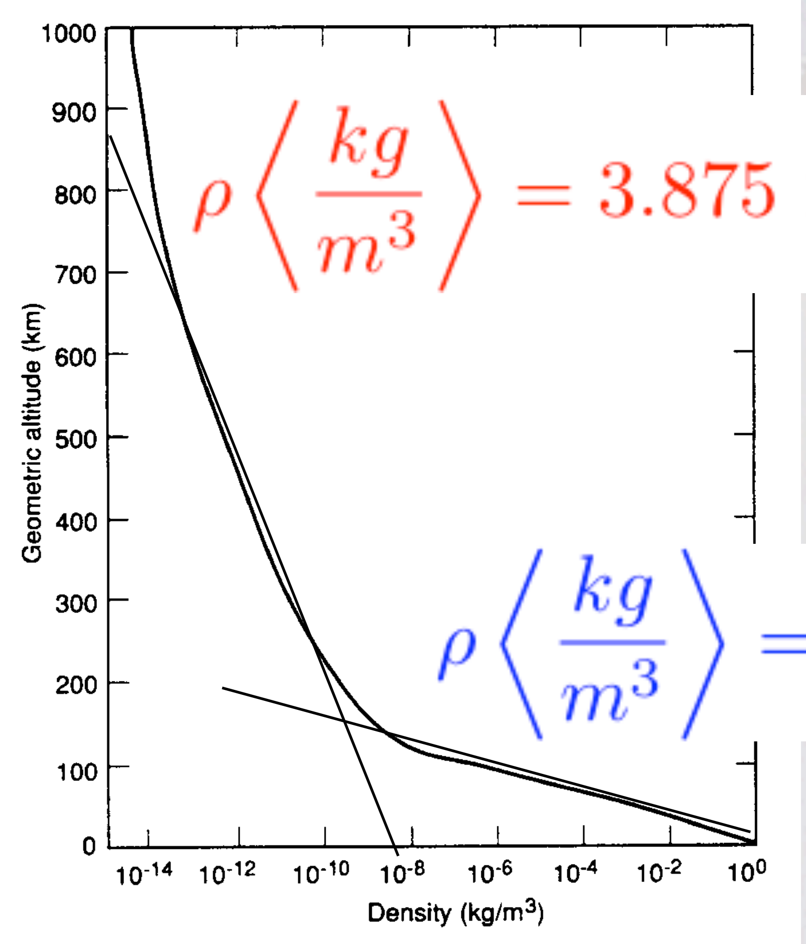
Ballistic Atmospheric Entry

- Standard atmospheres
- Orbital decay due to atmospheric drag
- Straight-line (no gravity) ballistic entry

Atmospheric Density with Altitude



$$\rho \left\langle \frac{kg}{m^3} \right\rangle = 3.875 \times 10^{-9} e^{-\frac{h\langle km \rangle}{59.06}}$$

$$\rho \left\langle \frac{kg}{m^3} \right\rangle = 1.226 e^{-\frac{h\langle km \rangle}{7.524}}$$

Ref: V. L. Pisacane and R. C. Moore, Fundamentals of Space Systems Oxford University Press, 1994

Energy Loss Due to Atmospheric Drag

$$\text{Drag } D \equiv \frac{1}{2}\rho v^2 A c_D \quad \text{Drag acceleration } a_d = \frac{D}{m} = \frac{\rho v^2}{2}\frac{A c_D}{m}$$

$$\beta = \frac{m}{c_D A} \qquad a_d = \frac{\rho v^2}{2\beta}$$

$$\text{orbital energy} \equiv E = -\frac{\mu}{2a}$$

$$\frac{dE}{dt} = \frac{\mu}{2a^2}\frac{da}{dt}$$

Since drag is highest at perigee, the first effect of atmospheric drag is to circularize the orbit (high perigee drag lowers apogee)

$$\frac{dE_{drag}}{dt} = a_d v \qquad v_{circ}^2 = \frac{\mu}{a} \qquad \frac{dE_{drag}}{dt} = -\frac{\rho v^2}{2\beta}\sqrt{\frac{\mu}{a}}$$

$$\frac{dE_{drag}}{dt} = -\sqrt{\frac{\mu}{a}}\frac{\rho}{2\beta}\frac{\mu}{a} = -\left(\frac{\mu}{a}\right)^{\frac{3}{2}}\frac{\rho}{2\beta}$$

Derivation of Orbital Decay Due to Drag

Set orbital energy variation equal to energy lost by drag

$$\frac{\mu}{2a^2}\frac{da}{dt} = -\frac{\rho}{2\beta}\left(\frac{\mu}{a}\right)^{\frac{3}{2}}$$

$$\frac{da}{dt} = -\frac{\rho}{\beta}\sqrt{\mu a}$$

$$\rho = \rho_o e^{-\frac{h}{h_s}} \qquad a = h + r_E \Longrightarrow \frac{da}{dt} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{\sqrt{\mu\left(h + r_E\right)}}{\beta}\rho_o e^{-\frac{h}{h_s}}$$

Derivation of Orbital Decay (2)

This is a separable differential equation...

$$\frac{1}{\sqrt{r_E + h}} e^{\frac{h}{h_s}} dh = -\frac{\sqrt{\mu}}{\beta} \rho_o dt$$

$$\int_{h_o}^{h} \frac{1}{\sqrt{r_E + h}} e^{\frac{h}{h_s}} dh = -\frac{\sqrt{\mu}}{\beta} \rho_o \int_{t_o}^{t} dt$$

$$\text{Assume } \sqrt{r_E + h} \sim \sqrt{r_E} \text{ for } r_E \gg h$$

$$\frac{1}{\sqrt{r_E}} \int_{h_o}^{h} e^{\frac{h}{h_s}} dh = -\frac{\sqrt{\mu}}{\beta} \rho_o (t - t_o)$$

Derivation of Orbital Decay (3)

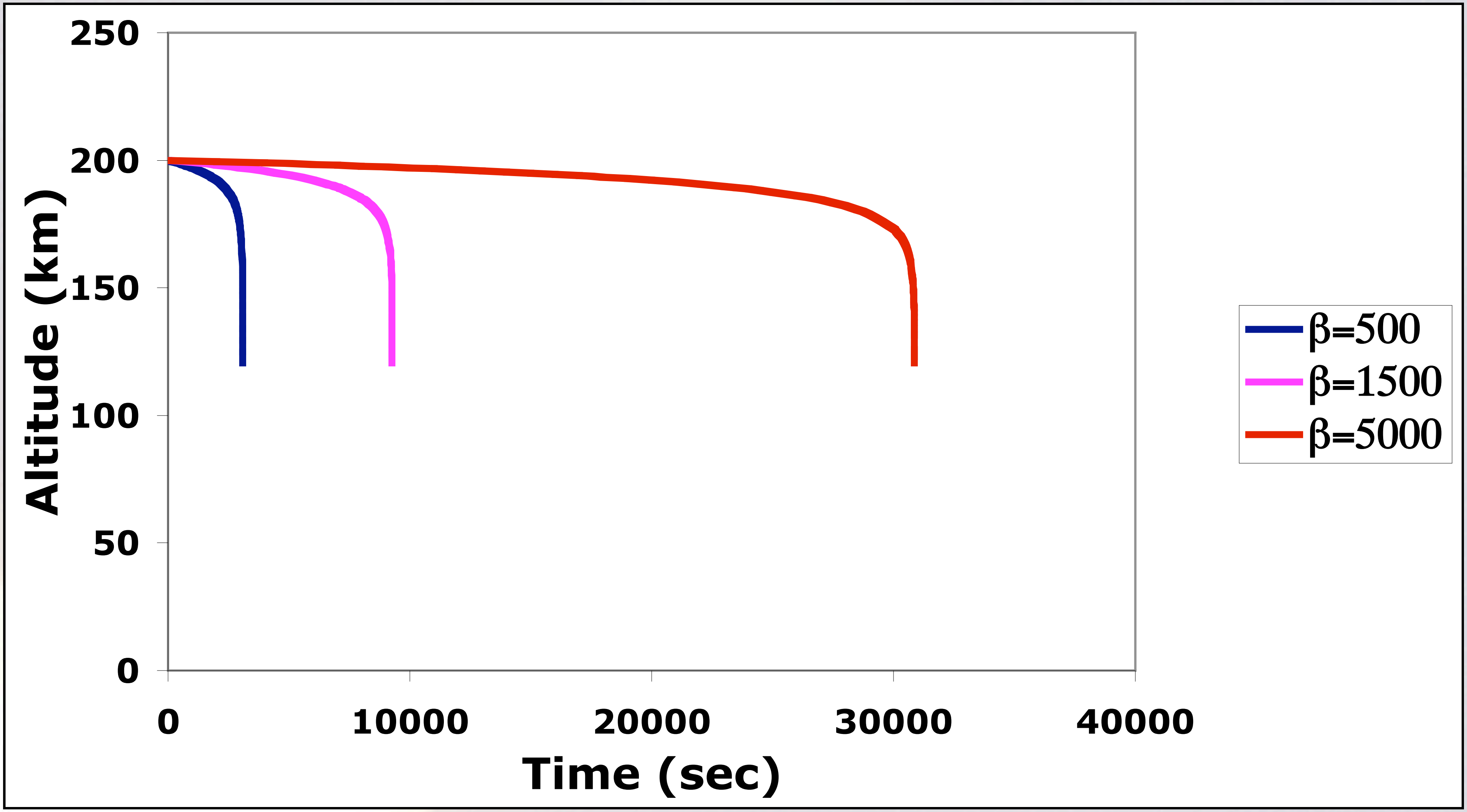
$$\frac{h_s}{\sqrt{r_E}}\left(e^{\frac{h}{h_s}} - e^{\frac{h_o}{h_s}}\right) = -\frac{\sqrt{\mu}}{\beta}\rho_o\left(t - t_o\right)$$

$$e^{\frac{h}{h_s}} - e^{\frac{h_o}{h_s}} = -\frac{\sqrt{\mu r_E}}{h_s\beta}\rho_o\left(t - t_o\right)$$

$$h(t) = h_s \ln\left[e^{\frac{h_o}{h_s}} - \frac{\sqrt{\mu r_E}}{h_s\beta}\rho_o\left(t - t_o\right)\right]$$

Note that some variables typically use km, and others are in meters - you have to make sure unit conversions are done properly to make this work out correctly!

Orbit Decay from Atmospheric Drag



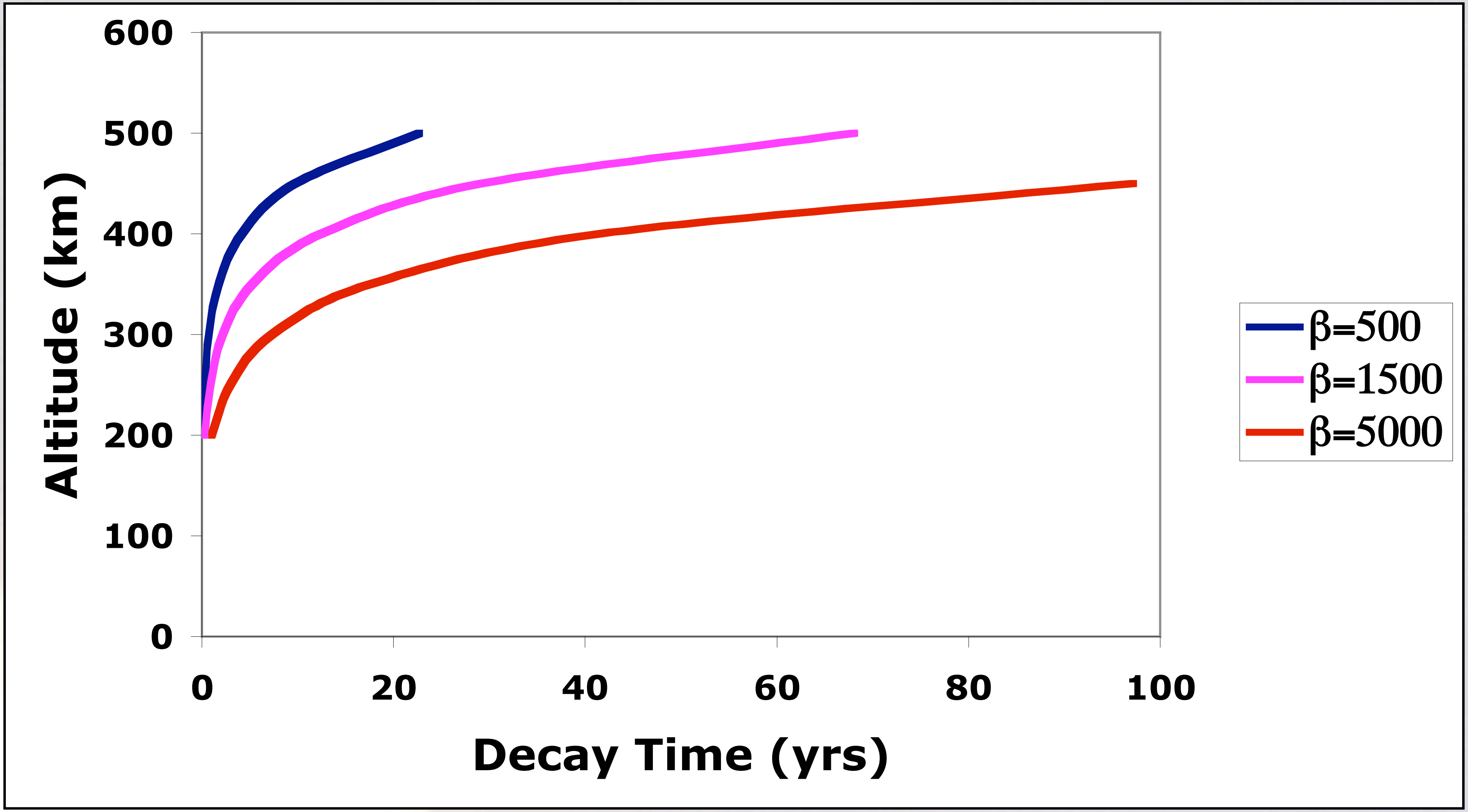
Time Until Orbital Decay

$$e^{\frac{h}{h_s}} - e^{\frac{h_o}{h_s}} = -\frac{\sqrt{\mu r_E}}{h_s\beta}\rho_o(t - t_o)$$

To find the time remaining (t_o=0) until the orbit reaches any given "critical" altitude, some algebra gives

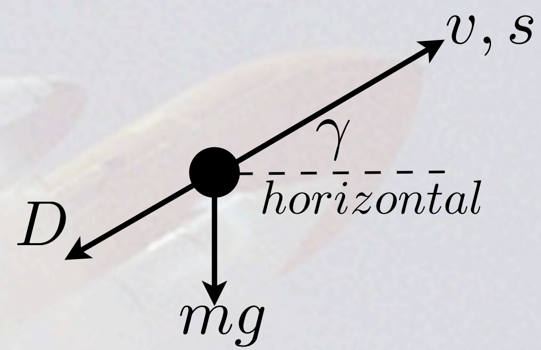
$$t(h) = \frac{h_s\beta}{\sqrt{\mu r_E}\rho_o}\left(e^{\frac{h_o}{h_s}} - e^{\frac{h}{h_s}}\right)$$

Decay Time to r=120 km



Ballistic Entry (no lift)

$s =$ distance along the flight path



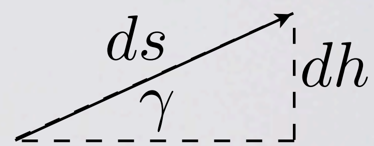
$$\frac{dv}{dt} = -g\sin\gamma - \frac{D}{m}$$

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = V\frac{dv}{ds} = \frac{1}{2}\frac{d(v^2)}{ds}$$

$$\frac{1}{2}\frac{d(v^2)}{ds} = -g\sin\gamma - \frac{D}{m}$$

Drag $D \equiv \frac{1}{2}\rho v^2 A c_D$

$$\frac{1}{2}\frac{d(v^2)}{ds} = -g\sin\gamma - \frac{\rho v^2}{2m}A c_D$$



$$ds = \frac{dh}{\sin\gamma}$$

$$\frac{\sin\gamma}{2}\frac{d(v^2)}{dh} = -g\sin\gamma - \frac{\rho v^2}{2m}A c_D$$

Ballistic Entry (2)

$$\text{Exponential atmosphere} \Rightarrow \rho = \rho_o e^{-\frac{h}{h_s}}$$

$$\frac{d\rho}{\rho_o} = e^{-\frac{h}{h_s}} \left(\frac{-dh}{h_s} \right) = \frac{\rho_o e^{-\frac{h}{h_s}}}{\rho_o} \left(\frac{-dh}{h_s} \right) = \frac{\rho}{\rho_o} \left(\frac{-dh}{h_s} \right)$$

$$dh = \frac{-h_s}{\rho} d\rho$$

$$\frac{\sin\gamma}{2} \frac{d(v^2)}{dh} = -g\sin\gamma - \frac{\rho v^2}{2m} A c_D$$

$$\frac{\sin\gamma}{2} \frac{d(v^2)}{d\rho} \left(\frac{-\rho}{h_s} \right) = -g\sin\gamma - \frac{\rho v^2}{2} \frac{A c_D}{m}$$

$$\frac{d(v^2)}{d\rho} = \frac{2 g h_s}{\rho} + \frac{h_s v^2}{\sin\gamma} \frac{A c_D}{m}$$

Ballistic Entry (3)

Let $\beta \equiv \dfrac{m}{c_D A} \Rightarrow$ Ballistic Coefficient

$$\frac{d(v^2)}{d\rho} - \frac{h_s}{\beta \sin\gamma} v^2 = \frac{2 g h_s}{\rho}$$

Assume $mg \ll D$ to get homogeneous ODE

$$\frac{d(v^2)}{d\rho} - \frac{h_s}{\beta \sin\gamma} v^2 = 0 \qquad \frac{d(v^2)}{v^2} = \frac{h_s}{\beta \sin\gamma} d\rho$$

Use (v^2) as integration variable

$$\int_{v_e}^{v} \frac{d(v^2)}{v^2} = \frac{h_s}{\beta \sin\gamma} \int_0^{\rho} d\rho \qquad v_e = \text{velocity at entry}$$

Ballistic Entry (4)

Note that the effect of ignoring gravity is that there is no force perpendicular to velocity vector ⇒ constant flight path angle γ ⇒ straight line trajectories

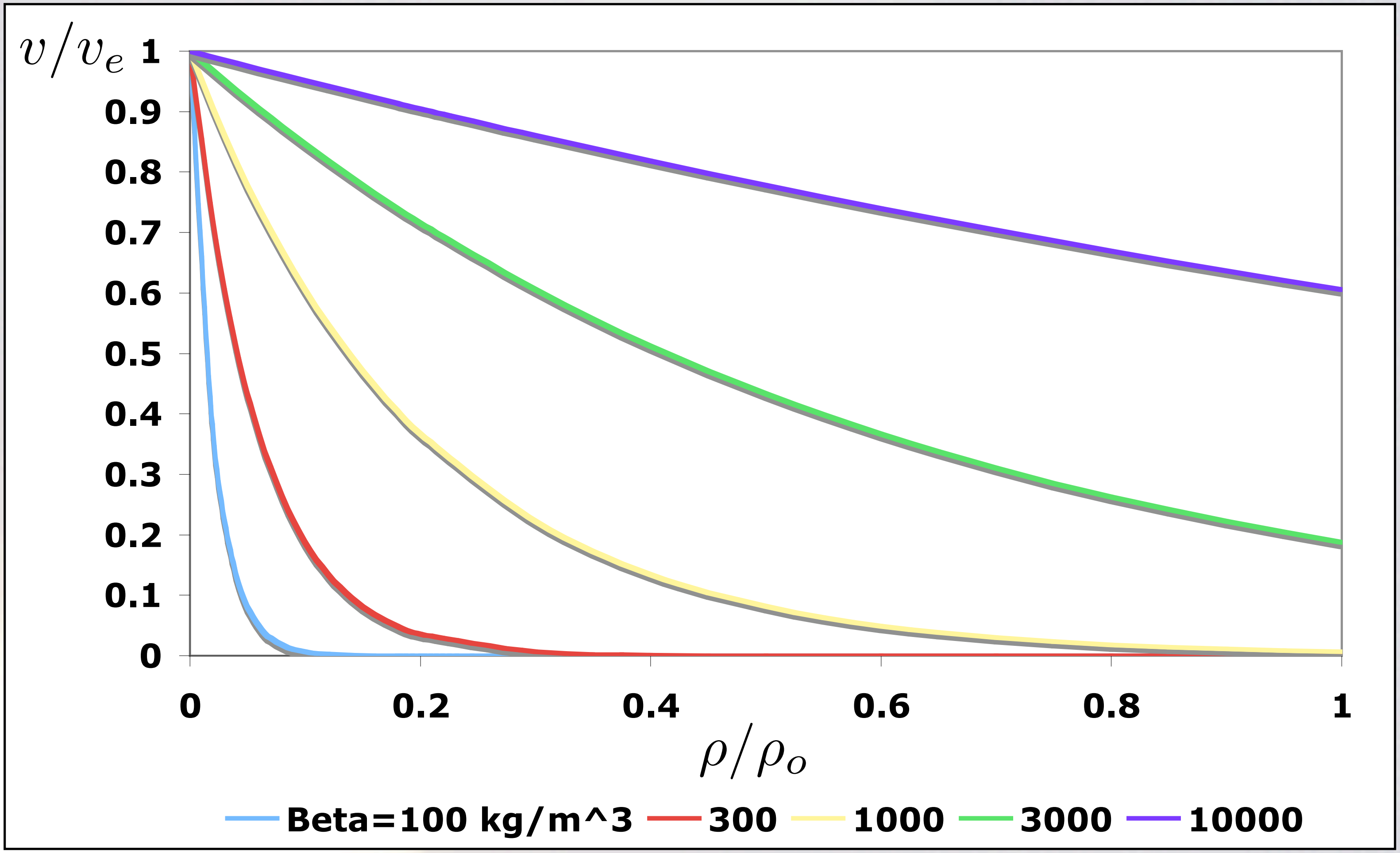
$$\ln \frac{v^2}{v_e^2} = 2 \ln \frac{v}{v_e} = \frac{h_s \rho}{\beta \sin \gamma}$$

$$\frac{v}{v_e} = \exp\left(\frac{h_s \rho}{2\beta \sin \gamma} \right)$$

$$\frac{v}{v_e} = \exp\left(\frac{h_s \rho_o}{2\beta \sin \gamma} \frac{\rho}{\rho_o} \right)$$

Check units: $\left(\frac{m \frac{kg}{m^3}}{\frac{kg}{m^2}} \right)$

Earth Entry, γ=-60°



What About Peak Deceleration?

$$n \equiv \frac{dv}{dt} = -\frac{\rho v^2}{2\beta}$$

To find n_{max}, set $\frac{d}{dt}\left(\frac{dv}{dt}\right) = \frac{d^2v}{dt^2} = 0$

$$\frac{d^2v}{dt^2} = -\frac{1}{2\beta}\left(2\rho v\frac{dv}{dt} + v^2\frac{d\rho}{dt}\right) = 0$$

$$\frac{d^2v}{dt^2} = -\frac{1}{2\beta}\left(-\frac{2\rho^2 v^3}{2\beta} + v^2\frac{d\rho}{dt}\right) = 0$$

$$\frac{\rho^2 v^3}{\beta} = v^2\frac{d\rho}{dt} \qquad \rho^2 v = \beta\frac{d\rho}{dt}$$

Peak Deceleration (2)

From exponential atmosphere,

$$\frac{d\rho}{dt} = -\frac{\rho_o}{h_s} e^{-\frac{h}{h_s}} \frac{dh}{dt} = -\frac{\rho}{h_s}\frac{dh}{dt}$$

From geometry, $\frac{dh}{dt} = v \sin\gamma$

$$\frac{d\rho}{dt} = -\frac{\rho v}{h_s}\sin\gamma \qquad \rho^2 v = \beta \frac{d\rho}{dt}$$

$$\rho^2 v = \beta\left(-\frac{\rho v}{h_s}\sin\gamma\right)$$

Remember that this refers to the conditions at max deceleration

$$\rho_{n_{max}} = -\frac{\beta}{h_s}\sin\gamma$$

Critical β for Deceleration Before Impact

At surface, $\rho = \rho_o$

$$\beta_{crit} = -\frac{\rho_o h_s}{\sin\gamma}$$ $\Leftarrow$ Value of β at which vehicle hits ground at point of maximum deceleration

How large is maximum deceleration?

$$\frac{dv}{dt} = \frac{\rho v^2}{2\beta} \qquad \Rightarrow \left|\frac{dv}{dt}\right|_{max} = \frac{\rho_{n_{max}} v^2}{2\beta}$$

$$\left|\frac{dv}{dt}\right|_{max} = \frac{v^2}{2\beta}\left(-\frac{\beta}{h_s}\sin\gamma\right) = -\frac{1}{2}\frac{v^2}{h_s}\sin\gamma$$

Note that this value of v is actually $v_{n_{max}}$

Peak Deceleration (3)

From page 13,

$$\frac{v}{v_e} = \exp\left(\frac{h_s \rho}{2\beta \sin\gamma}\right)$$

$$\frac{v_{n_{max}}}{v_e} = \exp\left[\frac{h_s}{2\beta \sin\gamma}\left(-\frac{\beta}{h_s}\sin\gamma\right)\right] = e^{-\frac{1}{2}}$$

$$\left|\frac{dv}{dt}\right|_{max} = -\frac{1}{2}\frac{\left(v_e e^{-\frac{1}{2}}\right)^2}{h_s}\sin\gamma = -\frac{v_e^2 \sin\gamma}{2h_s e}$$

Note that the velocity at which maximum deceleration occurs is always a fixed fraction of the entry velocity - it doesn't depend on ballistic coefficient, flight path angle, or anything else! Also, the magnitude of the maximum deceleration is not a function of ballistic coefficient - it is dependent on the entry trajectory (v_e and γ) but not spacecraft parameters (i.e., ballistic coefficient).

Terminal Velocity

Full form of ODE -

$$\frac{d\left(v^2\right)}{d\rho} - \frac{h_s}{\beta \sin\gamma} v^2 = \frac{2gh_s}{\rho}$$

At terminal velocity, $v = \text{constant} \equiv v_T$

$$-\frac{h_s}{\beta \sin\gamma} v_T^2 = \frac{2gh_s}{\rho}$$

$$v_T^2 = \sqrt{-\frac{2g\beta \sin\gamma}{\rho}}$$

"Cannon Ball" γ=-90° Ballistic Entry

6.75" diameter sphere, c_D=0.2, V_E=6000 m/sec

	Iron	Aluminum	Balsa Wood
Weight	40 lb	15.6 lb	14.5 oz
β_{md} (kg/m^2)	3938	1532	89
ρ_{md} (kg/m^3)	0.555	0.216	0.0125
h (m)	5600	12,300	32,500
V_{impact} (m/s)	1998	355	0*
V_{term} (m/sec)	251	156	38