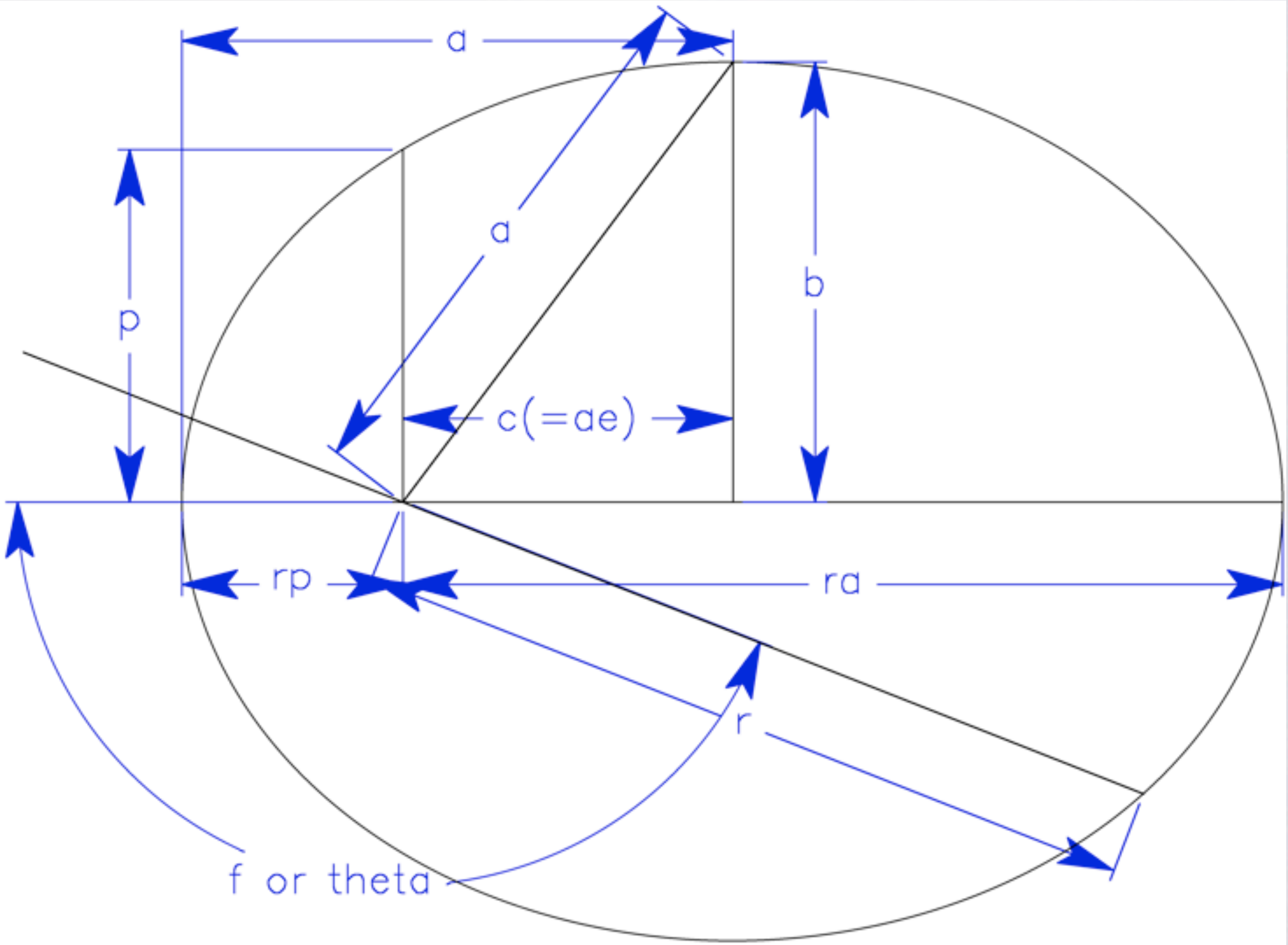
Rocket Performance

- The rest of orbital mechanics
- The rocket equation
- Mass ratio and performance
- Structural and payload mass fractions
- Regression analysis
- Multistaging
- Optimal ΔV distribution between stages
- Trade-off ratios
- Parallel staging
- Modular staging

Classical Parameters of Elliptical Orbits



Basic Orbital Parameters

- Semi-latus rectum (or parameter)

$$p = a(1 - e^2)$$

- Radial distance as function of orbital position

$$r = \frac{p}{1 + e\cos\theta}$$

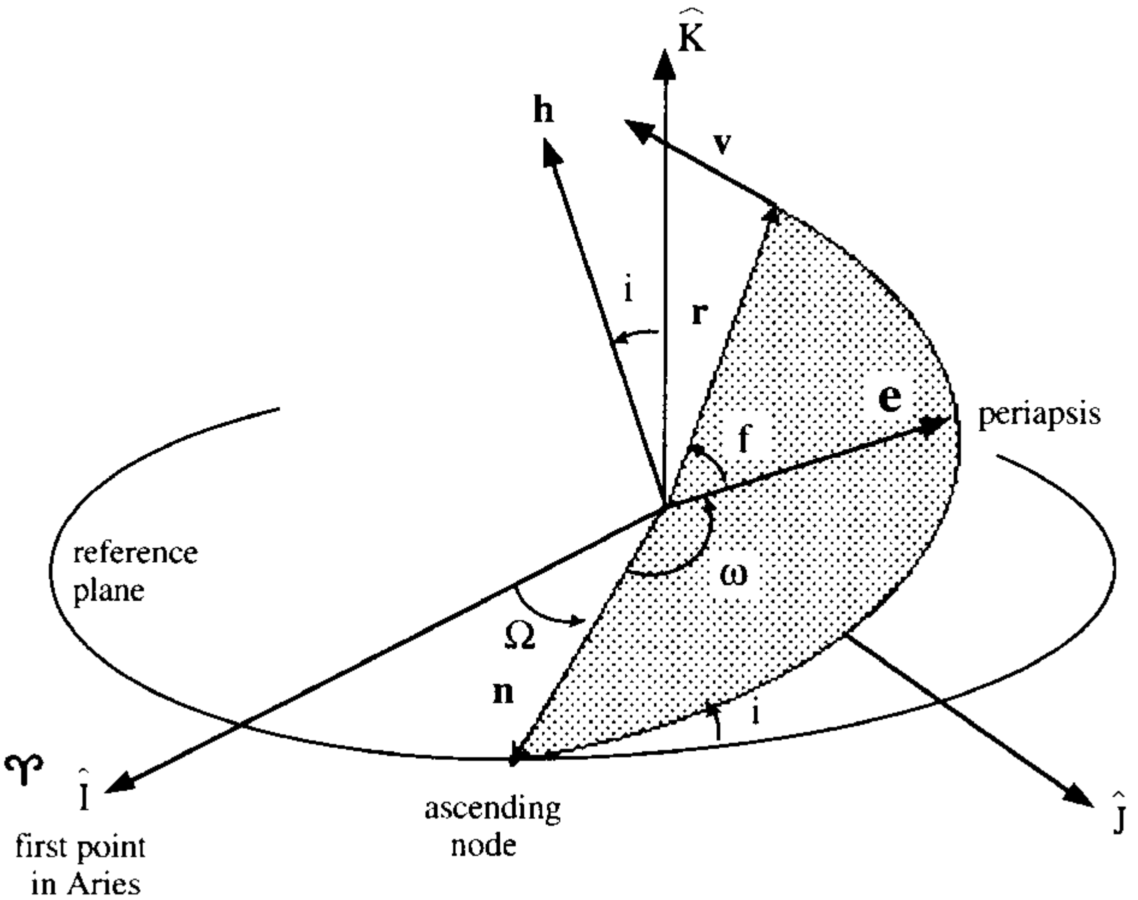
- Periapse and apoapse distances

$$r_p = a(1 - e) \qquad r_a = a(1 + e)$$

- Angular momentum

$$\vec{h} = \vec{r} \times \vec{v} \qquad h = \sqrt{\mu p}$$

The Classical Orbital Elements



Ω : longitude of the ascending node

ω : argument of periapsis

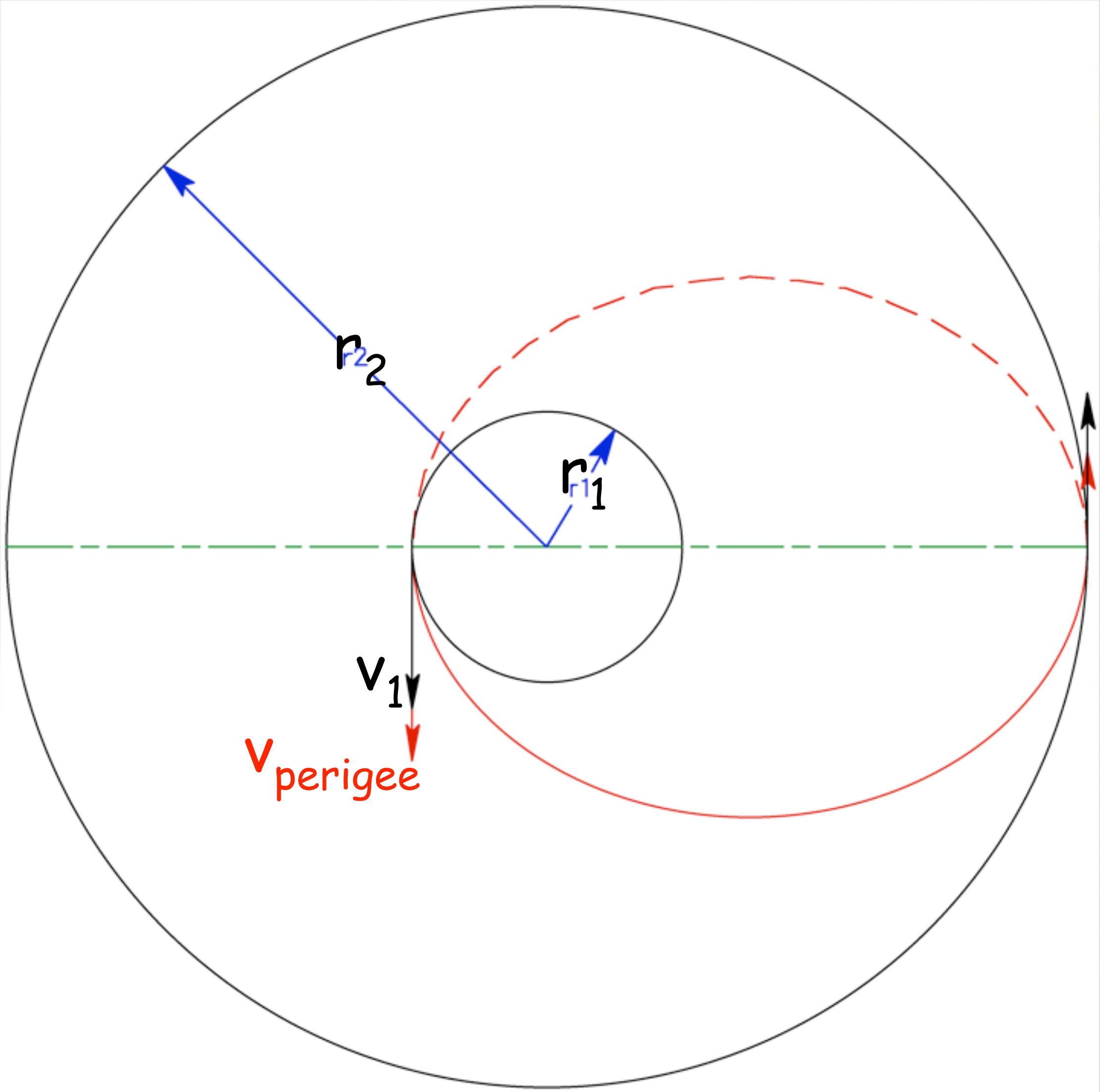
$\tilde{\omega} = \Omega + \omega$: longitude of periapsis

f : true anomaly

$L = \tilde{\omega} + f$: true longitude

Ref: J. E. Prussing and B. A. Conway, Orbital Mechanics Oxford University Press, 1993

The Hohmann Transfer



First Maneuver Velocities

- Initial vehicle velocity

$$v_1 = \sqrt{\frac{\mu}{r_1}}$$

- Needed final velocity

$$v_{perigee} = \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{2r_2}{r_1 + r_2}}$$

- Required ΔV

$$\Delta v_1 = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right)$$

Second Maneuver Velocities

- Initial vehicle velocity

$$v_{apogee} = \sqrt{\frac{\mu}{r_2}}\sqrt{\frac{2r_1}{r_1 + r_2}}$$

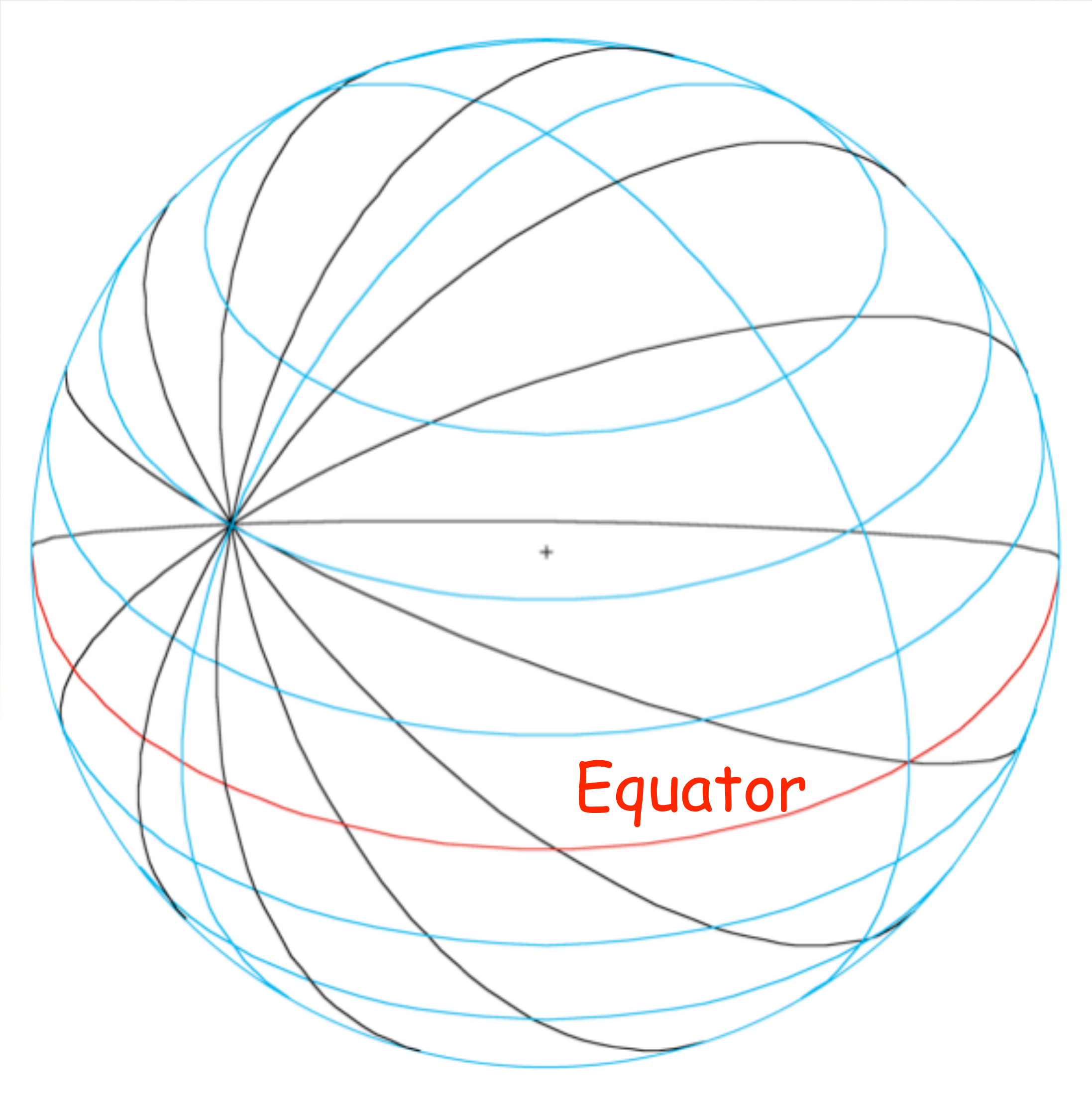
- Needed final velocity

$$v_2 = \sqrt{\frac{\mu}{r_2}}$$

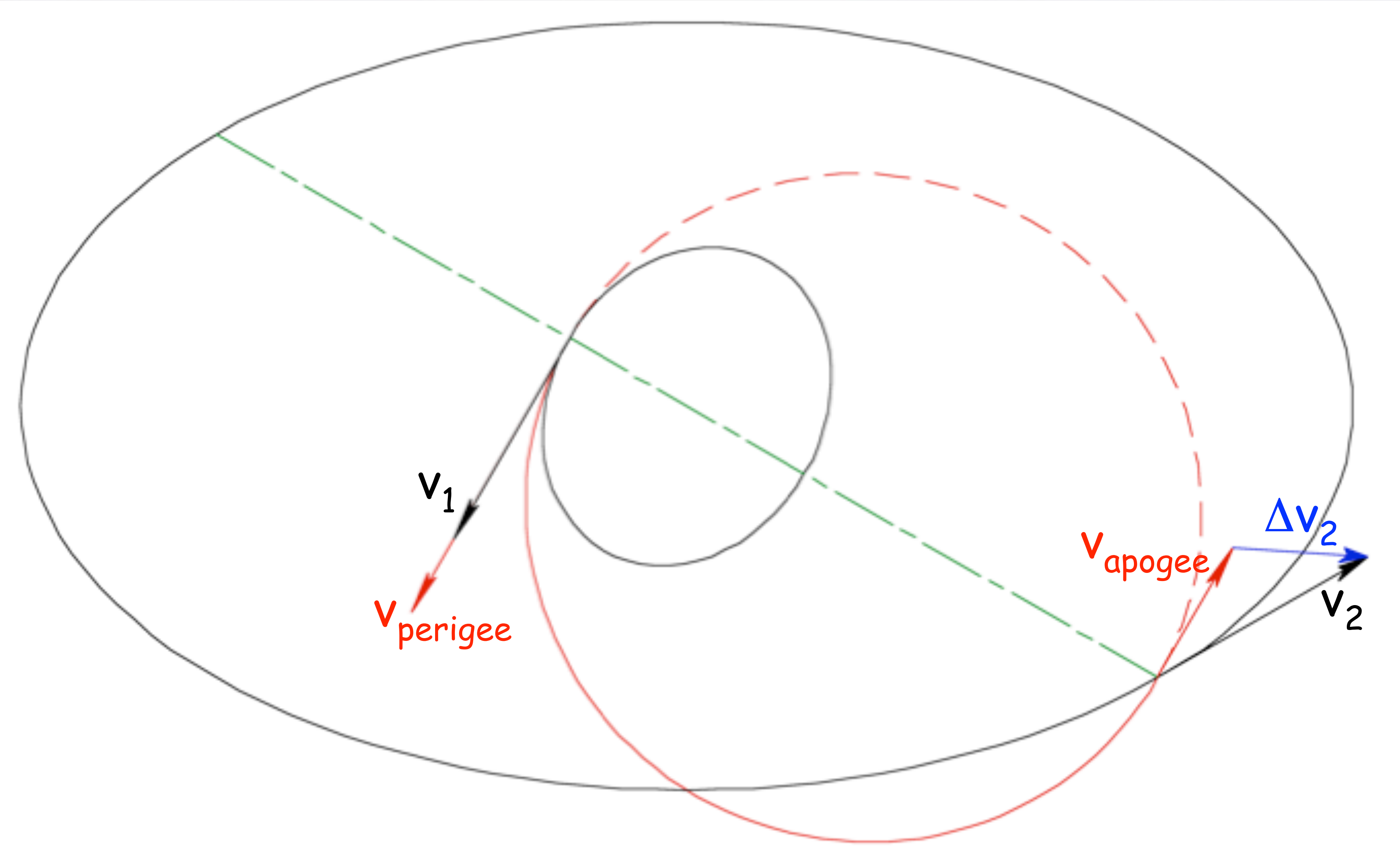
- Required ΔV

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right)$$

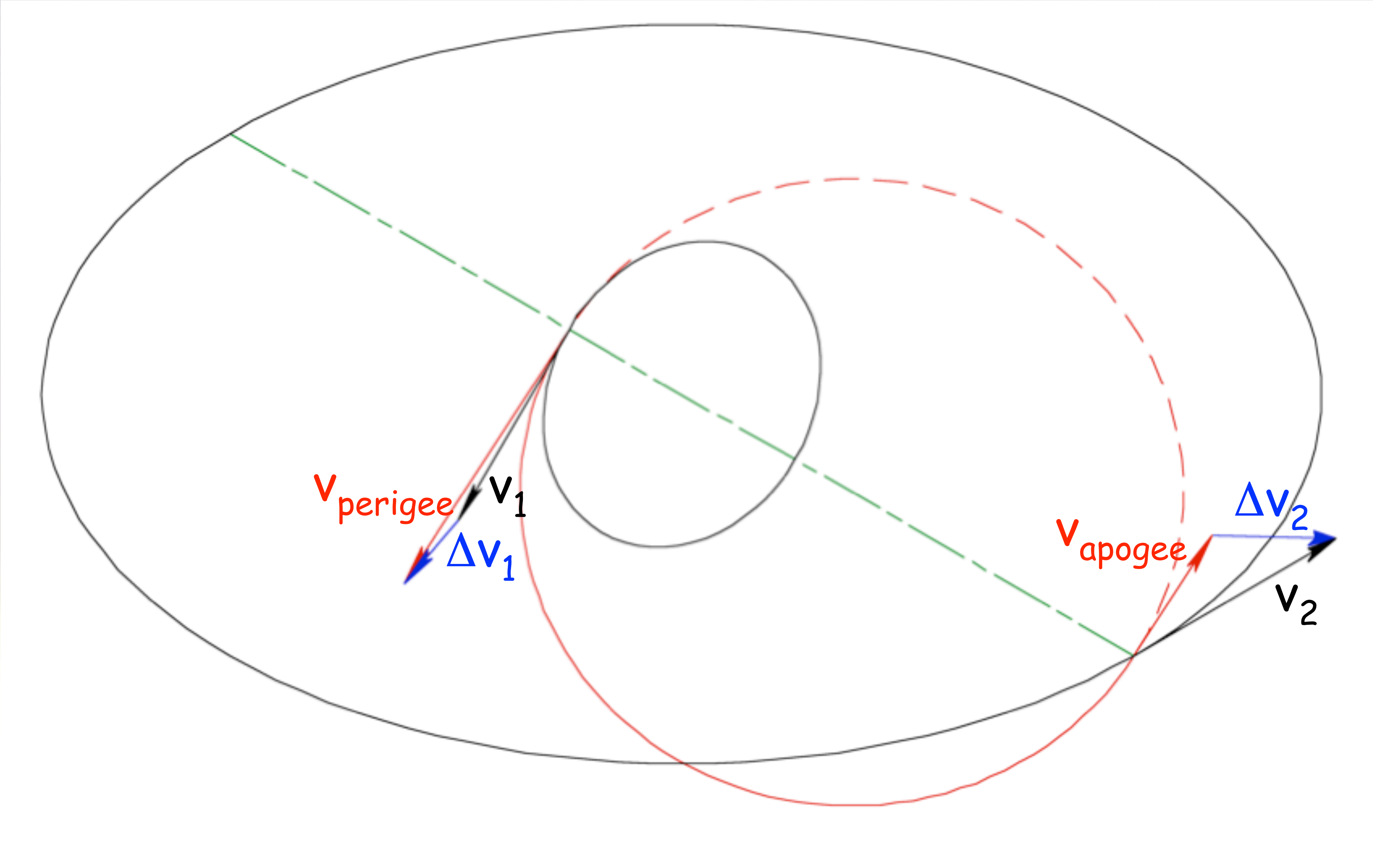
Limitations on Launch Inclinations



Simple Plane Change



Optimal Plane Change



First Maneuver with Plane Change Δi_1

- Initial vehicle velocity

$$v_1 = \sqrt{\frac{\mu}{r_1}}$$

- Needed final velocity

$$v_p = \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{2r_2}{r_1 + r_2}}$$

- Required ΔV

$$\Delta v_1 = \sqrt{v_1^2 + v_p^2 - 2v_1 v_p \cos \Delta i_1}$$

Second Maneuver with Plane Change Δi_2

- Initial vehicle velocity

$$v_a = \sqrt{\frac{\mu}{r_2}}\sqrt{\frac{2r_1}{r_1 + r_2}}$$

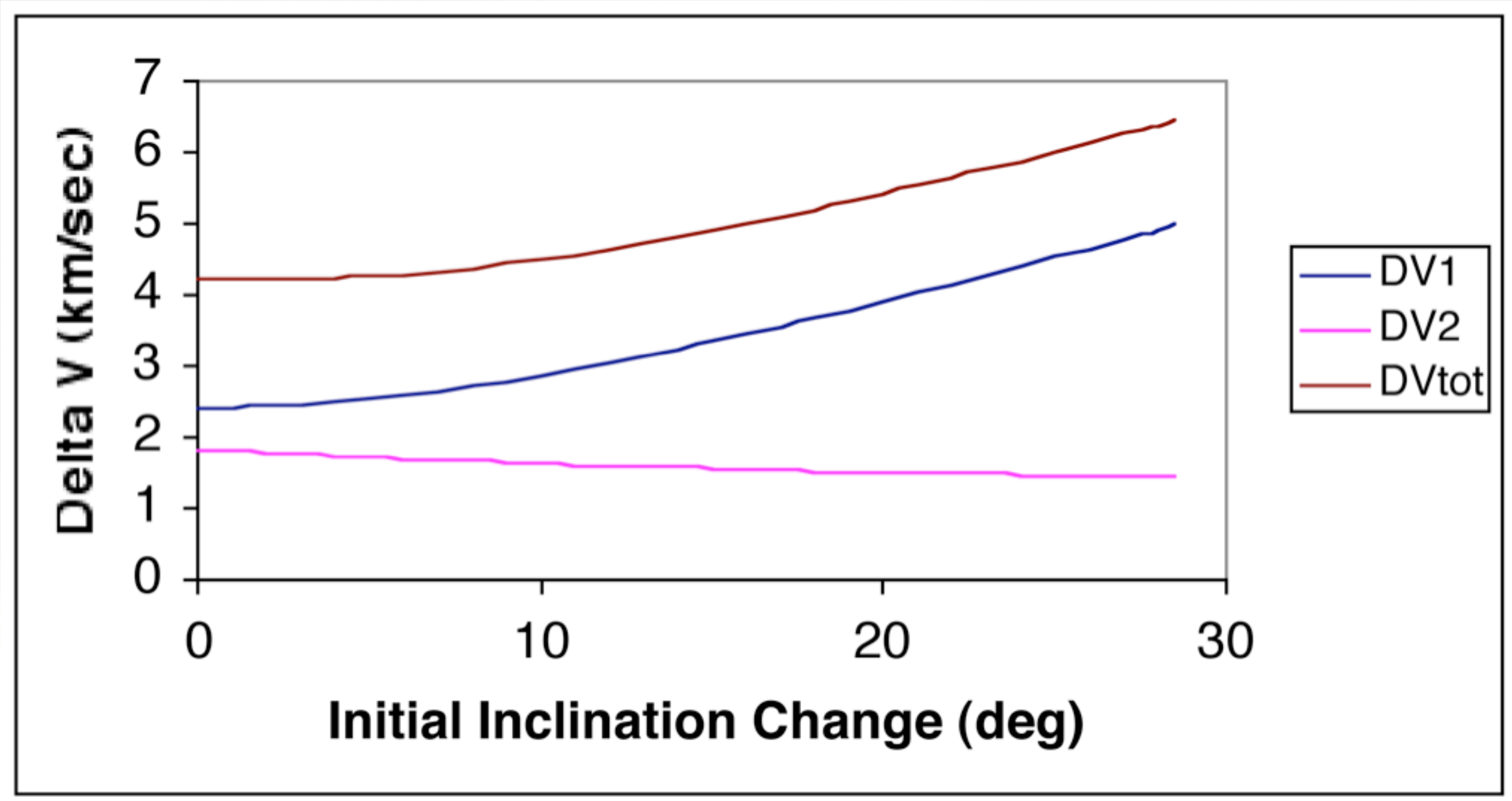
- Needed final velocity

$$v_2 = \sqrt{\frac{\mu}{r_2}}$$

- Required ΔV

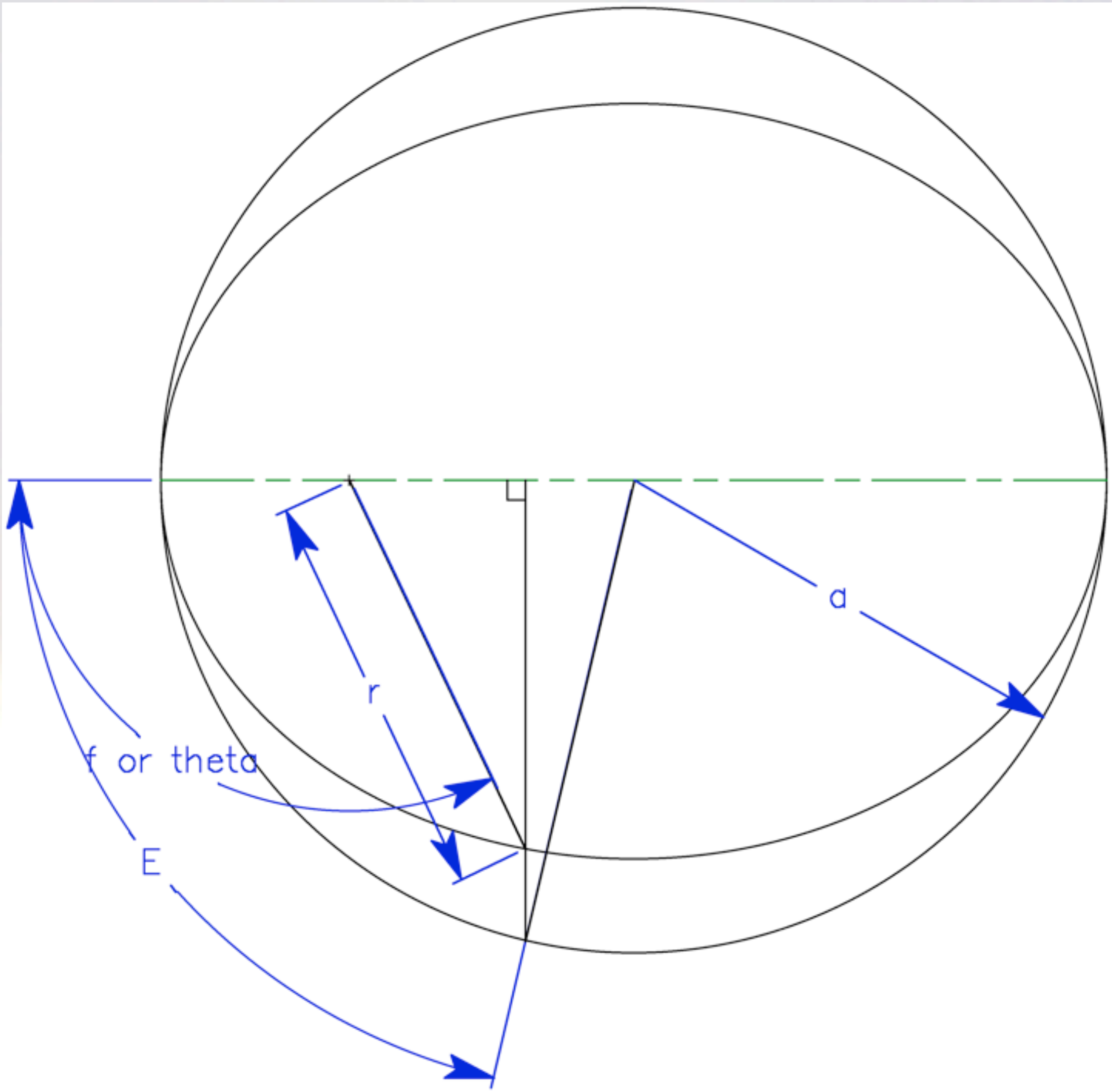
$$\Delta v_2 = \sqrt{v_2^2 + v_a^2 - 2v_2 v_a \cos \Delta i_2}$$

Sample Plane Change Maneuver



Optimum initial plane change = 2.20°

Calculating Time in Orbit



Time in Orbit

- Period of an orbit

$$P = 2\pi\sqrt{\frac{a^3}{\mu}}$$

- Mean motion (average angular velocity)

$$n = \sqrt{\frac{\mu}{a^3}}$$

- Time since pericenter passage

$$M = nt = E - e\sin E$$

➥M=mean anomaly

Dealing with the Eccentric Anomaly

- Relationship to orbit

$$r = a\left(1 - e\cos E\right)$$

- Relationship to true anomaly

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

- Calculating M from time interval: iterate

$$E_{i+1} = nt + e\sin E_i$$

until it converges

Example: Time in Orbit

- Hohmann transfer from LEO to GEO
  - h_1=300 km --> r_1=6378+300=6678 km
  - r_2=42240 km
- Time of transit (1/2 orbital period)

$$a = \frac{1}{2}(r_1 + r_2) = 24,459\ km$$

$$t_{transit} = \frac{P}{2} = \pi\sqrt{\frac{a^3}{\mu}} = 19,034\ sec = 5h17m14s$$

Example: Time-based Position

Find the spacecraft position 3 hours after perigee

$$n = \sqrt{\frac{\mu}{a^3}} = 1.650x10^{-4} \ \frac{rad}{sec}$$

$$e = 1 - \frac{r_p}{a} = 0.7270$$

$$E_{j+1} = nt + e \sin E_j = 1.783 + 0.7270 \sin E_j$$

E=0; 1.783; 2.494; 2.222; 2.361; 2.294; 2.328; 2.311; 2.320; 2.316; 2.318; 2.317; 2.317; 2.317

Example: Time-based Position (cont.)

$$E = 2.317$$

$$r = a(1 - e\cos E) = 12,387\ \text{km}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \Longrightarrow \theta = 160\ \text{deg}$$

Have to be sure to get the position in the proper quadrant - since the time is less than 1/2 the period, the spacecraft has yet to reach apogee --> $0° < \theta < 180°$

Velocity Components in Orbit

$$r = \frac{p}{1 + e\cos\theta}$$

$$v_r = \frac{dr}{dt} = \frac{d}{dt}\left(\frac{p}{1 + e\cos\theta}\right) = \frac{-p(-e\sin\theta\frac{d\theta}{dt})}{(1 + e\cos\theta)^2}$$

$$v_r = \frac{pe\sin\theta}{(1 + e\cos\theta)^2}\frac{d\theta}{dt}$$

$$1 + e\cos\theta = \frac{p}{r} \Rightarrow v_r = \frac{r^2\frac{d\theta}{dt}e\sin\theta}{p}$$

$$\overrightarrow{h} = \overrightarrow{r} \times \overrightarrow{v}$$

Velocity Components in Orbit (cont.)

$$\vec{h} = \vec{r} \times \vec{v}$$

$$h = rv\cos\gamma = r\left(r\frac{d\theta}{dt}\right) = r^2\frac{d\theta}{dt}$$

$$v_r = \frac{r^2\frac{d\theta}{dt}e\sin\theta}{p} = \frac{he\sin\theta}{p} = \frac{\sqrt{p\mu}}{p}e\sin\theta$$

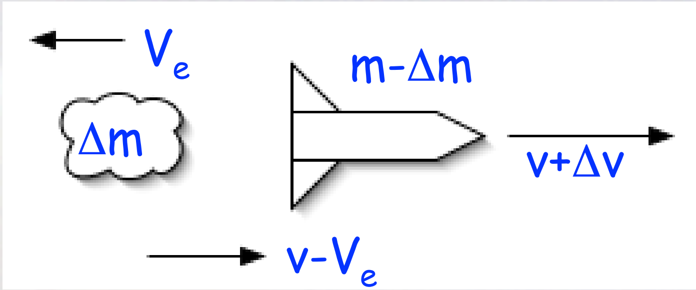
$$v_r = \sqrt{\frac{\mu}{p}}e\sin\theta$$

$$v_\theta = r\frac{d\theta}{dt} = r\frac{h}{r^2} = \frac{h}{r} = \frac{\sqrt{p\mu}}{r}$$

$$v_\theta = \sqrt{\frac{\mu}{p}}(1 + e\cos\theta)$$

$$\tan\gamma = \frac{v_r}{v_\theta} = \frac{e\sin\theta}{1 + e\cos\theta}$$

Derivation of the Rocket Equation



- Momentum at time t:

$$M = mv$$

- Momentum at time t+Δt:

$$M = (m - \Delta m)(V + \Delta v) + \Delta m(v - V_e)$$

- Some algebraic manipulation gives:

$$m\Delta v = -\Delta m V_e$$

- Take to limits and integrate:

$$\int_{m_{initial}}^{m_{final}} \frac{dm}{m} = -\int_{V_{initial}}^{V_{final}} \frac{dv}{V_e}$$

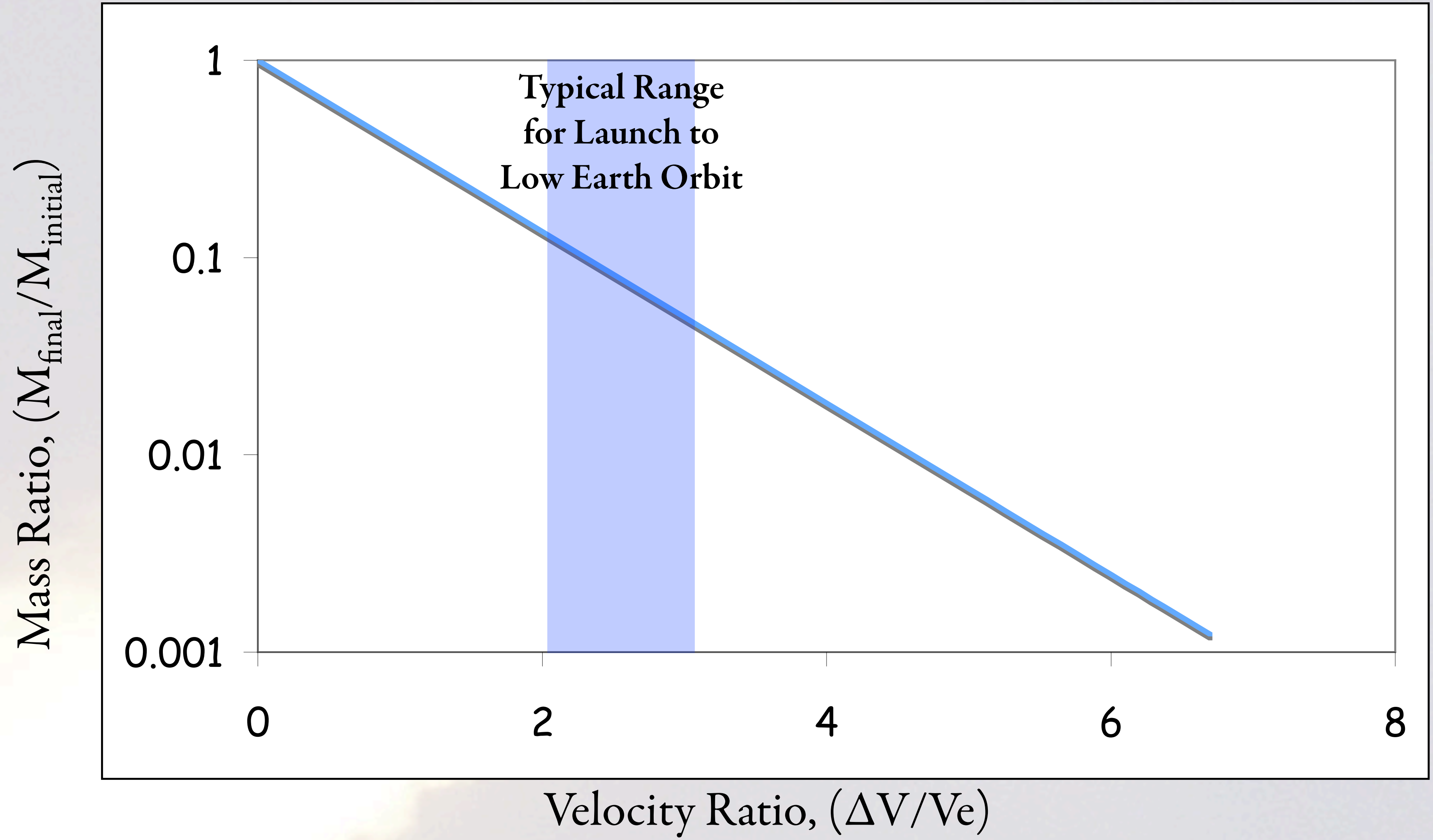
The Rocket Equation

- Alternate forms $$r \equiv \frac{m_{final}}{m_{initial}} = e^{-\frac{\Delta V}{V_e}}$$

$$\Delta v = -V_e \ln\left(\frac{m_{final}}{m_{initial}}\right) = -V_e \ln r$$

- Basic definitions/concepts
  – Mass ratio $$r \equiv \frac{m_{final}}{m_{initial}} \text{ or } \Re \equiv \frac{m_{initial}}{m_{final}}$$
  – Nondimensional velocity change
  "Velocity ratio" $$\nu \equiv \frac{\Delta V}{V_e}$$

Rocket Equation (First Look)



Sources and Categories of Vehicle Mass

Payload
Propellants
Structure
Propulsion
Avionics
Power
Mechanisms
Thermal
Etc.

Sources and Categories of Vehicle Mass

- Payload
- Propellants
- Inert Mass
  - Structure
  - Propulsion
  - Avionics
  - Power
  - Mechanisms
  - Thermal
  - Etc.

Basic Vehicle Parameters

- Basic mass summary

$$m_o = m_{pl} + m_{pr} + m_{in}$$

- Inert mass fraction

$$\delta \equiv \frac{m_{in}}{m_o} = \frac{m_{in}}{m_{pl} + m_{pr} + m_{in}}$$

- Payload fraction

$$\lambda \equiv \frac{m_{pl}}{m_o} = \frac{m_{pl}}{m_{pl} + m_{pr} + m_{in}}$$

- Parametric mass ratio

$$r = \lambda + \delta$$

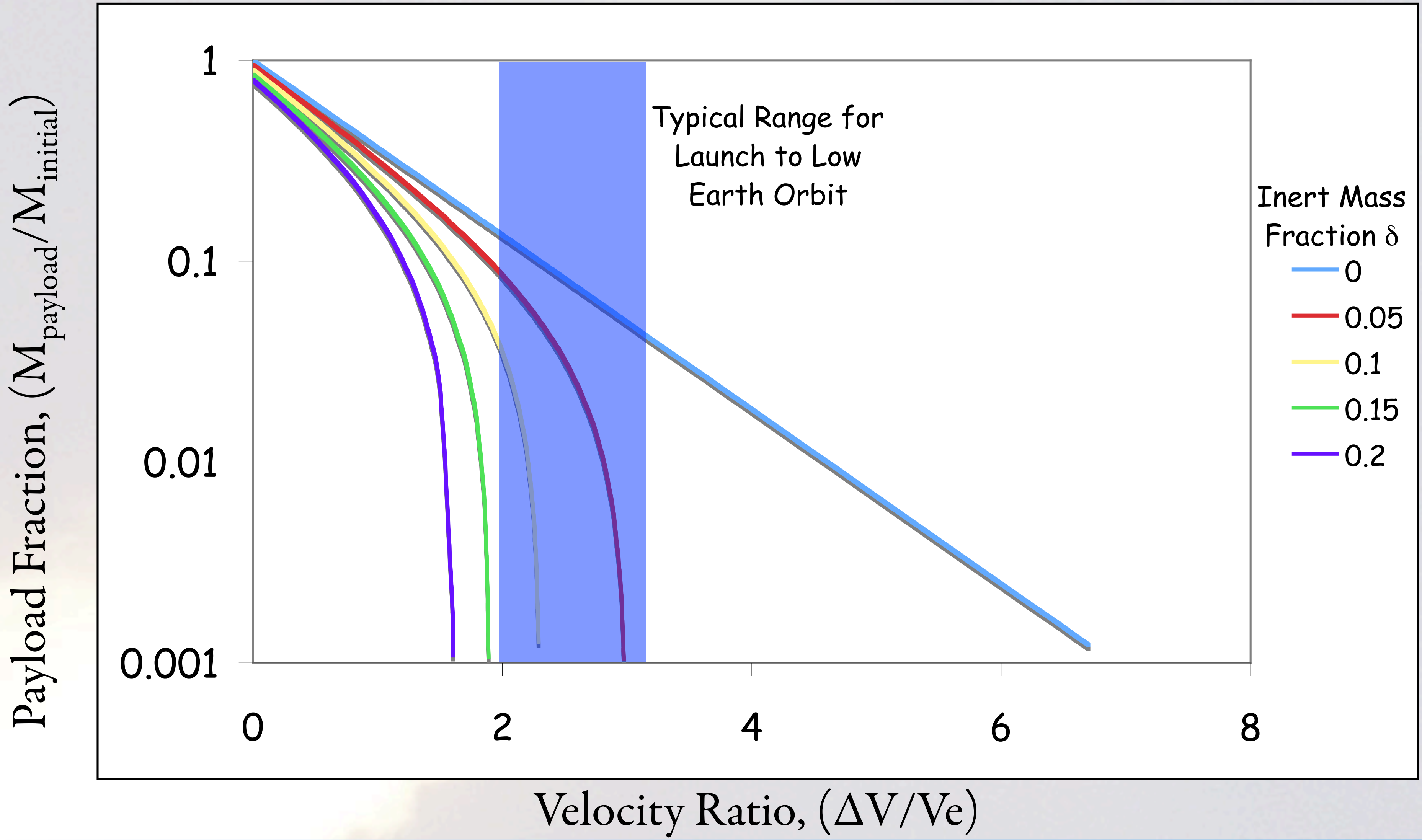
m_o =initial mass

m_{pl} =payload mass

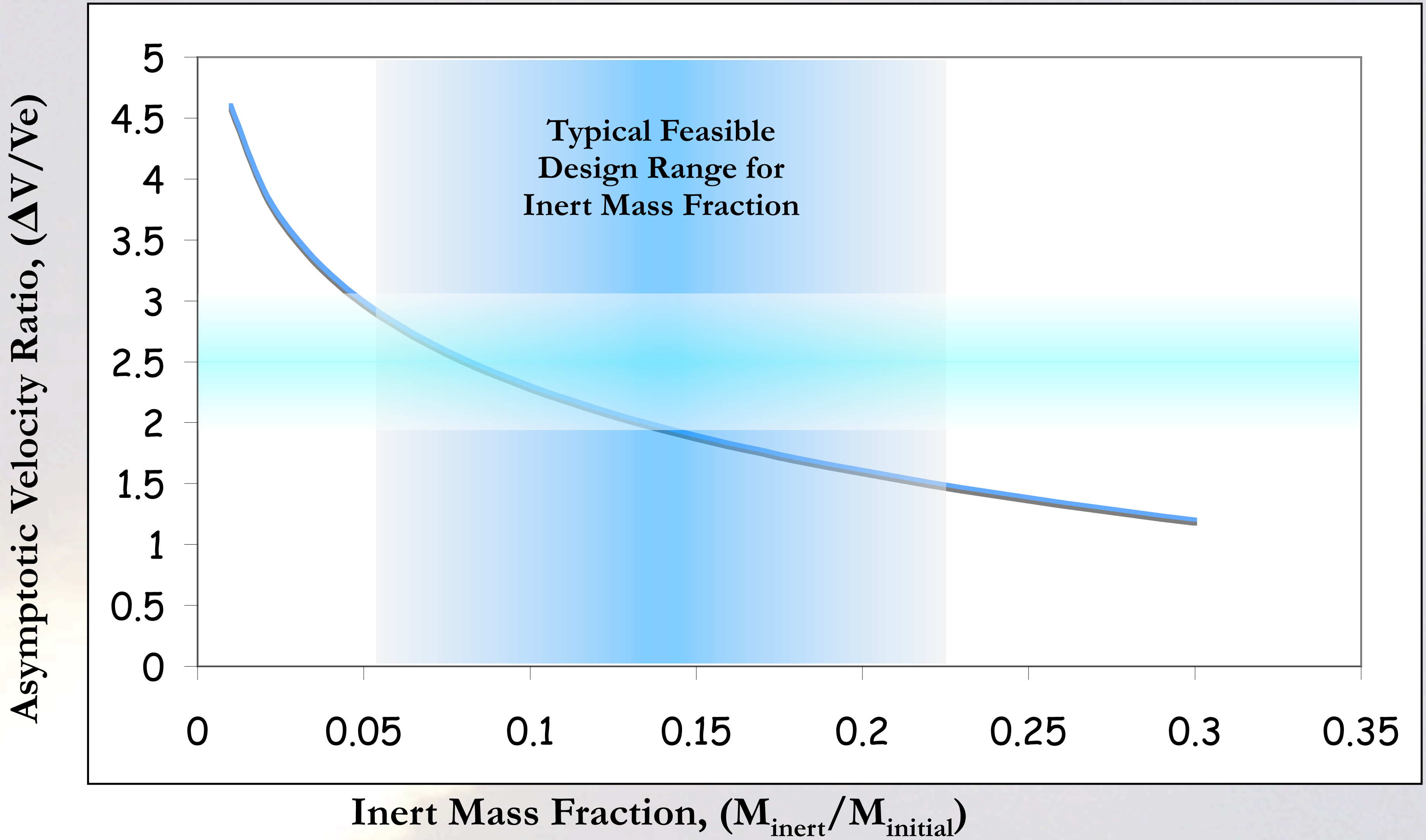
m_{pr} =propellant mass

m_{in} =inert mass

Rocket Equation (including Inert Mass)



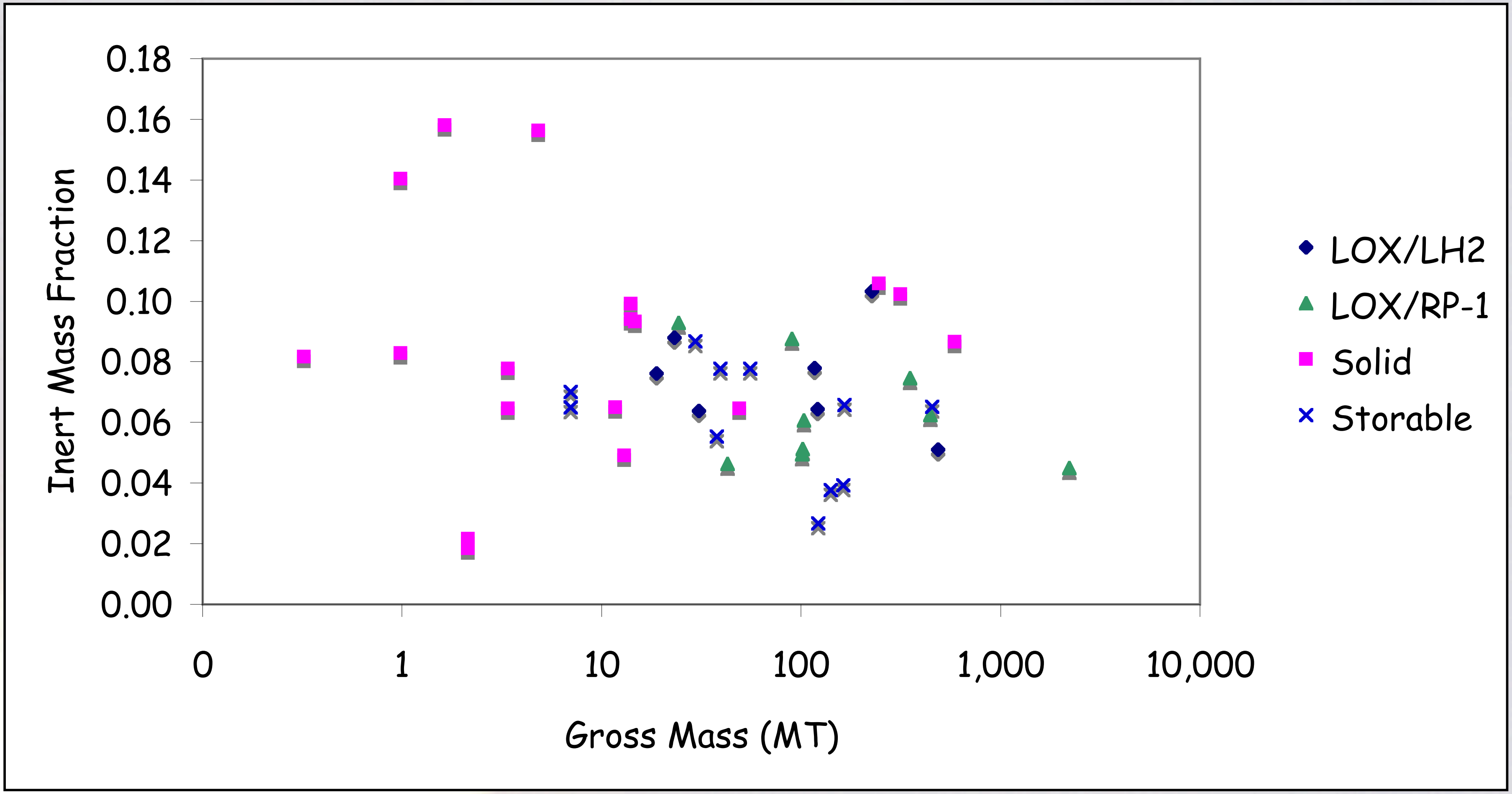
Limiting Performance Based on Inert Mass



Regression Analysis of Existing Vehicles

Veh/Stage	prop mass (lbs)	gross mass (lbs)	Type	Propellants	Isp vac (sec)	isp sl (sec)	sigma	eps	delta
Delta 6925 Stage 2	13,367	15,394	Storab	N2O4-A50	319.4		0.152	0.132	0.070
Delta 7925 Stage 2	13,367	15,394	Storab	N2O4-A50	319.4		0.152	0.132	0.065
Titan II Stage 2	59,000	65,000	Storab	N2O4-A50	316.0		0.102	0.092	0.087
Titan III Stage 2	77,200	83,600	Storab	N2O4-A50	316.0		0.083	0.077	0.055
Titan IV Stage 2	77,200	87,000	Storab	N2O4-A50	316.0		0.127	0.113	0.078
Proton Stage 3	110,000	123,000	Storab	N2O4-A50	315.0		0.118	0.106	0.078
Titan II Stage 1	260,000	269,000	Storab	N2O4-A50	296.0		0.035	0.033	0.027
Titan III Stage 1	294,000	310,000	Storab	N2O4-A50	302.0		0.054	0.052	0.038
Titan IV Stage 1	340,000	359,000	Storab	N2O4-A50	302.0		0.056	0.053	0.039
Proton Stage 2	330,000	365,000	Storab	N2O4-A50	316.0		0.106	0.096	0.066
Proton Stage 1	904,000	1,004,000	Storab	N2O4-A50	316.0	285.0	0.111	0.100	0.065
average					312.2	285.0	0.100	0.089	0.061
standard deviation					8.1		0.039	0.033	0.019

Inert Mass Fraction Data for Existing LVs



Regression Analysis

- Given a set of N data points (x_i, y_i)
- Linear curve fit: y=Ax+B
  - find A and B to minimize sum squared error

$$\text{error} = \sum_{i=1}^{N} (Ax_i + B - y_i)^2$$

  - Analytical solutions exist, or use Solver in Excel
- Power law fit: $y = Bx^A$

$$\text{error} = \sum_{i=1}^{N} \left[A \log(x_i) + B - \log(y_i) \right]^2$$

- Polynomial, exponential, many other fits possible

Solution of Least-Squares Linear Regression

$$\frac{\partial(\text{error})}{\partial A} = 2\sum_{i=1}^{N}(Ax_i + B - y_i)x_i = 0$$

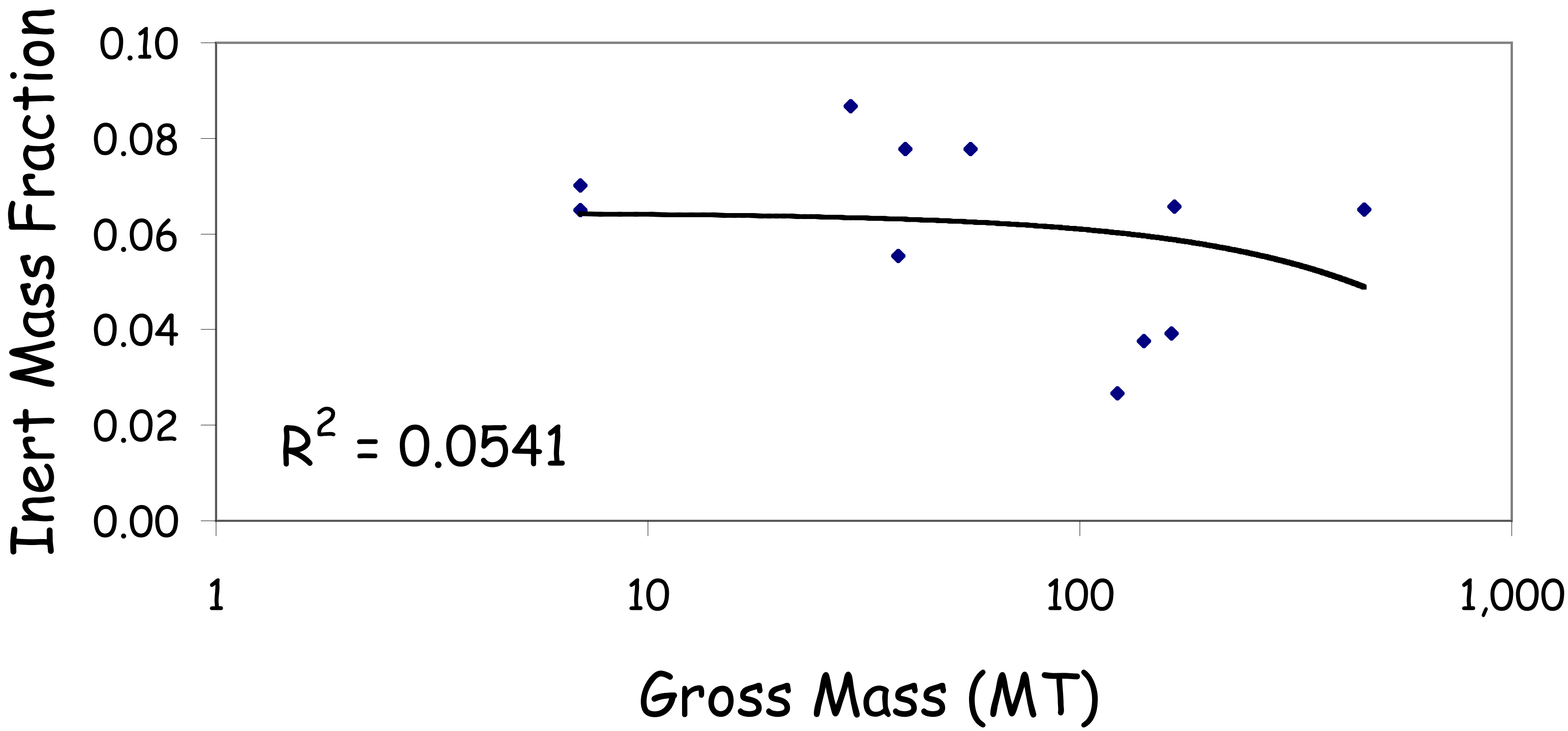
$$\frac{\partial(\text{error})}{\partial B} = 2\sum_{i=1}^{N}(Ax_i + B - y_i) = 0$$

$$A\sum_{i=1}^{N}x_i^2 + B\sum_{i=1}^{N}x_i - \sum_{i=1}^{N}x_i y_i = 0$$

$$A\sum_{i=1}^{N}x_i + NB - \sum_{i=1}^{N}y_i = 0$$

$$A = \frac{N\sum x_i y_i - \sum x_i \sum y_i}{N\sum x_i^2 - \left(\sum x_i\right)^2} \qquad B = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{N\sum x_i^2 - \left(\sum x_i\right)^2}$$

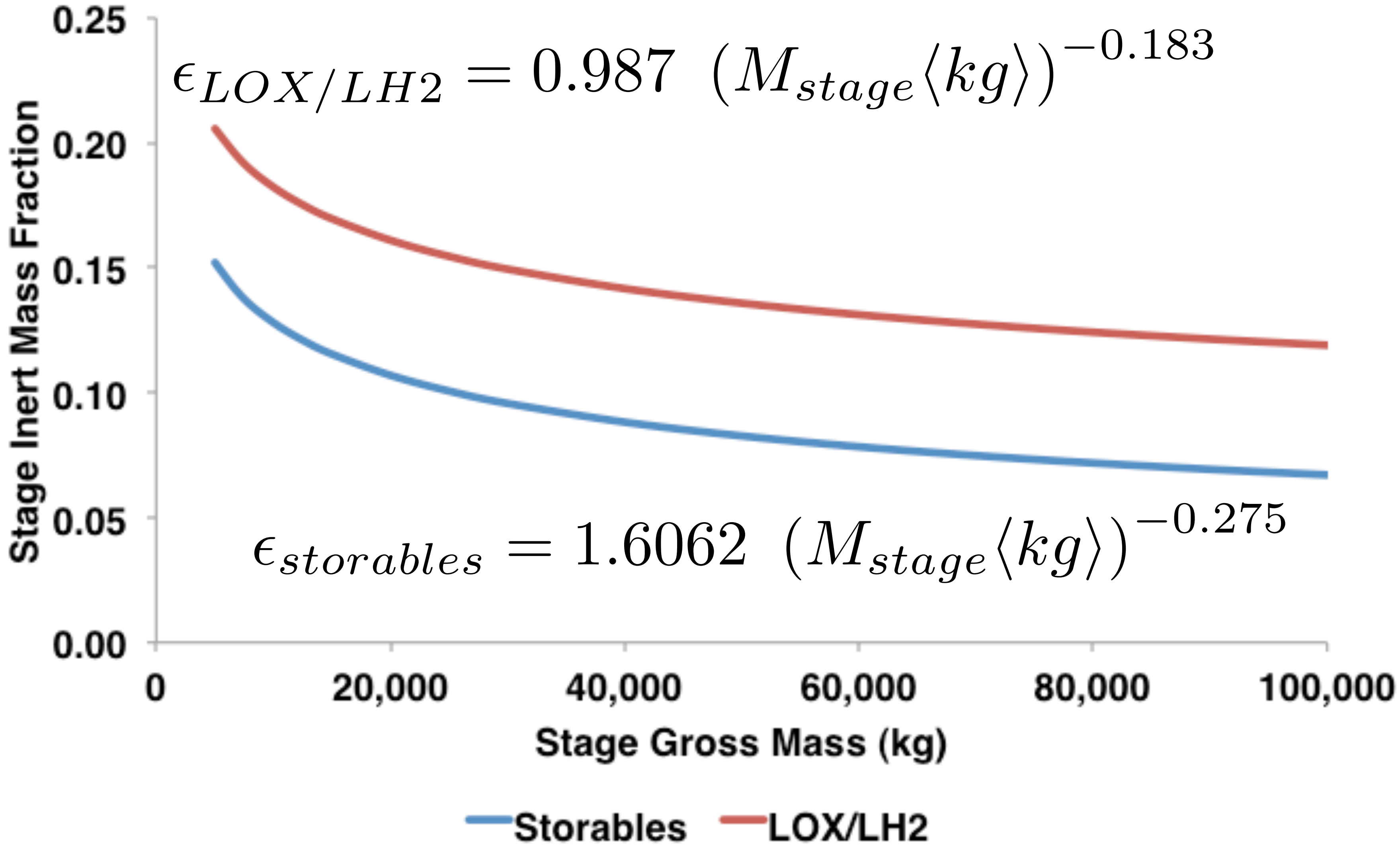
Regression Analysis - Storables



Regression Values for Design Parameters

	Vacuum Ve (m/sec)	Inert Mass Fraction	Max ΔV (m/sec)
LOX/LH2	4273	0.075	11,070
LOX/RP-1	3136	0.063	8664
Storables	3058	0.061	8575
Solids	2773	0.087	6783

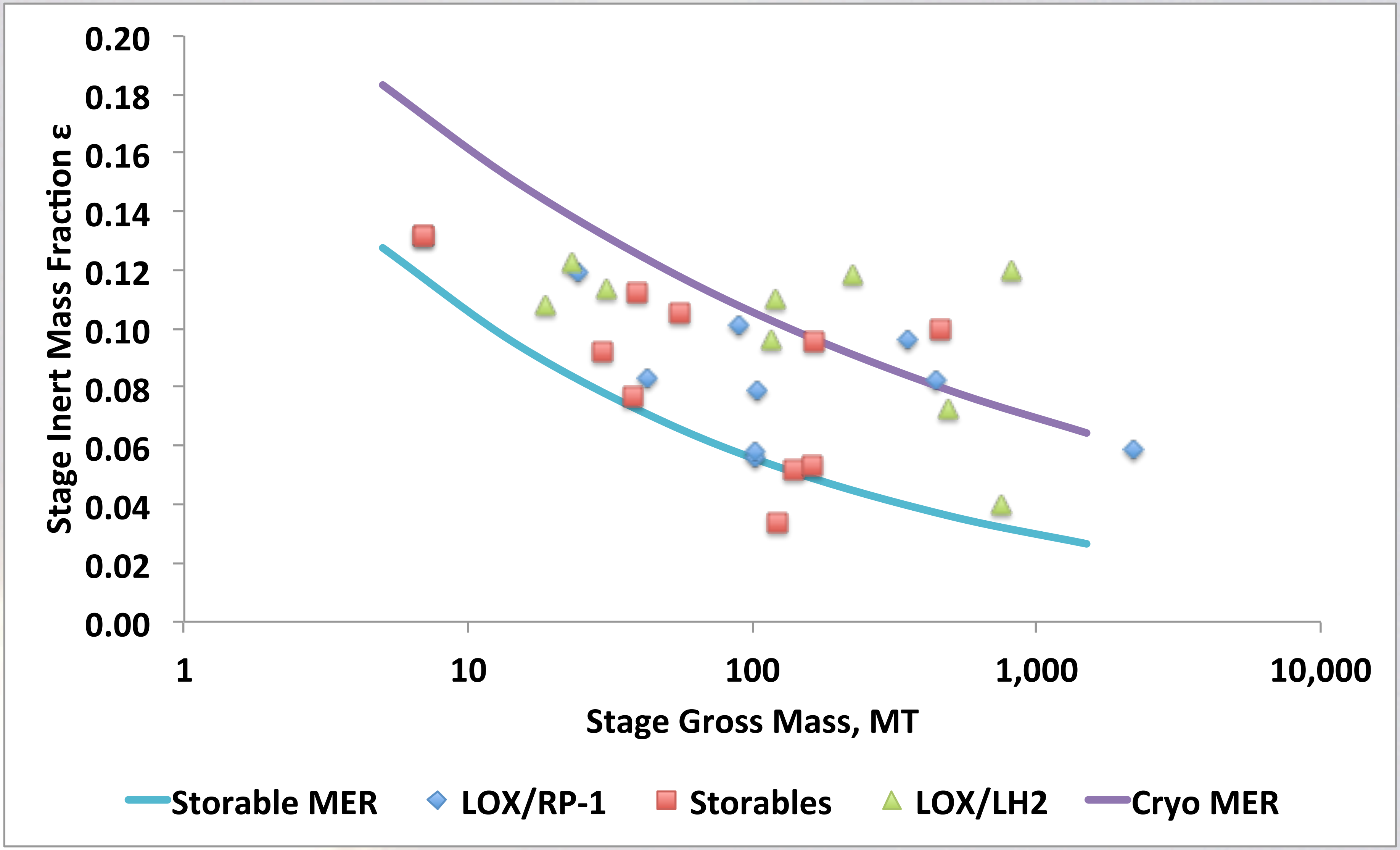
Stage Inert Mass Fraction Estimation



$$\epsilon_{LOX/LH2} = 0.987 \ \left(M_{stage}\langle kg \rangle\right)^{-0.183}$$

$$\epsilon_{storables} = 1.6062 \ \left(M_{stage}\langle kg \rangle\right)^{-0.275}$$

Economy of Scale for Stage Size



The Rocket Equation for Multiple Stages

- Assume two stages

$$\Delta V_1 = -V_{e1} \ln\left(\frac{m_{final1}}{m_{initial1}}\right) = -V_{e1} \ln(r_1)$$

$$\Delta V_2 = -V_{e2} \ln\left(\frac{m_{final2}}{m_{initial2}}\right) = -V_{e2} \ln(r_2)$$

- Assume $V_{e1}=V_{e2}=V_e$

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1) - V_e \ln(r_2) = -V_e \ln(r_1 r_2)$$

Continued Look at Multistaging

- There's a historical tendency to define $r_0 = r_1 r_2$

$$\Delta V_1 + \Delta V_2 = -V_e \ln \left(r_1 r_2 \right) = -V_e \ln \left(r_0 \right)$$

- But it's important to remember that it's really

$$\Delta V_1 + \Delta V_2 = -V_e \ln \left(r_1 r_2 \right) = -V_e \ln \left(\frac{m_{final1}}{m_{initial1}} \frac{m_{final2}}{m_{initial2}} \right)$$

- And that r_0 has *no* physical significance, since

$$m_{final1} \neq m_{initial2} \Rightarrow r_0 \neq \frac{m_{final2}}{m_{initial1}}$$

Multistage Inert Mass Fraction

- Total inert mass fraction

$$\delta_0 = \frac{m_{in,1} + m_{in,2} + m_{in,3}}{m_0} = \frac{m_{in,1}}{m_0} + \frac{m_{in,2}}{m_0} + \frac{m_{in,3}}{m_0}$$

$$\delta_0 = \frac{m_{in,1}}{m_0} + \frac{m_{in,2}}{m_{0,2}}\frac{m_{0,2}}{m_0} + \frac{m_{in,3}}{m_{0,3}}\frac{m_{0,3}}{m_{0,2}}\frac{m_{0,2}}{m_0}$$

- Convert to dimensionless parameters

$$\delta_0 = \delta_1 + \delta_2\lambda_1 + \delta_3\lambda_2\lambda_1$$

- General form of the equation

$$\delta_0 = \sum_{j=1}^{n \text{ stages}} [\delta_j \prod_{\ell=1}^{j-1} \lambda_\ell]$$

Multistage Payload Fraction

- Total payload fraction (3 stage example)

$$\lambda_0 = \frac{m_{pl}}{m_0} = \frac{m_{pl}}{m_{0,3}} \frac{m_{0,3}}{m_{0,2}} \frac{m_{0,2}}{m_0}$$

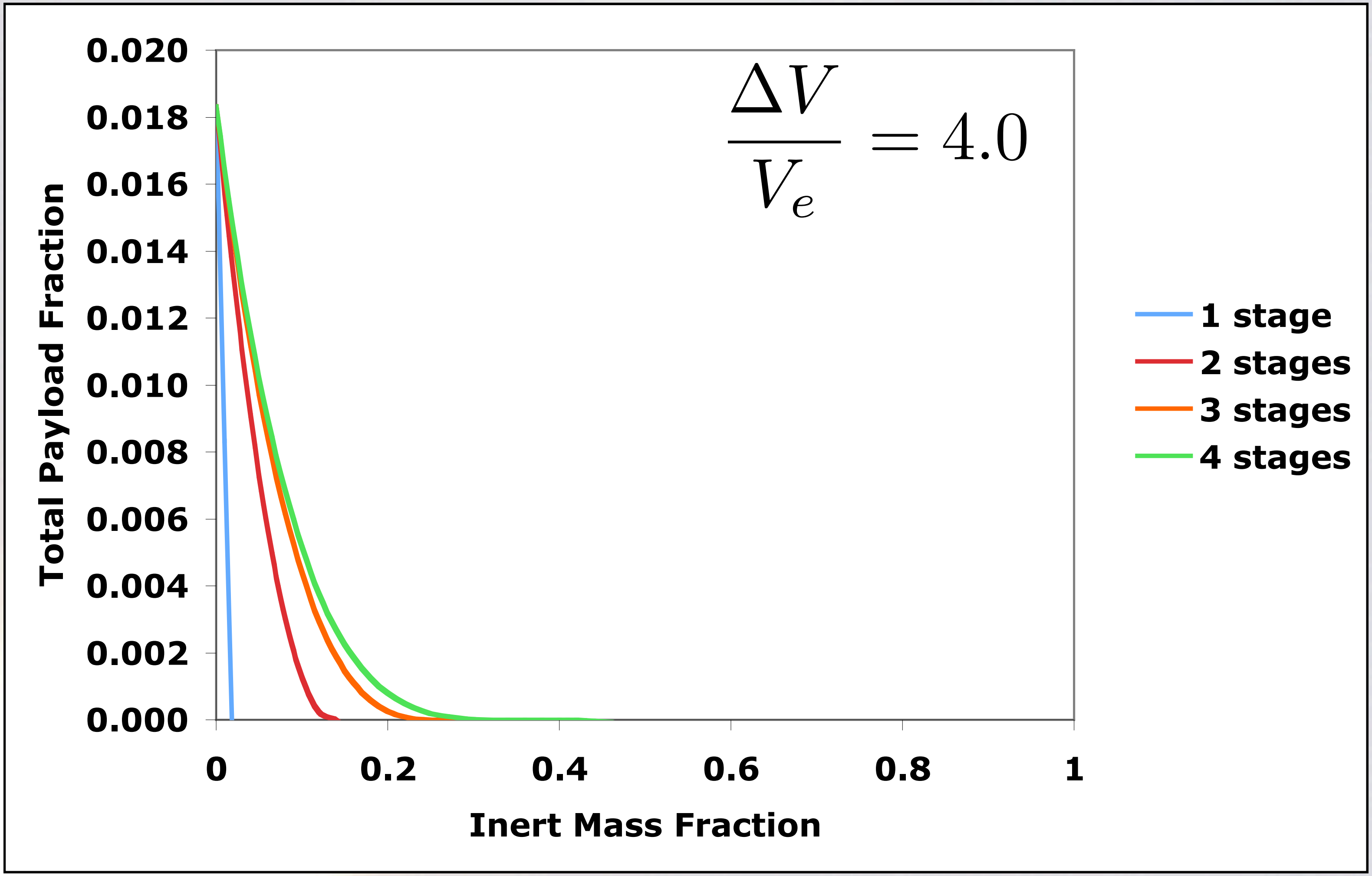
- Convert to dimensionless parameters

$$\lambda_0 = \lambda_3 \lambda_2 \lambda_1$$

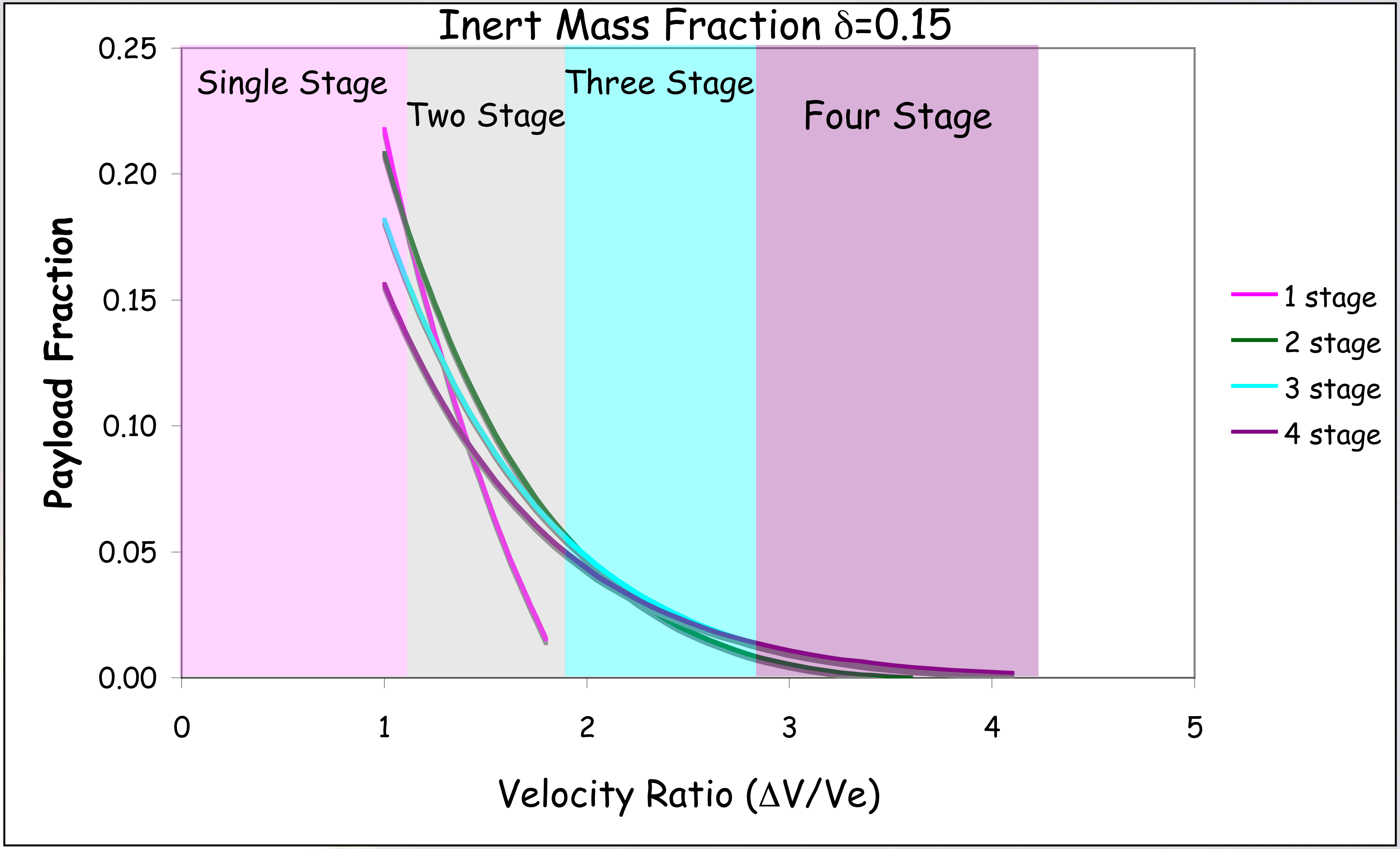
- Generic form of the equation

$$\lambda_0 = \prod_{j=1}^{n \text{ stages}} \lambda_j$$

Effect of δ and ΔV/Ve on Payload



Effect of Staging



Trade Space for Number of Stages