Rocket Performance

- The rocket equation
- Mass ratio and performance
- Structural and payload mass fractions
- Regression analysis
- Multistaging
- Optimal $\Delta V$ distribution between stages
- Trade-off ratios
- Parallel staging
- Modular staging
Derivation of the Rocket Equation

- Momentum at time $t$:
  \[ M = m v \]

- Momentum at time $t + \Delta t$:
  \[ M = (m - \Delta m)(V + \Delta v) + \Delta m(v - V_e) \]

- Some algebraic manipulation gives:
  \[ m \Delta v = -\Delta m V_e \]

- Take to limits and integrate:
  \[
  \int_{m_{\text{initial}}}^{m_{\text{final}}} \frac{dm}{m} = -\int_{V_{\text{initial}}}^{V_{\text{final}}} \frac{dv}{V_e}
  \]
The Rocket Equation

• Alternate forms

\[ r \equiv \frac{m_{\text{final}}}{m_{\text{initial}}} = e^{-\frac{\Delta V}{V_e}} \]

\[ \Delta v = -V_e \ln \left( \frac{m_{\text{final}}}{m_{\text{initial}}} \right) = -V_e \ln r \]

• Basic definitions/concepts

– Mass ratio

\[ r \equiv \frac{m_{\text{final}}}{m_{\text{initial}}} \quad \text{or} \quad \mathcal{R} \equiv \frac{m_{\text{initial}}}{m_{\text{final}}} \]

– Nondimensional velocity change

“Velocity ratio”

\[ \nu \equiv \frac{\Delta V}{V_e} \]
Rocket Equation (First Look)

![Diagram showing mass ratio vs. velocity ratio for typical range for launch to Low Earth Orbit.]

- Mass Ratio, \( \frac{M_{\text{final}}}{M_{\text{initial}}} \)
- Velocity Ratio, \( \frac{\Delta V}{V_e} \)

Typical Range for Launch to Low Earth Orbit
Sources and Categories of Vehicle Mass

Payload
Propellants
Inert Mass
Structure
Propulsion
Avionics
Power
Mechanisms
Thermal
Etc.
Basic Vehicle Parameters

- Basic mass summary
  \[ m_o = m_{pl} + m_{pr} + m_{in} \]

- Inert mass fraction
  \[ \delta \equiv \frac{m_{in}}{m_o} = \frac{m_{in}}{m_{pl} + m_{pr} + m_{in}} \]

- Payload fraction
  \[ \lambda \equiv \frac{m_{pl}}{m_o} = \frac{m_{pl}}{m_{pl} + m_{pr} + m_{in}} \]

- Parametric mass ratio
  \[ r = \lambda + \delta \]

- \( m_o \) = initial mass
- \( m_{pl} \) = payload mass
- \( m_{pr} \) = propellant mass
- \( m_{in} \) = inert mass
Rocket Equation (including Inert Mass)

Payload Fraction, \( \frac{M_{\text{payload}}}{M_{\text{initial}}} \)

Velocity Ratio, \( \frac{\Delta V}{V_e} \)

Typical Range for Launch to Low Earth Orbit

Inert Mass Fraction \( \delta \)
- 0
- 0.05
- 0.1
- 0.15
- 0.2
Limiting Performance Based on Inert Mass

Asymptotic Velocity Ratio, \((\Delta V/V_e)\)

Typical Feasible Design Range for Inert Mass Fraction

Inert Mass Fraction, \((M_{\text{inert}}/M_{\text{initial}})\)
# Regression Analysis of Existing Vehicles

<table>
<thead>
<tr>
<th>Veh/Stage</th>
<th>prop mass (lbs)</th>
<th>gross mass (lbs)</th>
<th>Type</th>
<th>Propellants</th>
<th>Isp vac (sec)</th>
<th>Isp sl (sec)</th>
<th>sigma</th>
<th>eps</th>
<th>delta</th>
</tr>
</thead>
<tbody>
<tr>
<td>Delta 6925 Stage 2</td>
<td>13,367</td>
<td>15,394</td>
<td>Storable</td>
<td>N2O4-A50</td>
<td>319.4</td>
<td></td>
<td>0.152</td>
<td>0.132</td>
<td>0.070</td>
</tr>
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<td></td>
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<td>0.132</td>
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<tr>
<td>Titan II Stage 2</td>
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<td>N2O4-A50</td>
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<td>0.102</td>
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<tr>
<td>Titan III Stage 2</td>
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<td>0.083</td>
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<td>Proton Stage 3</td>
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<td>0.118</td>
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<td>0.052</td>
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<td>N2O4-A50</td>
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<td>0.056</td>
<td>0.053</td>
<td>0.039</td>
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<td>365,000</td>
<td>Storable</td>
<td>N2O4-A50</td>
<td>316.0</td>
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<td>1,004,000</td>
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<td>0.111</td>
<td>0.100</td>
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<td><strong>average</strong></td>
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<td></td>
<td></td>
<td><strong>312.2</strong></td>
<td><strong>285.0</strong></td>
<td><strong>0.100</strong></td>
<td><strong>0.089</strong></td>
<td><strong>0.061</strong></td>
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<tr>
<td><strong>standard deviation</strong></td>
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<td></td>
<td></td>
<td></td>
<td><strong>8.1</strong></td>
<td><strong>0.039</strong></td>
<td><strong>0.033</strong></td>
<td><strong>0.019</strong></td>
<td></td>
</tr>
</tbody>
</table>
Inert Mass Fraction Data for Existing LVs

- LOX/LH2
- LOX/RP-1
- Solid
- Storable

Gross Mass (MT) vs. Inert Mass Fraction
Regression Analysis

- Given a set of N data points \((x_i, y_i)\)
- Linear curve fit: \(y = Ax + B\)
  - find A and B to minimize sum squared error
    \[
    \text{error} = \sum_{i=1}^{N} (Ax_i + B - y_i)^2
    \]
    - Analytical solutions exist, or use Solver in Excel
- Power law fit: \(y = Bx^A\)
  \[
  \text{error} = \sum_{i=1}^{N} [A \log(x_i) + B - \log(y_i)]^2
  \]
- Polynomial, exponential, many other fits possible
Solution of Least-Squares Linear Regression

\[
\frac{\partial \text{error}}{\partial A} = 2 \sum_{i=1}^{N} (Ax_i + B - y_i)x_i = 0
\]

\[
\frac{\partial \text{error}}{\partial B} = 2 \sum_{i=1}^{N} (Ax_i + B - y_i) = 0
\]

\[
A \sum_{i=1}^{N} x_i^2 + B \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} x_i y_i = 0
\]

\[
A \sum_{i=1}^{N} x_i + NB - \sum_{i=1}^{N} y_i = 0
\]

\[
A = \frac{N \sum x_i y_i - \sum x_i \sum y_i}{N \sum x_i^2 - (\sum x_i)^2}
\]

\[
B = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{N \sum x_i^2 - (\sum x_i)^2}
\]
Regression Analysis - Storables

Inert Mass Fraction

Gross Mass (MT)

$R^2 = 0.0541$
## Regression Values for Design Parameters

<table>
<thead>
<tr>
<th></th>
<th>Vacuum Ve (m/sec)</th>
<th>Inert Mass Fraction</th>
<th>Max ΔV (m/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LOX/LH2</td>
<td>4273</td>
<td>0.075</td>
<td>11,070</td>
</tr>
<tr>
<td>LOX/RP-1</td>
<td>3136</td>
<td>0.063</td>
<td>8664</td>
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<td>Storables</td>
<td>3058</td>
<td>0.061</td>
<td>8575</td>
</tr>
<tr>
<td>Solids</td>
<td>2773</td>
<td>0.087</td>
<td>6783</td>
</tr>
</tbody>
</table>
The Rocket Equation for Multiple Stages

- Assume two stages

\[
\Delta V_1 = -V_{e1} \ln \left( \frac{m_{final1}}{m_{initial1}} \right) = -V_{e1} \ln(r_1)
\]

\[
\Delta V_2 = -V_{e2} \ln \left( \frac{m_{final2}}{m_{initial2}} \right) = -V_{e2} \ln(r_2)
\]

- Assume \( V_{e1} = V_{e2} = V_e \)

\[
\Delta V_1 + \Delta V_2 = -V_e \ln(r_1) - V_e \ln(r_2) = -V_e \ln(r_1 r_2)
\]
Continued Look at Multistaging

- There’s a historical tendency to define $r_0 = r_1 r_2$
  \[ \Delta V_1 + \Delta V_2 = -V_e \ln (r_1 r_2) = -V_e \ln (r_0) \]

- But it’s important to remember that it’s really
  \[ \Delta V_1 + \Delta V_2 = -V_e \ln (r_1 r_2) = -V_e \ln \left( \frac{m_{\text{final1}}}{m_{\text{initial1}}} \frac{m_{\text{final2}}}{m_{\text{initial2}}} \right) \]

- And that $r_0$ has no physical significance, since
  \[ m_{\text{final1}} \neq m_{\text{initial2}} \Rightarrow r_0 \neq \frac{m_{\text{final2}}}{m_{\text{initial1}}} \]
Multistage Inert Mass Fraction

- Total inert mass fraction

\[ \delta_0 = \frac{m_{\text{in},1} + m_{\text{in},2} + m_{\text{in},3}}{m_0} = \frac{m_{\text{in},1}}{m_0} + \frac{m_{\text{in},2}}{m_0} + \frac{m_{\text{in},3}}{m_0} \]

\[ \delta_0 = \frac{m_{\text{in},1}}{m_0} + \frac{m_{\text{in},2}}{m_{0,2}} \cdot \frac{m_0}{m_0} + \frac{m_{\text{in},3}}{m_{0,3}} \cdot \frac{m_0}{m_{0,2}} \cdot \frac{m_{0,2}}{m_0} \]

- Convert to dimensionless parameters

\[ \delta_0 = \delta_1 + \delta_2 \lambda_1 + \delta_3 \lambda_2 \lambda_1 \]

- General form of the equation

\[ \delta_0 = \sum_{j=1}^{n \text{ stages}} \left[ \delta_j \prod_{\ell=1}^{j-1} \lambda_\ell \right] \]
Multistage Payload Fraction

- Total payload fraction (3 stage example)
  \[ \lambda_0 = \frac{m_{pl}}{m_0} = \frac{m_{pl}}{m_{0,3}} \frac{m_{0,3}}{m_{0,2}} \frac{m_{0,2}}{m_0} \]

- Convert to dimensionless parameters
  \[ \lambda_0 = \lambda_3 \lambda_2 \lambda_1 \]

- Generic form of the equation
  \[ \lambda_0 = \prod_{j=1}^{n \text{ stages}} \lambda_j \]
Effect of $\delta$ and $\Delta V/V_e$ on Payload

\[
\frac{\Delta V}{V_e} = 4.0
\]

1 stage
2 stages
3 stages
4 stages

Inert Mass Fraction
Total Payload Fraction

1 stage
2 stages
3 stages
4 stages
Effect of Staging

Inert Mass Fraction $\delta=0.15$

Payload Fraction

Velocity Ratio ($\Delta V/V_e$)

Single Stage
Two Stage
Three Stage
Four Stage

1 stage
2 stage
3 stage
4 stage
Trade Space for Number of Stages

Inert Mass Fraction

Velocity Ratio DV/Ve

Solids

LOX/RP-1

Storables

LOX/LH2

Single Stage

Two Stage

Three Stage

Four Stage

0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4

Inert Mass Fraction

0 0.5 1 1.5 2 2.5 3 3.5 4

Velocity Ratio DV/Ve
Effect of $\Delta V$ Distribution

![Diagram showing the effect of $\Delta V$ distribution on normalized mass for a rocket with 1st Stage: Solids and 2nd Stage: LOX/LH2. The graph plots Stage 2 $\Delta V$ (m/sec) on the x-axis and Normalized Mass (kg/kg of payload) on the y-axis. The graph includes lines for Total mass, Stage 2 mass, and Stage 1 mass.]
ΔV Distribution and Design Parameters

![Graph showing ΔV distribution for 1st and 2nd stages.](image)

- **1st Stage: Solids**
- **2nd Stage: LOX/LH2**

- **Stage 2 ΔV (m/sec)**

- **Delta_0**
- **Lambda_0**
- **Lambda/delta**
Lagrange Multipliers

- Given an objective function

\[ y = f(x) \]

subject to constraint function

\[ z = g(x) \]

- Create a new objective function

\[ z = f(x) + \lambda [g(x) - z] \]

- Solve simultaneous equations

\[ \frac{\partial y}{\partial x} = 0 \quad \frac{\partial y}{\partial \lambda} = 0 \]
Optimum $\Delta V$ Distribution Between Stages

- Maximize payload fraction (2 stage case)
  \[ \lambda_0 = \lambda_1 \lambda_2 = (r_1 - \delta_1)(r_2 - \delta_2) \]
  subject to constraint function
  \[ \Delta V_{total} = \Delta V_1 + \Delta V_2 \]
- Create a new objective function
  \[ \lambda_o = \left( e^{\frac{-\Delta V_1}{V_{e,1}}} - \delta_1 \right) \left( e^{\frac{-\Delta V_2}{V_{e,2}}} - \delta_2 \right) + K \left[ \Delta V_1 + \Delta V_2 - \Delta V_{total} \right] \]

⇒ Very messy for partial derivatives!
Optimum $\Delta V$ Distribution (continued)

- Use substitute objective function

$$\max (\lambda_o) \iff \max [\ln (\lambda_o)]$$

- Create a new constrained objective function

$$\ln (\lambda_o) = \ln (r_1 - \delta_1) + \ln (r_2 - \delta_2) + K [\Delta V_1 + \Delta V_2 - \Delta V_{\text{total}}]$$

- Take partials and set equal to zero

$$\frac{\partial [\ln (\lambda_o)]}{\partial r_1} = 0 \quad \frac{\partial [\ln (\lambda_o)]}{\partial r_2} = 0 \quad \frac{\partial [\ln (\lambda_o)]}{\partial K} = 0$$
Optimum $\Delta V$ Special Cases

- “Generic” partial of objective function

$$\frac{\partial [\ln (\lambda_o)]}{\partial r_i} = \frac{1}{r_i - \delta_i} + K \frac{V_{e,i}}{r_i} = 0$$

- Special case: $\delta_1 = \delta_2$ $V_{e,1} = V_{e,2}$

$$r_1 = r_2 \implies \Delta V_1 = \Delta V_2 = \frac{\Delta V_{total}}{2}$$

- Same principle holds for $n$ stages

$$r_1 = r_2 = \cdots = r_n \implies$$

$$\Delta V_1 = \Delta V_2 = \cdots = \Delta V_n = \frac{\Delta V_{total}}{n}$$
Sensitivity to Inert Mass

$\Delta V$ for multistaged rocket

\[
\Delta V_{tot} = \sum_{k=1}^{n \text{ stages}} \Delta V_k = \sum_{k=1}^{n} V_{e,k} \ln \left( \frac{m_{o,k}}{m_{f,k}} \right)
\]

where

\[
m_{o,k} = m_{pl} + m_{pr,k} + m_{in,k} + \sum_{j=k+1}^{n} (m_{pr,j} + m_{in,j})
\]

\[
m_{f,k} = m_{pl} + m_{in,k} + \sum_{j=k+1}^{n} (m_{pr,j} + m_{in,j})
\]
Finding Payload Sensitivity to Inert Mass

• Given the equation linking mass to \( \Delta V \), take

\[
\frac{\partial (\Delta V_{tot})}{\partial m_{pl}} dm_{pl} + \frac{\partial (\Delta V_{tot})}{\partial m_{in, j}} dm_{in, j} = 0
\]

and solve to find

\[
\left. \frac{dm_{pl}}{dm_{in, k}} \right|_{\partial (\Delta V_{tot}) = 0} = - \sum_{j=1}^{k} V_{e, j} \left( \frac{1}{m_{o, j}} - \frac{1}{m_{f, j}} \right)
\]

\[
\sum_{\ell=1}^{N} V_{e, \ell} \left( \frac{1}{m_{o, \ell}} - \frac{1}{m_{f, \ell}} \right)
\]

• This equation shows the “trade-off ratio” - \( \Delta \)payload resulting from a change in inert mass for stage \( k \) (for a vehicle with \( N \) total stages)
## Trade-off Ratio Example: Gemini-Titan II

<table>
<thead>
<tr>
<th></th>
<th>Stage 1</th>
<th>Stage 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Mass (kg)</td>
<td>150,500</td>
<td>32,630</td>
</tr>
<tr>
<td>Final Mass (kg)</td>
<td>39,370</td>
<td>6099</td>
</tr>
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<td>Ve (m/sec)</td>
<td>2900</td>
<td>3097</td>
</tr>
<tr>
<td>$\frac{dm_{pl}}{dm_{in,k}}$</td>
<td>-0.1164</td>
<td>-1</td>
</tr>
</tbody>
</table>
Payload Sensitivity to Propellant Mass

- In a similar manner, solve to find

\[
\frac{dm_{pl}}{dm_{pr,k}} \bigg|_{\partial(\Delta V_{tot})=0} = \frac{- \sum_{j=1}^{k} V_{e,j} \left( \frac{1}{m_{o,j}} \right)}{\sum_{\ell=1}^{N} V_{e,\ell} \left( \frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}} \right)}
\]

- This equation gives the change in payload mass as a function of additional propellant mass (all other parameters held constant)
## Trade-off Ratio Example: Gemini-Titan II

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<td>$V_e$ (m/sec)</td>
<td>2900</td>
<td>3097</td>
</tr>
<tr>
<td>$\frac{dm_{pl}}{dm_{in,k}}$</td>
<td>-0.1164</td>
<td>-1</td>
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<tr>
<td>$\frac{dm_{pl}}{dm_{pr,k}}$</td>
<td>0.04124</td>
<td>0.2443</td>
</tr>
</tbody>
</table>
Payload Sensitivity to Exhaust Velocity

- Use the same technique to find the change in payload resulting from additional exhaust velocity for stage \( k \)

\[
\frac{dm_{pl}}{dV_{e,k}} \bigg|_{\partial(\Delta V_{tot})=0} = \frac{\sum_{j=1}^{k} \ln \left( \frac{m_{o,j}}{m_{f,j}} \right)}{\sum_{\ell=1}^{N} V_{e,\ell} \left( \frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}} \right)}
\]

- This trade-off ratio (unlike the ones for inert and propellant masses) has units - kg/(m/sec)
## Trade-off Ratio Example: Gemini-Titan II

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<tr>
<td>$\frac{dm_{pl}}{dm_{pr,k}}$</td>
<td>0.04124</td>
<td>0.2443</td>
</tr>
<tr>
<td>$\frac{dm_{pl}}{dV_e,k}$ (kg/m/sec)</td>
<td>2.870</td>
<td>6.459</td>
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</tbody>
</table>
Parallel Staging

- Multiple dissimilar engines burning simultaneously
- Frequently a result of upgrades to operational systems
- General case requires “brute force” numerical performance analysis
Parallel-Staging Rocket Equation

- Momentum at time $t$:
  \[ M = m v \]

- Momentum at time $t + \Delta t$:
  (subscript “b”=boosters; “c”=core vehicle)
  \[ M = (m - \Delta m_b - \Delta m_c)(v + \Delta v) \]
  \[ + \Delta m_b(v - V_{e,b}) + \Delta m_c(v - V_{e,c}) \]

- Assume thrust (and mass flow rates) constant
Parallel-Staging Rocket Equation

- Rocket equation during booster burn

\[ \Delta V = -\bar{V}_e \ln \left( \frac{m_{\text{final}}}{m_{\text{initial}}} \right) = -\bar{V}_e \ln \left( \frac{m_{\text{in},b} + m_{\text{in},c} + \chi m_{\text{pr},c} + m_0,2}{m_{\text{in},b} + m_{\text{pr},b} + m_{\text{in},c} + m_{\text{pr},c} + m_0,2} \right) \]

where \( \chi = \text{fraction of core propellant remaining after booster burnout, and where} \)

\[ \bar{V}_e = \frac{V_{e,b} \dot{m}_b + V_{e,c} \dot{m}_c}{\dot{m}_b + \dot{m}_c} = \frac{V_{e,b} m_{pr,b} + V_{e,c} (1-\chi) m_{pr,c}}{m_{pr,b} + (1-\chi) m_{pr,c}} \]
Analyzing Parallel-Staging Performance

Parallel stages break down into pseudo-serial stages:

- **Stage “0” (boosters and core)**
  \[
  \Delta V_0 = -\bar{V}_e \ln \left( \frac{m_{in,b} + m_{in,c} + \chi m_{pr,c} + m_{0,2}}{m_{in,b} + m_{pr,b} + m_{in,c} + m_{pr,c} + m_{0,2}} \right)
  \]

- **Stage “1” (core alone)**
  \[
  \Delta V_1 = -V_{e,c} \ln \left( \frac{m_{in,c} + m_{0,2}}{m_{in,c} + \chi m_{pr,c} + m_{0,2}} \right)
  \]

- **Subsequent stages are as before**
Parallel Staging Example: Space Shuttle

- 2 x solid rocket boosters (data below for single SRB)
  - Gross mass 589,670 kg
  - Empty mass 86,183 kg
  - Ve 2636 m/sec
  - Burn time 124 sec

- External tank (space shuttle main engines)
  - Gross mass 750,975 kg
  - Empty mass 29,930 kg
  - Ve 4459 m/sec
  - Burn time 480 sec

- “Payload” (orbiter + P/L) 125,000 kg
Shuttle Parallel Staging Example

\[ V_{e,b} = 2636 \frac{m}{sec} \]

\[ V_{e,c} = 4459 \frac{m}{sec} \]

\[ \chi = \frac{480 - 124}{480} = 0.7417 \]

\[ V_e = \frac{2636(1,007,000) + 4459(721,000)(1 - 0.7417)}{1,007,000 + 721,000(1 - 0.7417)} = 2921 \frac{m}{sec} \]

\[ \Delta V_0 = -2921 \ln \frac{862,000}{3,062,000} = 3702 \frac{m}{sec} \]

\[ \Delta V_1 = -4459 \ln \frac{154,900}{689,700} = 6659 \frac{m}{sec} \]

\[ \Delta V_{tot} = 10,360 \frac{m}{sec} \]
Modular Staging

- Use identical modules to form multiple stages
- Have to cluster modules on lower stages to make up for nonideal ΔV distributions
- Advantageous from production and development cost standpoints
Module Analysis

- All modules have the same inert mass and propellant mass
- Because $\delta$ varies with payload mass, not all modules have the same $\delta$!
- Introduce two new parameters

\[
\varepsilon \equiv \frac{m_{in}}{m_{in} + m_{pr}} \quad \sigma \equiv \frac{m_{in}}{m_{pr}}
\]

- Conversions

\[
\varepsilon = \frac{\delta}{1 - \lambda} \quad \sigma = \frac{\delta}{1 - \delta - \lambda}
\]
Rocket Equation for Modular Boosters

- Assuming $n$ modules in stage 1,

$$r_1 = \frac{n(m_{in}) + m_{o2}}{n(m_{in} + m_{pr}) + m_{o2}} = \frac{n\varepsilon + \frac{m_{o2}}{m_{mod}}}{n + \frac{m_{o2}}{m_{mod}}}$$

- If all 3 stages use same modules, $n_j$ for stage $j$,

$$r_1 = \frac{n_1\varepsilon + n_2 + n_3 + \rho_{pl}}{n_1 + n_2 + n_3 + \rho_{pl}}$$

where

$$\rho_{pl} \equiv \frac{m_{pl}}{m_{mod}}; \quad m_{mod} = m_{in} + m_{pr}$$
Example: Conestoga 1620 (EER)

- Small launch vehicle (1 flight, 1 failure)
- Payload 900 kg
- Module gross mass 11,400 kg
- Module empty mass 1,400 kg
- Exhaust velocity 2754 m/sec
- Staging pattern
  - 1st stage - 4 modules
  - 2nd stage - 2 modules
  - 3rd stage - 1 module
  - 4th stage - Star 48V (gross mass 2200 kg, empty mass 140 kg, $V_e$ 2842 m/sec)
Conestoga 1620 Performance

- 4th stage $\Delta V$

$$\Delta V_4 = -V_{e4} \ln \frac{m_{f4}}{m_{o4}} = -2842 \ln \frac{900 + 140}{900 + 2200} = 3104 \text{ m/sec}$$

- Treat like three-stage modular vehicle; $M_{pl} = 3100$ kg

$$\epsilon = \frac{m_{in}}{m_{mod}} = \frac{1400}{11400} = 0.1228$$

$$\rho_{pl} = \frac{m_{pl}}{m_{mod}} = \frac{3100}{11400} = 0.2719$$

$$n_1 = 4; \quad n_2 = 2; \quad n_3 = 1$$
\[ r_1 = \frac{n_1 \epsilon + n_2 + n_3 + \rho_{pl}}{n_1 + n_2 + n_3 + \rho_{pl}} = \frac{4 \times 0.1228 + 2 + 1 + 0.2719}{4 + 2 + 1 + 0.2719} = 0.5175 \]

\[ r_2 = \frac{n_2 \epsilon + n_3 + \rho_{pl}}{n_2 + n_3 + \rho_{pl}} = \frac{2 \times 0.1228 + 1 + 0.2719}{2 + 1 + 0.2719} = 0.4638 \]

\[ r_3 = \frac{n_3 \epsilon + \rho_{pl}}{n_3 + \rho_{pl}} = \frac{1 \times 0.1228 + 0.2719}{1 + 0.2719} = 0.3103 \]

\[ V_1 = 1814 \text{ m/sec}; \quad V_2 = 2116 \text{ m/sec} \]

\[ V_3 = 3223 \text{ m/sec}; \quad V_4 = 3104 \text{ m/sec} \]

\[ V_{total} = 10,257 \text{ m/sec} \]
Discussion about Modular Vehicles

- Modularity has several advantages
  - Saves money (smaller modules cost less to develop)
  - Saves money (larger production run = lower cost/module)
  - Allows resizing launch vehicles to match payloads

- Trick is to optimize number of stages, number of modules/stage to minimize total number of modules

- Generally close to optimum by doubling number of modules at each lower stage

- Have to worry about packing factors, complexity
OTRAG - 1977-1983
Modular Example

- Let’s build a launch vehicle out of seven Space Shuttle Solid Rocket Boosters
  - $M_{in} = 86,180\, \text{kg}$
  - $M_{pr} = 503,500\, \text{kg}$

$$\varepsilon \equiv \frac{m_{in}}{m_{in} + m_{pr}} = 0.1461 \quad \sigma \equiv \frac{m_{in}}{m_{pr}} = 0.1711$$

- Look at possible approaches to sequential firing
Modular Sequencing - SRB Example

- Assume no payload
- All seven firing at once - $\Delta V_{\text{tot}}=5138$ m/sec
- 3-3-1 sequence - $\Delta V_{\text{tot}}=9087$ m/sec
- 4-2-1 sequence - $\Delta V_{\text{tot}}=9175$ m/sec
- 2-2-2-1 sequence - $\Delta V_{\text{tot}}=9250$ m/sec
- 2-1-1-1-1-1 sequence - $\Delta V_{\text{tot}}=9408$ m/sec
- 1-1-1-1-1-1-1 sequence - $\Delta V_{\text{tot}}=9418$ m/sec
- Sequence limited by need to balance thrust laterally