ENAE 791 - Spring, 2008
Problem Set 1 Solutions

1. A spacecraft in orbit around the Earth has the current position $\vec{r}$=(0, 7680 km, 0) and velocity of $\vec{v}$=(-7.204 km/sec, 1.441 km/sec, 0). At the current time, calculate

a) Angular momentum $\vec{h}$

$\vec{h} = \vec{r} \times \vec{v}$=(0, 0, 55,330 km$^2$/sec)

b) Semi-major axis a

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \Rightarrow a = \left(\frac{2}{r} - \frac{v^2}{\mu}\right)^{-1} \Rightarrow a = \left(\frac{2}{7680} - \frac{7.204^2 + 1.441^2}{398,604}\right)^{-1}$$
$$a = 8000 \text{ km}$$

c) Eccentricity e

$$p = a(1 - e^2) = \frac{h^2}{\mu} \Rightarrow e = \sqrt{1 - \frac{h^2}{a\mu}} = \sqrt{1 - \frac{55330^2}{(8000)(398604)}}$$
$$e = 0.200$$

d) Parameter p

$$p = a(1 - e^2) = 8000(1 - 0.2^2) = 7680 \text{ km}$$

e) Perigee radius $r_p$

$$r_p = a(1 - e) = 8000(1 - 0.2) = 6400 \text{ km}$$

f) Apogee radius $r_a$

$$r_a = a(1 + e) = 8000(1 + 0.2) = 9600 \text{ km}$$

g) Flight path angle $\gamma$

$$\gamma = \tan^{-1}\frac{v_r}{v_\theta}$$

Because of the specific geometry here (r aligned with y axis), we can write simply

$$\gamma = \tan^{-1}\frac{1.441}{7.204} = 11.31 \text{ deg}$$

h) True anomaly $\theta$

$$r = \frac{p}{1 - e\cos\theta} \Rightarrow \theta = \cos^{-1}\left[\frac{1}{e}\left(1 - \frac{p}{r}\right)\right] = \cos^{-1}\left[\frac{1}{0.2}\left(1 - \frac{7680}{7680}\right)\right] = \pm\frac{\pi}{2}$$

Since $v_r$ is positive with respect to r, the radial distance is increasing, so the spacecraft is on the first half of the ellipse, so $\theta = 90$ deg

i) Orbital period P

$$P = 2\pi\sqrt{\frac{a^3}{\mu}} = 2\pi\sqrt{\frac{8000^3}{398604}} = 7121 \text{ sec}$$

j) Time since perigee passage $t_p$

First calculate the eccentric anomaly

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2} = \sqrt{\frac{1-0.2}{1+0.2}}\tan\frac{\pi/2}{2} \Rightarrow E = 78.46 \text{ deg} = 1.369 \text{ rad}$$

$$n = \sqrt{\frac{\mu}{a^3}} = \sqrt{\frac{398604}{8000^3}} = 8.823 \times 10^{-4} \ \frac{\text{rad}}{\text{sec}}$$

$$t = \frac{1}{n}(E - e\sin E) = \frac{1}{8.823 \times 10^{-4}}(1.369 - 0.2\sin 1.369) = 1330 \text{ sec}$$

The current orbit will intersect the Earth's atmosphere at an altitude of 122 km. At that point, calculate

k) True anomaly $\theta$

$$r = r_E + h = 6378 + 122 = 6500 \text{ km}$$

$$\theta = \pm\cos^{-1}\left[\frac{1}{e}\left(\frac{p}{r} - 1\right)\right] = \pm\cos^{-1}\left[\frac{1}{0.2}\left(\frac{7680}{6500} - 1\right)\right] = \pm 24.8 \text{ deg}$$

Since the spacecraft is descending, the correct value must be $\theta = -24.8 \text{ deg} = 335.2 \text{ deg}$

l) Scalar velocity v

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} = \sqrt{398604\left(\frac{2}{6500} - \frac{1}{8000}\right)} = 8.534 \ \frac{\text{km}}{\text{sec}}$$

m) Radial velocity $v_r$

$$v_r = \sqrt{\frac{\mu}{p}}e\sin\theta = \sqrt{\frac{398604}{7860}}(0.2)\sin(335.2) = -0.6046 \ \frac{\text{km}}{\text{sec}}$$

n) Tangential velocity $v_\theta$

$$v_\theta = \sqrt{\frac{\mu}{p}}\,(1 + e\cos\theta) = \sqrt{\frac{398604}{7680}}\,(1 + 0.2\cos 335.2) = 8.512\ \frac{\text{km}}{\text{sec}}$$

o) Flight path angle $\gamma$

$$\gamma = \tan^{-1}\frac{v_r}{v_\theta} = \tan^{-1}\frac{-0.6046}{8.512} = -4.06\ \text{deg}$$

p) Time from the initial point specified above

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2} = \sqrt{\frac{1-0.2}{1+0.2}}\tan\frac{335.2}{2} \Rightarrow E = 349.8\ \text{deg} = 6.105\ \text{rad}$$

$$t = \frac{1}{n}(E - e\sin E) = \frac{1}{8.823\times 10^{-4}}(6.105 - 0.2\sin 349.8) = 6960\ \text{sec since perigee}$$

Time since the initial point $= 6960 - 1330 = 5630$ sec

2. Consider a two-stage vehicle for launching 50,000 kg of payload to Earth orbit ($\Delta$V-9200 m/sec), and three choices for propellants in each stage: LOX/LH2, storables, and solids. Using the mean values for $\delta$ and Ve from the lecture notes, find the optimum $\Delta$V distribution for each possible propellant combination to find the one that minimizes $\delta/\lambda$. What is the gross vehicle mass and inert masses for each of the two stages for each case? (Hint: find numerical solutions, not analytic ones. Some form of mechanization (spreadsheet or Matlab) is highly recommended.)

Using the parameters for $V_e$ and $\delta$ from page 14 of the Lecture #02 notes, and choosing an arbitrary initial value for $\Delta v_2$, I set up the following equations in a spreadsheet:

$$r_1 = e^{-\frac{\Delta v_1}{v_e}} \qquad r_2 = e^{-\frac{\Delta v_2}{v_e}}$$

$$\lambda_1 = r_1 - \delta_1 \qquad \lambda_2 = r_2 - \delta_2$$

$$M_{o2} = \frac{M_{pl}}{\lambda_2} \qquad M_o = \frac{M_{o2}}{\lambda_1}$$

$$M_{in1} = \delta_1 M_o \qquad M_{in2} = \delta_2 M_{o2}$$

$$\delta_o = \frac{M_{in1} + M_{in2}}{M_o} \qquad \lambda_o = \frac{M_{pl}}{M_o}$$

and used Solver in Excel to find the value of $\Delta v_2$ that minimizes $\delta_o/\lambda_o$. By repeating this for each of the nine cases, I generated the data in the attached table. The preferred system is one where both stages are LOX/LH2, and the $\Delta$V split is 5824/3376 m/sec.

| | First Stage | Second Stage | First Stage | Second Stage | First Stage | Second Stage |
|---|---|---|---|---|---|---|
| **Propellants** | Solid | Solid | Solid | Storables | Solid | Cryo |
| $\Delta v$ | 4747 | 4453 | 3881 | 5319 | 2546 | 6654 |
| $r$ | 0.1805 | 0.2007 | 0.2467 | 0.1756 | 0.3992 | 0.2107 |
| $\delta_{stage}$ | 0.087 | 0.087 | 0.087 | 0.061 | 0.087 | 0.075 |
| $\lambda_{stage}$ | 0.0935 | 0.1137 | 0.1597 | 0.1146 | 0.3122 | 0.1357 |
| $M_o$ | 4,700,932 | 439,692 | 2,731,243 | 436,129 | 1,179,842 | 368,341 |
| $M_{inert}$ | 408,981 | 38,253 | 237,618 | 26,604 | 102,646 | 27,626 |
| $\delta_0$ | | 0.0951 | | 0.0967 | | 0.1104 |
| $\lambda_0$ | | 0.0106 | | 0.0183 | | 0.0424 |
| $\delta_0/\lambda_o$ | | 8.945 | | 5.284 | | 2.605 |
| **Propellants** | Storables | Solids | Storables | Storables | Storables | Cryo |
| $\Delta v$ | 5908 | 3292 | 5084 | 4116 | 3736 | 5464 |
| $r$ | 0.1449 | 0.3051 | 0.1897 | 0.2603 | 0.2947 | 0.2784 |
| $\delta_{stage}$ | 0.061 | 0.087 | 0.061 | 0.061 | 0.061 | 0.075 |
| $\lambda_{stage}$ | 0.0839 | 0.2181 | 0.1287 | 0.1993 | 0.2337 | 0.2034 |
| $M_o$ | 2,733,865 | 229,294 | 1,950,087 | 250,893 | 1,051,814 | 245,819 |
| $M_{inert}$ | 166,766 | 19,949 | 118,955 | 15,304 | 64,161 | 18,436 |
| $\delta_0$ | | 0.0683 | | 0.0688 | | 0.0785 |
| $\lambda_0$ | | 0.0183 | | 0.0256 | | 0.0475 |
| $\delta_0/\lambda_o$ | | 3.734 | | 2.685 | | 1.652 |
| **Propellants** | Cryo | Solids | Cryo | Storables | Cryo | Cryo |
| $\Delta v$ | 7984 | 1216 | 7314 | 1886 | 5824 | 3376 |
| $r$ | 0.1544 | 0.645 | 0.1805 | 0.5398 | 0.2559 | 0.4538 |
| $\delta_{stage}$ | 0.075 | 0.087 | 0.075 | 0.061 | 0.075 | 0.075 |
| $\lambda_{stage}$ | 0.0794 | 0.558 | 0.1055 | 0.4788 | 0.1809 | 0.3788 |
| $M_o$ | 1,129,141 | 89,602 | 989,470 | 104,435 | 729,665 | 131,978 |
| $M_{inert}$ | 84,686 | 7,795 | 74,210 | 6,371 | 54,725 | 9,898 |
| $\delta_0$ | | 0.0819 | | 0.0814 | | 0.0886 |
| $\lambda_0$ | | 0.0443 | | 0.0505 | | 0.0685 |
| $\delta_0/\lambda_o$ | | 1.85 | | 1.612 | | 1.292 |

3. The Falcon 1e launch vehicle by SpaceX has the following properties:

1st stage empty mass - 4000 lb
1st stage propellant mass - 69,000 lb
1st stage exhaust velocity - 2979 m/sec
1st stage nominal burn time - 169 sec
2nd stage empty mass - 1125 lb
2nd stage propellant mass - 8881 lb
2nd stage exhaust velocity - 3205 m/sec
Payload mass - 1529 lb
Payload fairing mass - 300 lb

a) Calculate the ΔV contributions of each of the stages. Assume the payload fairing is jettisoned at the same time as the first stage

$$r_1 = \frac{4000 + 1125 + 8881 + 1529 + 300}{4000 + 69000 + 1125 + 8881 + 1529 + 300} = 0.1867$$

$$\Delta V_1 = -v_e \ln r_1 = -2979 \ln(0.1867) = 5000 \ \frac{\text{m}}{\text{sec}}$$

$$r_2 = \frac{1125 + 1529}{1125 + 8881 + 1529} = 0.2301$$

$$\Delta V_2 = -v_e \ln r_2 = -3205 \ln(0.2301) = 4709 \ \frac{\text{m}}{\text{sec}}$$

b) Calculate the change in total ΔV if the payload fairing is not jettisoned, but stays with the payload all the way to orbit

$$r_2 = \frac{1125 + 1529 + 300}{1125 + 8881 + 1529 + 300} = 0.2496$$

$$\Delta V_2 = -v_e \ln r_2 = -3205 \ln(0.2496) = 4448 \ \frac{\text{m}}{\text{sec}}$$

Change in $\Delta v = 261$ m/sec

c) How much payload would you have to give up to achieve the same total ΔV as in (a) with the payload fairing retained throughout the flight?

Need to have $r_2 = \dfrac{1125 + m_{pl} + 300}{1125 + 8881 + m_{pl} + 300} = 0.2301$

$m_{pl} = 1229$ kg $\Rightarrow$ loss of (surprisingly enough) 300 kg!!!

d) Find the three trade-off ratios for both the first and second stage, assuming the fairing is jettisoned with the first stage.

$$m_{o,1} = 4000 + 69000 + 1125 + 8881 + 1529 + 300 = 84,835 \text{ kg}$$
$$m_{f,1} = 4000 + 1125 + 8881 + 1529 + 300 = 15,835 \text{ kg}$$

$m_{o,2} = 1125 + 8881 + 1529 = 11,535$ kg
$m_{f,2} = 1125 + 1529 = 2654$ kg

$$\left.\frac{\partial m_{pl}}{\partial m_{in,1}}\right|_{\partial(\Delta V_{tot})=0} = \frac{-V_{e,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right)}{V_{e,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + V_{e,2}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)}$$

$$\left.\frac{\partial m_{pl}}{\partial m_{in,1}}\right|_{\partial(\Delta V_{tot})=0} = \frac{-2979\left(\frac{1}{84835} - \frac{1}{15835}\right)}{2979\left(\frac{1}{84835} - \frac{1}{15835}\right) + 3205\left(\frac{1}{11535} - \frac{1}{2654}\right)} = -0.1413$$

$$\left.\frac{\partial m_{pl}}{\partial m_{in,2}}\right|_{\partial(\Delta V_{tot})=0} = \frac{-V_{e,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) - V_{e,2}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)}{V_{e,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + V_{e,2}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)} = -1$$

$$\left.\frac{\partial m_{pl}}{\partial m_{pr,1}}\right|_{\partial(\Delta V_{tot})=0} = \frac{-V_{e,1}\left(\frac{1}{m_{o,1}}\right)}{V_{e,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + V_{e,2}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)}$$

$$\left.\frac{\partial m_{pl}}{\partial m_{pr,1}}\right|_{\partial(\Delta V_{tot})=0} = \frac{-2979\left(\frac{1}{84835}\right)}{2979\left(\frac{1}{84835} - \frac{1}{15835}\right) + 3205\left(\frac{1}{11535} - \frac{1}{2654}\right)} = 0.03243$$

$$\left.\frac{\partial m_{pl}}{\partial m_{pr,2}}\right|_{\partial(\Delta V_{tot})=0} = \frac{-V_{e,1}\left(\frac{1}{m_{o,1}}\right) - V_{e,2}\left(\frac{1}{m_{o,2}}\right)}{V_{e,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + V_{e,2}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)}$$

$$\left.\frac{\partial m_{pl}}{\partial m_{pr,2}}\right|_{\partial(\Delta V_{tot})=0} = \frac{-2979\left(\frac{1}{84835}\right) - 3205\left(\frac{1}{11535}\right)}{2979\left(\frac{1}{84835} - \frac{1}{15835}\right) + 3205\left(\frac{1}{11535} - \frac{1}{2654}\right)} = 0.2825$$

$$\left.\frac{\partial m_{pl}}{\partial V_{e,1}}\right|_{\partial(\Delta V_{tot})=0} = \frac{\ln\left(\frac{m_{o,1}}{m_{f,1}}\right)}{V_{e,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + V_{e,2}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)}$$

$$\left.\frac{\partial m_{pl}}{\partial V_{e,2}}\right|_{\partial(\Delta V_{tot})=0} = \frac{\ln\left(\frac{84835}{15835}\right)}{2979\left(\frac{1}{84835} - \frac{1}{15835}\right) + 3205\left(\frac{1}{11535} - \frac{1}{2654}\right)} = -1.550\ \frac{\text{kg}}{\text{m/sec}}$$

$$\left.\frac{\partial m_{pl}}{\partial V_{e,2}}\right|_{\partial(\Delta V_{tot})=0} = \frac{\ln\left(\frac{m_{o,1}}{m_{f,1}}\right) + \ln\left(\frac{m_{o,2}}{m_{f,2}}\right)}{V_{e,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + V_{e,2}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)}$$

$$\left.\frac{\partial m_{pl}}{\partial V_{e,2}}\right|_{\partial(\Delta V_{tot})=0} = \frac{\ln\left(\frac{84835}{15835}\right) + \ln\left(\frac{11535}{2654}\right)}{2979\left(\frac{1}{84835} - \frac{1}{15835}\right) + 3205\left(\frac{1}{11535} - \frac{1}{2654}\right)} = -2.907\ \frac{\text{kg}}{\text{m/sec}}$$

Note: since I didn't change the masses to kg, the units of the last two terms should be $\frac{\text{lb}}{\text{m/sec}}$.

With this system, I would have to retype the entire thing to fix it, so I won't.

e) You wish to augment the payload by adding strap-on solid rocket motors to the first stage. You choose the Castor IV-A solid rocket motor, which has a total mass of 25,800 lb, a propellant mass of 22,300 lbs, an exhaust velocity of 2325 m/sec, and a burning time of 56.2 seconds. Two of these motors will be added to the first stage and ignited at lift-off in parallel with the Falcon 1e first stage engine. Find the payload capacity of this system in order to reach the same ΔV as in (a).

$$\chi = \frac{169-56.2}{169} = 0.6675$$

$$\bar{V}_e = \frac{V_{e,b}m_{pr,b}+V_{e,c}(1-\chi)m_{pr,c}}{m_{pr,b}+(1-\chi)m_{pr,c}} = \frac{2325(44600)+2979(1-0.6675)69000}{44600+(1-0.6675)69000} = 2547\ \frac{\text{m}}{\text{sec}}$$

$$m_{o,2} = 1125 + 8881 + m_{pl} + 300 = 10,306 + m_{pl}$$

$$\Delta V_0 = -\bar{V}_e \ln\left(\frac{m_{in,b}+m_{in,c}+\chi m_{pr,c}+m_{0,2}}{m_{in,b}+m_{pr,b}+m_{in,c}+m_{pr,c}+m_{0,2}}\right) = -2547\ln\left(\frac{7000+4000+0.6675(69000)+10306+m_{pl}}{7000+44600+4000+69000+10306+m_{pl}}\right)$$

$$\Delta V_0 = -2547\ln\left(\frac{67364+m_{pl}}{134900+m_{pl}}\right)$$

$$\Delta V_1 = -V_{e,c}\ln\left(\frac{m_{in,c}+m_{0,2}}{m_{in,c}+\chi m_{pr,c}+m_{0,2}}\right)$$

Notice this corrects a typo in the notes!

$$\Delta V_1 = -2979\ln\left(\frac{4000+10306+m_{pl}}{4000+0.6675(69000)+10306+m_{pl}}\right) = -2979\ln\left(\frac{14306+m_{pl}}{60364+m_{pl}}\right)$$

$$\Delta V_0 + \Delta V_1 = 5000\ \frac{\text{m}}{\text{sec}}$$

A minute with Solver gives $m_{pl} = 2389 \text{ lb} = 1084 \text{ kg}$

f) You need to put 5800 kg of payload into orbit [same ΔV as (a)]. You plan to mass-produce Falcon 1e first stages to build a modular launch vehicle to carry at least this much payload into orbit. Find the configuration (modules/stage and number of stages) which will accomplish this with the minimum total number of modules. You may use up to four stages. (To simplify this calculation, assume the payload fairing is carried all the way to orbit.)

| Problem 3f | Module stats | Stage 1 | Stage 2 | Stage 3 | Stage 4 | Payload |
|---|---|---|---|---|---|---|
| **Inert mass** | 1,815 | | | | | |
| **Propellant Mass** | 31,307 | | | | | |
| **Payload** | | | | | | 5800 |
| **Fairing mass** | | | | | | 136 |
| **Ve** | 2,979 | | | | | |
| **Number of modules** | | 4 | 2 | 1 | 0 | |
| **Mtotals** | 33,122 | | | | | 5936 |
| **Minit** | | 237,787 | 105,301 | 39,058 | 5,936 | |
| **Mfinal** | | 112,560 | 42,687 | 7,751 | 5,936 | |
| **r** | | 0.4734 | 0.4054 | 0.1984 | 1.0000 | |
| **delta-V** | | 2,228 | 2,690 | 4,818 | 0 | **9,736** |