Ballistic Atmospheric Entry

- Standard atmospheres
- Orbital decay due to atmospheric drag
- Straight-line (no gravity) ballistic entry based on atmospheric density

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Atmospheric Density with Altitude

\[ \rho \left\langle \frac{kg}{m^3} \right\rangle = 3.875 \times 10^{-9} e^{-\frac{h(km)}{59.06}} \]

Energy Loss Due to Atmospheric Drag

Drag \( D \equiv \frac{1}{2} \rho v^2 Ac_D \)

Drag acceleration \( a_d = \frac{D}{m} = \frac{\rho v^2}{2} \frac{Ac_D}{m} \)

\[ \beta \equiv \frac{m}{c_D A} \] <= Ballistic Coefficient

\[ a_d = \frac{\rho v^2}{2\beta} \]

orbital energy \( \equiv E = -\frac{\mu}{2a} \)

\[ \frac{dE}{dt} = \frac{\mu}{2a^2} \frac{da}{dt} \]
Energy Loss Due to Atmospheric Drag

Since drag is highest at perigee, the first effect of atmospheric drag is to circularize the orbit (high perigee drag lowers apogee)

\[
\frac{dE_{drag}}{dt} = a_d v
\]

\[
\nu_{circ}^2 = \frac{\mu}{a} \quad \frac{dE_{drag}}{dt} = -\frac{\rho v^2}{2\beta} \sqrt{\frac{\mu}{a}}
\]

\[
\frac{dE_{drag}}{dt} = -\sqrt{\frac{\mu}{a}} \frac{\rho}{2\beta} \frac{\mu}{a} = -\left(\frac{\mu}{a}\right)^{\frac{3}{2}} \frac{\rho}{2\beta}
\]
Derivation of Orbital Decay Due to Drag

Set orbital energy variation equal to energy lost by drag

\[
\frac{\mu}{2a^2} \frac{da}{dt} = - \frac{\rho}{2\beta} \left( \frac{\mu}{a} \right)^{\frac{3}{2}}
\]

\[
\frac{da}{dt} = - \frac{\rho}{\beta} \sqrt{\mu a}
\]

\[
\rho = \rho_0 e^{-\frac{h}{h_s}} \quad a = h + r_E \quad \Rightarrow \quad \frac{da}{dt} = \frac{dh}{dt}
\]

\[
\frac{dh}{dt} = - \frac{\sqrt{\mu (h + r_E)}}{\beta} \rho_0 e^{-\frac{h}{h_s}}
\]
Derivation of Orbital Decay (2)

This is a separable differential equation...

$$\frac{1}{\sqrt{r_E + h}} e^{\frac{h}{h_s}} dh = -\frac{\sqrt{\mu}}{\beta} \rho_o dt$$

$$\int_{h_o}^{h} \frac{1}{\sqrt{r_E + h}} e^{\frac{h}{h_s}} dh = -\frac{\sqrt{\mu}}{\beta} \rho_o \int_{t_o}^{t} dt$$

Assume $\sqrt{r_E + h} \sim \sqrt{r_E}$ for $r_E \gg h$

$$\frac{1}{\sqrt{r_E}} \int_{h_o}^{h} e^{\frac{h}{h_s}} dh = -\frac{\sqrt{\mu}}{\beta} \rho_o (t - t_o)$$
Derivation of Orbital Decay (3)

\[
\frac{h_s}{\sqrt{rE}} \left( e^{\frac{h}{h_s}} - e^{\frac{h_o}{h_s}} \right) = -\frac{\sqrt{\mu}}{\beta} \rho_o (t - t_o)
\]

\[
e^{\frac{h}{h_s}} - e^{\frac{h_o}{h_s}} = -\frac{\sqrt{\mu rE}}{h_s\beta} \rho_o (t - t_o)
\]

\[
h(t) = h_s \ln \left[ e^{\frac{h_o}{h_s}} - \frac{\sqrt{\mu rE}}{h_s\beta} \rho_o (t - t_o) \right]
\]

Note that some variables typically use km, and others are in meters - you have to make sure unit conversions are done properly to make this work out correctly!
Orbit Decay from Atmospheric Drag

![Graph showing orbit decay from atmospheric drag with time on the x-axis and altitude on the y-axis, with different curves for β = 500, 1500, and 5000]
Time Until Orbital Decay

\[ e^{\frac{h}{h_s}} - e^{\frac{h_o}{h_s}} = -\frac{\sqrt{\mu r E}}{h_s \beta} \rho_o (t - t_o) \]

To find the time remaining \((t_o = 0)\) until the orbit reaches any given “critical” altitude, some algebra gives

\[ t(h) = \frac{h_s \beta}{\sqrt{\mu r E \rho_o}} \left( e^{\frac{h_o}{h_s}} - e^{\frac{h}{h_s}} \right) \]
Decay Time to $r=120$ km

- $\beta = 500$
- $\beta = 1500$
- $\beta = 5000$

Altitude (km) vs. Decay Time (yrs)
Ballistic Entry (no lift)

\[ s = \text{distance along the flight path} \]

\[ \frac{dv}{dt} = -g \sin \gamma - \frac{D}{m} \]

\[ \frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt} = V \frac{dv}{ds} = \frac{1}{2} \frac{d(v^2)}{ds} \]

\[ \frac{1}{2} \frac{d(v^2)}{ds} = -g \sin \gamma - \frac{D}{m} \]

Drag \( D \equiv \frac{1}{2} \rho v^2 A c_D \)

\[ \frac{1}{2} \frac{d(v^2)}{ds} = -g \sin \gamma - \frac{\rho v^2}{2m} A c_D \]

\[ \sin \gamma \frac{d(v^2)}{dh} = -g \sin \gamma - \frac{\rho v^2}{2m} A c_D \]
Ballistic Entry (2)

Exponential atmosphere $\Rightarrow \rho = \rho_o e^{-\frac{h}{h_s}}$

\[
\frac{d\rho}{\rho_o} = e^{-\frac{h}{h_s}} \left( -\frac{dh}{h_s} \right) = \rho_o e^{-\frac{h}{h_s}} \left( -\frac{dh}{h_s} \right) = \frac{\rho}{\rho_o} \left( -\frac{dh}{h_s} \right)
\]

\[
dh = \frac{-h_s}{\rho} d\rho
\]

\[
\frac{\sin \gamma}{2} \frac{d(v^2)}{dh} = -g \sin \gamma - \frac{\rho v^2}{2m} A_D
\]

\[
\frac{\sin \gamma}{2} \frac{d(v^2)}{d\rho} \left( -\frac{\rho}{h_s} \right) = -g \sin \gamma - \frac{\rho v^2}{2} \frac{A_D}{m}
\]

\[
\frac{d(v^2)}{d\rho} = \frac{2gh_s}{\rho} + \frac{h_s v^2}{\sin \gamma} \frac{A_D}{m}
\]
Ballistic Entry (3)

Let \( \beta \equiv \frac{m}{c_D A} \Rightarrow \) Ballistic Coefficient

\[
\frac{d(v^2)}{d\rho} - \frac{h_s}{\beta \sin \gamma} v^2 = \frac{2gh_s}{\rho}
\]

Assume \( mg \ll D \) to get homogeneous ODE

\[
\frac{d(v^2)}{d\rho} - \frac{h_s}{\beta \sin \gamma} v^2 = 0
\]

\[
\frac{d(v^2)}{v^2} = \frac{h_s}{\beta \sin \gamma} d\rho
\]

Use \((v^2)\) as integration variable

\[
\int_{v_e}^{v} \frac{d(v^2)}{v^2} = \frac{h_s}{\beta \sin \gamma} \int_0^{\rho} d\rho \quad v_e = \text{velocity at entry}
\]
Ballistic Entry (4)

Note that the effect of ignoring gravity is that there is no force perpendicular to velocity vector $\Rightarrow$ constant flight path angle $\gamma$ $\Rightarrow$ straight line trajectories

\[
\ln \frac{v^2}{v_e^2} = 2 \ln \frac{v}{v_e} = \frac{h_s \rho}{\beta \sin \gamma}
\]

\[
\frac{v}{v_e} = \exp \left( \frac{h_s \rho}{2 \beta \sin \gamma} \right)
\]

Check units: \( \frac{m}{k g m^3} \)
Earth Entry, $\gamma = -60^\circ$

\[
v/v_e = 1 - \frac{C}{\beta} \left( \frac{\rho}{\rho_0} \right)
\]

\[
C = \frac{1}{\gamma} \left( \frac{\beta}{M} \right)
\]

\[
M = \frac{1}{C} \left( \frac{\beta}{\gamma} \right)
\]

\[
\beta = 100 \text{ kg/m}^3
\]

\[
\rho_0
\]

\[
v_e
\]
What About Peak Deceleration?

To find $n_{max}$, set

$$\frac{d}{dt} \left( \frac{dv}{dt} \right) = \frac{d^2v}{dt^2} = 0$$

$$\frac{d^2v}{dt^2} = -\frac{1}{2\beta} \left( 2\rho v \frac{dv}{dt} + v^2 \frac{d\rho}{dt} \right) = 0$$

$$\frac{d^2v}{dt^2} = -\frac{1}{2\beta} \left( -\frac{2\rho^2v^3}{2\beta} + v^2 \frac{d\rho}{dt} \right) = 0$$

$$\frac{\rho^2v^3}{\beta} = v^2 \frac{d\rho}{dt}$$

$$\rho^2v = \beta \frac{d\rho}{dt}$$
Peak Deceleration (2)

From exponential atmosphere,

\[ \frac{d\rho}{dt} = -\frac{\rho_0}{h_s} e^{-\frac{h}{h_s}} \frac{dh}{dt} = -\frac{\rho}{h_s} \frac{dh}{dt} \]

From geometry, \( \frac{dh}{dt} = v \sin \gamma \)

\[ \frac{d\rho}{dt} = -\frac{\rho v}{h_s} \sin \gamma \quad \rho^2 v = \beta \frac{d\rho}{dt} \]

\[ \rho^2 v = \beta \left( -\frac{\rho v}{h_s} \sin \gamma \right) \]

Remember that this refers to the conditions at max deceleration

\[ \rho_{n_{\text{max}}} = -\frac{\beta}{h_s} \sin \gamma \]
Critical $\beta$ for Deceleration Before Impact

At surface, $\rho = \rho_o$

$\beta_{crit} = -\frac{\rho_o h_s}{\sin \gamma}$  \( \Leftarrow \) Value of $\beta$ at which vehicle hits ground at point of maximum deceleration

How large is maximum deceleration?

$$\frac{dv}{dt} = \frac{\rho v^2}{2\beta} \quad \Rightarrow \quad \left| \frac{dv}{dt} \right|_{max} = \frac{\rho n_{max} v^2}{2\beta}$$

$$\left| \frac{dv}{dt} \right|_{max} = \frac{v^2}{2\beta} \left( -\frac{\beta}{h_s} \sin \gamma \right) = -\frac{1}{2} \frac{v^2}{h_s} \sin \gamma$$

Note that this value of $v$ is actually $v_{n_{max}}$
Peak Deceleration (3)

From page 14,

\[
\frac{v}{v_e} = \exp \left( \frac{h_s \rho}{2 \beta \sin \gamma} \right)
\]

\[
\frac{v_{n\text{max}}}{v_e} = \exp \left[ \frac{h_s}{2 \beta \sin \gamma} \left( -\frac{\beta}{h_s} \sin \gamma \right) \right] = e^{-\frac{1}{2}}
\]

\[
\left| \frac{dv}{dt} \right|_{\text{max}} = -\frac{1}{2} \frac{(v_e e^{-\frac{1}{2}})^2}{h_s} \sin \gamma = -\frac{v_e^2 \sin \gamma}{2h_s e}
\]

Note that the velocity at which maximum deceleration occurs is always a fixed fraction of the entry velocity - it doesn’t depend on ballistic coefficient, flight path angle, or anything else! Also, the magnitude of the maximum deceleration is not a function of ballistic coefficient - it is dependent on the entry trajectory \((v_e \text{ and } \gamma)\) but not spacecraft parameters (i.e., ballistic coefficient).
Terminal Velocity

Full form of ODE -

\[
\frac{d (v^2)}{d\rho} \ - \ \frac{h_s}{\beta \sin \gamma} v^2 = \frac{2gh_s}{\rho}
\]

At terminal velocity, \( v = \text{constant} \equiv v_T \)

\[
- \frac{h_s}{\beta \sin \gamma} v_T^2 = \frac{2gh_s}{\rho}
\]

\[
v_T^2 = \sqrt{- \frac{2g\beta \sin \gamma}{\rho}}
\]
“Cannon Ball” γ=-90° Ballistic Entry

6.75” diameter sphere, c_D=0.2, V_E=6000 m/sec

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<th></th>
<th>Iron</th>
<th>Aluminum</th>
<th>Balsa Wood</th>
</tr>
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<tbody>
<tr>
<td>Weight</td>
<td>40 lb</td>
<td>15.6 lb</td>
<td>14.5 oz</td>
</tr>
<tr>
<td>β_md (kg/m^2)</td>
<td>3938</td>
<td>1532</td>
<td>89</td>
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<tr>
<td>ρ_md (kg/m^3)</td>
<td>0.555</td>
<td>0.216</td>
<td>0.0125</td>
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<tr>
<td>h_md (m)</td>
<td>5600</td>
<td>12,300</td>
<td>32,500</td>
</tr>
<tr>
<td>V_impact (m/s)</td>
<td>1998</td>
<td>355</td>
<td>0*</td>
</tr>
<tr>
<td>V_term (m/sec)</td>
<td>251</td>
<td>156</td>
<td>38</td>
</tr>
</tbody>
</table>

*Artifact of assumption that D ≫ mg
Atmospheric Density with Altitude

Pressure = the integral of the atmospheric density in the column above the reference area

\[ P_o = \int_0^\infty \rho_g dh = \rho_o g \int_0^\infty e^{-\frac{h}{h_s}} dh = -\rho_o g h_s \left[ e^{-\frac{h}{h_s}} \right]_0^\infty \]

\[ = -\rho_o g h_s [0 - 1] \]

Earth: \( \rho_o = 1.226 \frac{kg}{m^3} \); \( h_s = 7524 m \);

\( P_o (calc) = 90,400 \ Pa \); \( P_o (act) = 101,300 \ Pa \)
Nondimensional Ballistic Coefficient

\[ \frac{v}{v_e} = \exp\left(\frac{h_s \rho_o}{2\beta \sin \gamma \rho_o} \right) = \exp\left(\frac{P_o}{2\beta g \sin \gamma \rho_o} \right) \]

Let \( \hat{\beta} \equiv \frac{\beta g}{P_o} \) (Nondimensional form of ballistic coefficient)

Note that we are using the estimated value of \( P_o = \rho_o h_s \), not the actual surface pressure.

\[ \frac{v}{v_e} = \exp\left(\frac{1}{2\hat{\beta} \sin \gamma \rho_o} \right) \]

\[ \beta_{crit} = -\frac{\rho_o h_s}{\sin \gamma} \quad \hat{\beta}_{crit} = -\frac{1}{\sin \gamma} \]
Entry Velocity Trends, $\gamma = -90^\circ$