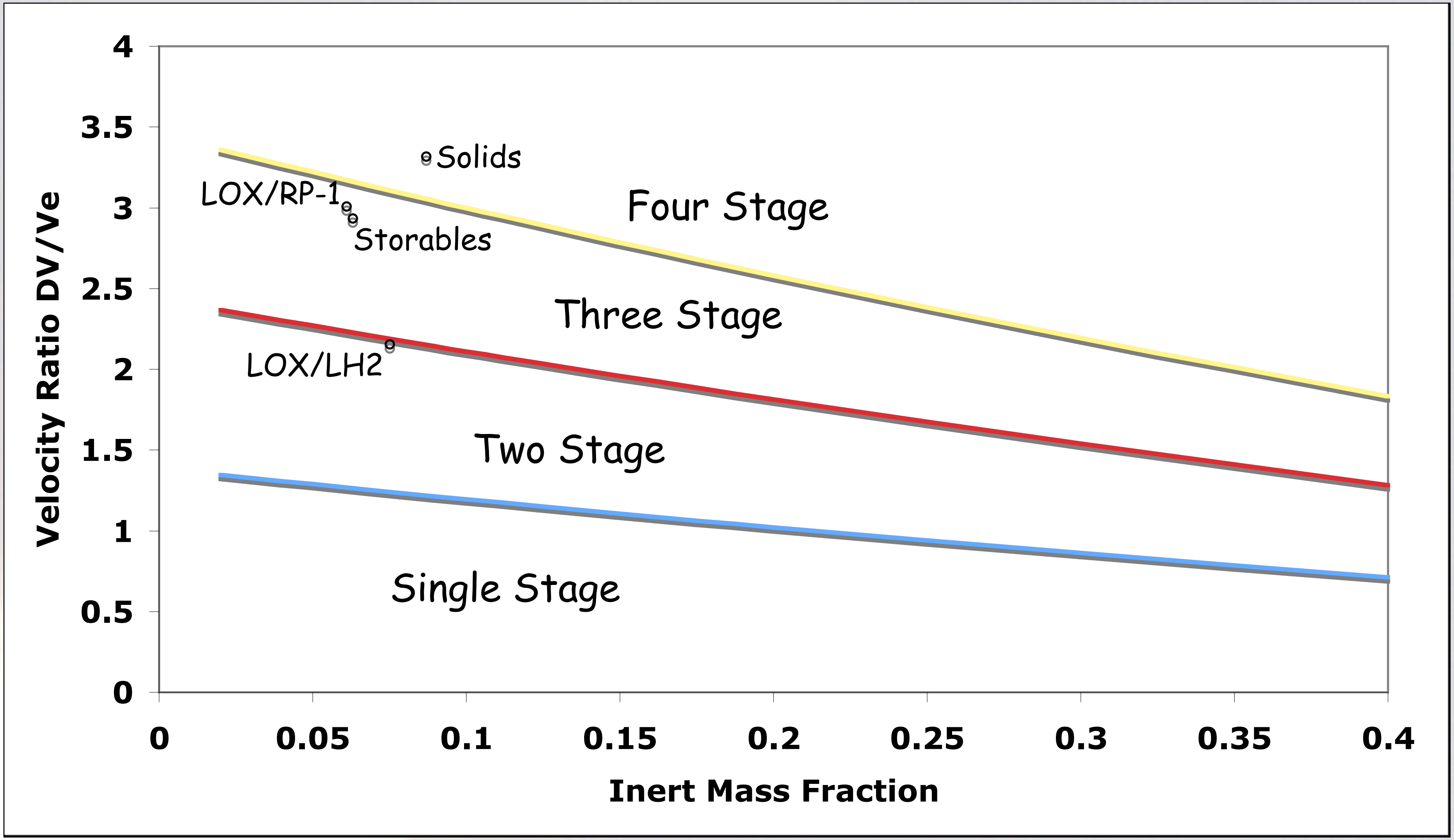
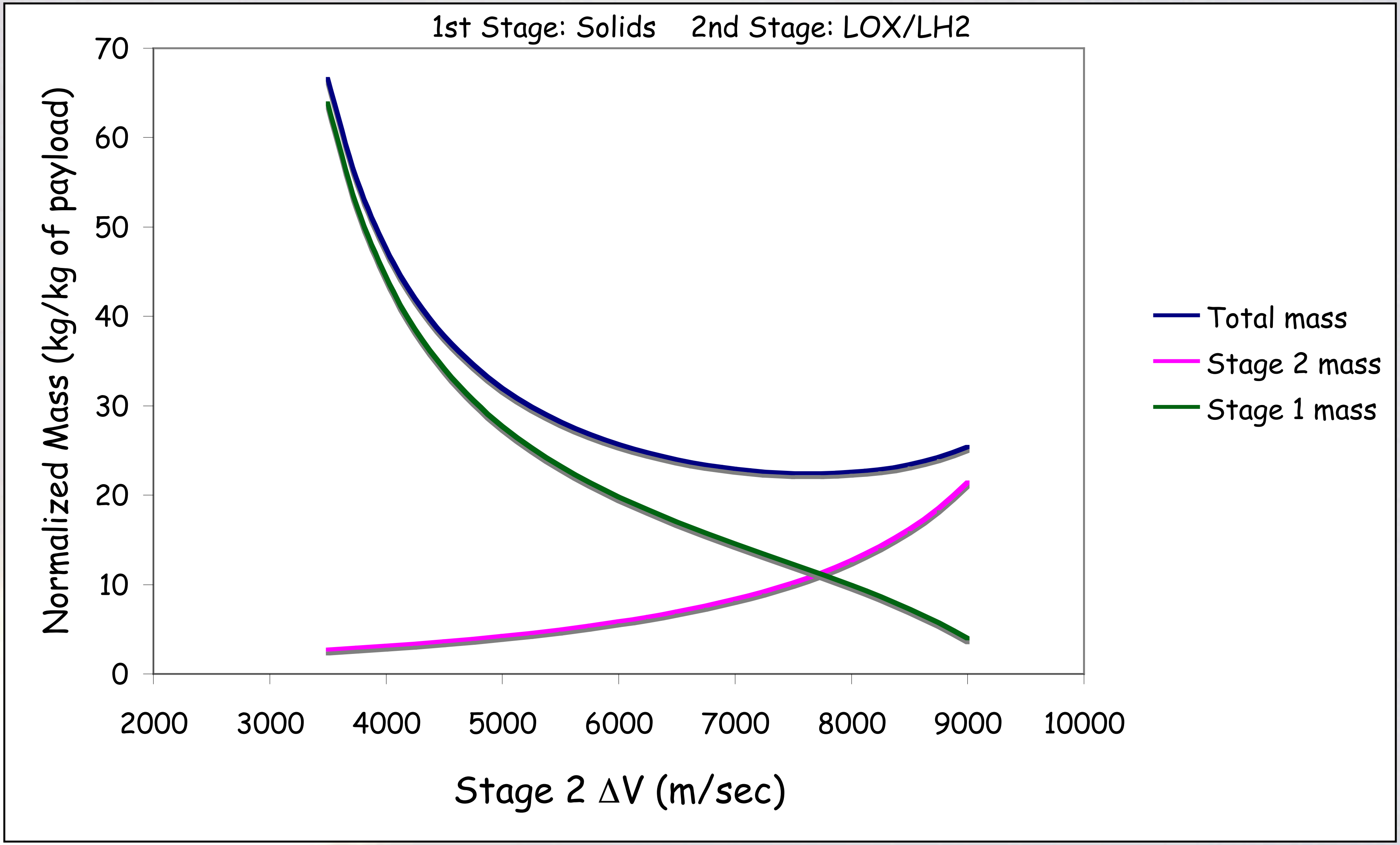
Ballistic Atmospheric Entry

- Continuation of rocket performance lecture
- Standard atmospheres
- Orbital decay due to atmospheric drag
- Straight-line (no gravity) ballistic entry based on atmospheric density

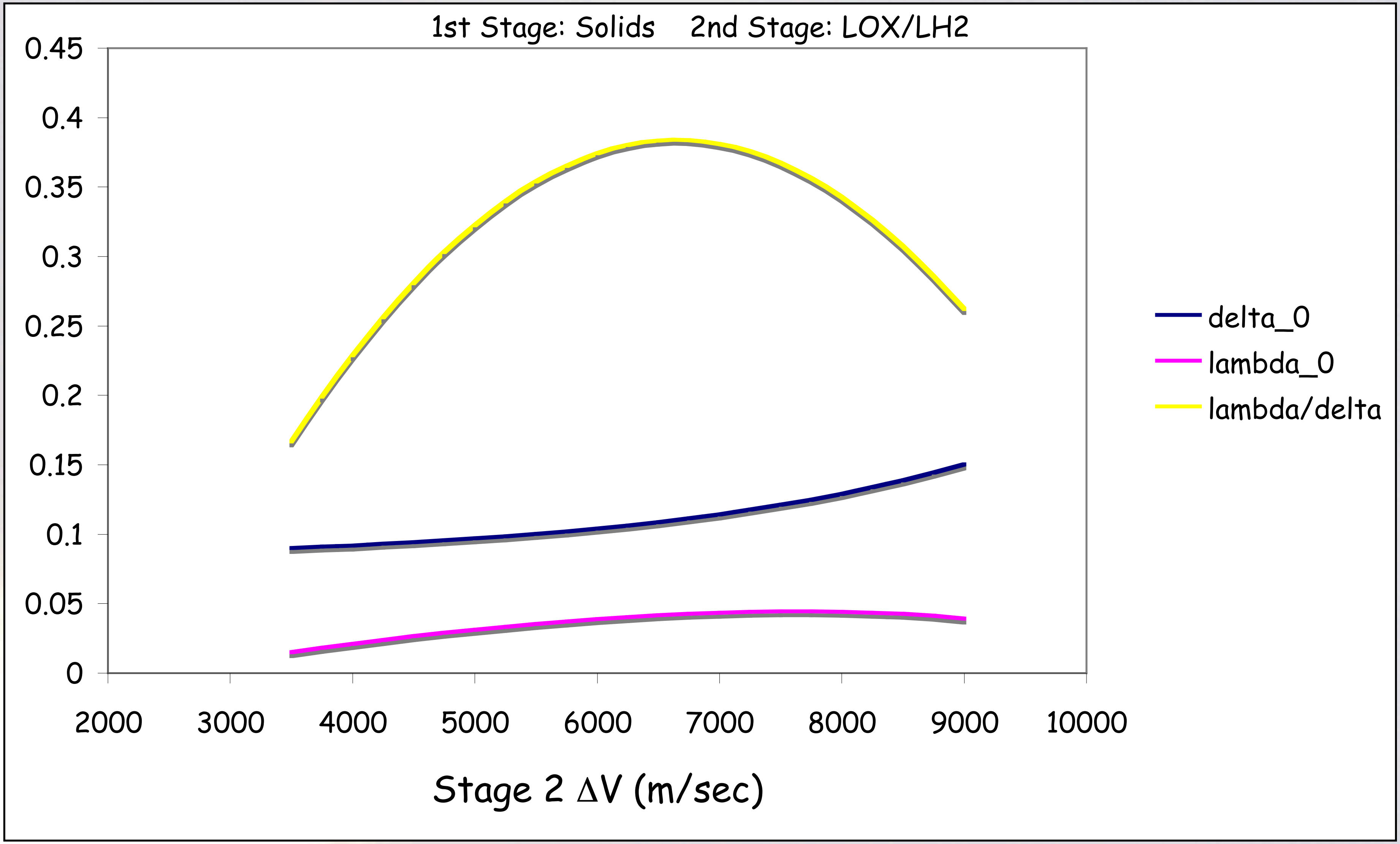
Trade Space for Number of Stages



Effect of ΔV Distribution



ΔV Distribution and Design Parameters



Lagrange Multipliers

- Given an objective function

$$y = f(x)$$

subject to constraint function

$$z = g(x)$$

- Create a new objective function

$$z = f(x) + \lambda[g(x) - z]$$

- Solve simultaneous equations

$$\frac{\partial y}{\partial x} = 0 \qquad \frac{\partial y}{\partial \lambda} = 0$$

Optimum ΔV Distribution Between Stages

- Maximize payload fraction (2 stage case)

$$\lambda_0 = \lambda_1 \lambda_2 = (r_1 - \delta_1)(r_2 - \delta_2)$$

subject to constraint function

$$\Delta V_{total} = \Delta V_1 + \Delta V_2$$

- Create a new objective function

$$\lambda_o = \left(e^{\frac{-\Delta V_1}{V_{e,1}}} - \delta_1 \right) \left(e^{\frac{-\Delta V_2}{V_{e,2}}} - \delta_2 \right) + K \left[\Delta V_1 + \Delta V_2 - \Delta V_{total} \right]$$

➡Very messy for partial derivatives!

Optimum ΔV Distribution (continued)

- Use substitute objective function

$$max\left(\lambda_o\right) \Longleftrightarrow max\left[ln\left(\lambda_o\right)\right]$$

- Create a new constrained objective function

$$ln\left(\lambda_o\right) = ln\left(r_1 - \delta_1\right) + ln\left(r_2 - \delta_2\right) + K\left[\Delta V_1 + \Delta V_2 - \Delta V_{total}\right]$$

- Take partials and set equal to zero

$$\frac{\partial\left[ln\left(\lambda_o\right)\right]}{\partial r_1} = 0 \qquad \frac{\partial\left[ln\left(\lambda_o\right)\right]}{\partial r_2} = 0 \qquad \frac{\partial\left[ln\left(\lambda_o\right)\right]}{\partial K} = 0$$

Optimum ΔV Special Cases

- "Generic" partial of objective function

$$\frac{\partial\left[ln\left(\lambda_o\right)\right]}{\partial r_i} = \frac{1}{r_i - \delta_i} + K\frac{V_{e,i}}{r_i} = 0$$

- Special case: $\delta_1 = \delta_2$ $V_{e,1} = V_{e,2}$

$$r_1 = r_2 \Longrightarrow \Delta V_1 = \Delta V_2 = \frac{\Delta V_{total}}{2}$$

- Same principle holds for n stages

$$r_1 = r_2 = \cdots = r_n \Longrightarrow$$

$$\Delta V_1 = \Delta V_2 = \cdots = \Delta V_n = \frac{\Delta V_{total}}{n}$$

Sensitivity to Inert Mass

ΔV for multistaged rocket

$$\Delta V_{tot} = \sum_{k=1}^{n\ stages} \Delta V_k = \sum_{k=1}^{n} V_{e,k} \ln\left(\frac{m_{o,k}}{m_{f,k}}\right)$$

where

$$m_{o,k} = m_{pl} + m_{pr,k} + m_{in,k} + \sum_{j=k+1}^{n} \left(m_{pr,j} + m_{in,j}\right)$$

$$m_{f,k} = m_{pl} + m_{in,k} + \sum_{j=k+1}^{n} \left(m_{pr,j} + m_{in,j}\right)$$

Finding Payload Sensitivity to Inert Mass

- Given the equation linking mass to ΔV, take

$$\frac{\partial(\Delta V_{tot})}{\partial m_{pl}} dm_{pl} + \frac{\partial(\Delta V_{tot})}{\partial m_{in,j}} dm_{in,j} = 0$$

and solve to find

$$\left.\frac{dm_{pl}}{dm_{in,k}}\right|_{\partial(\Delta V_{tot})=0} = \frac{-\sum_{j=1}^{k} V_{e,j}\left(\frac{1}{m_{o,j}} - \frac{1}{m_{f,j}}\right)}{\sum_{\ell=1}^{N} V_{e,\ell}\left(\frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}}\right)}$$

- This equation shows the "trade-off ratio" - Δpayload resulting from a change in inert mass for stage k (for a vehicle with N total stages)

Trade-off Ratio Example: Gemini-Titan II

	Stage 1	Stage 2
Initial Mass (kg)	150,500	32,630
Final Mass (kg)	39,370	6099
Ve (m/sec)	2900	3097
$dm_{pl}/dm_{in,k}$	-0.1164	-1

Payload Sensitivity to Propellant Mass

- In a similar manner, solve to find

$$\left.\frac{dm_{pl}}{dm_{pr,k}}\right|_{\partial(\Delta V_{tot})=0} = \frac{-\sum_{j=1}^{k} V_{e,j}\left(\frac{1}{m_{o,j}}\right)}{\sum_{\ell=1}^{N} V_{e,\ell}\left(\frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}}\right)}$$

- This equation gives the change in payload mass as a function of additional propellant mass (all other parameters held constant)

Trade-off Ratio Example: Gemini-Titan II

	Stage 1	Stage 2
Initial Mass (kg)	150,500	32,630
Final Mass (kg)	39,370	6099
Ve (m/sec)	2900	3097
$dm_{pl}/dm_{in,k}$	-0.1164	-1
$dm_{pl}/dm_{pr,k}$	0.04124	0.2443

Payload Sensitivity to Exhaust Velocity

- Use the same technique to find the change in payload resulting from additional exhaust velocity for stage k

$$\left.\frac{dm_{pl}}{dV_{e,k}}\right|_{\partial(\Delta V_{tot})=0} = \frac{\sum_{j=1}^{k} \ln\left(\frac{m_{o,j}}{m_{f,j}}\right)}{\sum_{\ell=1}^{N} V_{e,\ell}\left(\frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}}\right)}$$

- This trade-off ratio (unlike the ones for inert and propellant masses) has units - kg/(m/sec)

Trade-off Ratio Example: Gemini-Titan II

	Stage 1	Stage 2
Initial Mass (kg)	150,500	32,630
Final Mass (kg)	39,370	6099
Ve (m/sec)	2900	3097
$dm_{pl}/dm_{in,k}$	-0.1164	-1
$dm_{pl}/dm_{pr,k}$	0.04124	0.2443
$dm_{pl}/dV_{e,k}$ (kg/m/sec)	2.870	6.459

Parallel Staging

- Multiple dissimilar engines burning simultaneously
- Frequently a result of upgrades to operational systems
- General case requires "brute force" numerical performance analysis

Parallel-Staging Rocket Equation

- Momentum at time t:

$$M = mv$$

- Momentum at time t+Δt:
(subscript "b"=boosters; "c"=core vehicle)

$$M = (m - \Delta m_b - \Delta m_c)(v + \Delta v)$$
$$+\Delta m_b(v - V_{e,b}) + \Delta m_c(v - V_{e,c})$$

- Assume thrust (and mass flow rates) constant

Parallel-Staging Rocket Equation

- Rocket equation during booster burn

$$\Delta V = -\bar{V}_e \ln\left(\frac{m_{final}}{m_{initial}}\right) = -\bar{V}_e \ln\left(\frac{m_{in,b}+m_{in,c}+\chi m_{pr,c}+m_{0,2}}{m_{in,b}+m_{pr,b}+m_{in,c}+m_{pr,c}+m_{0,2}}\right)$$

where χ = fraction of core propellant remaining after booster burnout, and where

$$\bar{V}_e = \frac{V_{e,b}\dot{m}_b+V_{e,c}\dot{m}_c}{\dot{m}_b+\dot{m}_c} = \frac{V_{e,b}m_{pr,b}+V_{e,c}(1-\chi)m_{pr,c}}{m_{pr,b}+(1-\chi)m_{pr,c}}$$

Analyzing Parallel-Staging Performance

Parallel stages break down into pseudo-serial stages:

- Stage "0" (boosters and core)

$$\Delta V_0 = -\bar{V}_e \ln\left(\frac{m_{in,b}+m_{in,c}+\chi m_{pr,c}+m_{0,2}}{m_{in,b}+m_{pr,b}+m_{in,c}+m_{pr,c}+m_{0,2}}\right)$$

- Stage "1" (core alone)

$$\Delta V_1 = -V_{e,c} \ln\left(\frac{m_{in,c}+m_{0,2}}{m_{in,c}+\chi m_{pr,c}+m_{0,2}}\right)$$

- Subsequent stages are as before

Parallel Staging Example: Space Shuttle

- 2 x solid rocket boosters (data below for single SRB)
  - Gross mass 589,670 kg
  - Empty mass 86,183 kg
  - Ve 2636 m/sec
  - Burn time 124 sec
- External tank (space shuttle main engines)
  - Gross mass 750,975 kg
  - Empty mass 29,930 kg
  - Ve 4459 m/sec
  - Burn time 480 sec
- "Payload" (orbiter + P/L) 125,000 kg

Shuttle Parallel Staging Example

$$V_{e,b} = 2636 \frac{m}{sec} \qquad V_{e,c} = 4459 \frac{m}{sec}$$

$$\chi = \frac{480 - 124}{480} = 0.7417$$

$$\bar{V}_e = \frac{2636(1,007,000) + 4459(721,000)(1 - .7417)}{1,007,000 + 721,000(1 - .7417)} = 2921 \frac{m}{sec}$$

$$\Delta V_0 = -2921 \ln \frac{862,000}{3,062,000} = 3702 \frac{m}{sec}$$

$$\Delta V_1 = -4459 \ln \frac{154,900}{689,700} = 6659 \frac{m}{sec}$$

$$\Delta V_{tot} = 10,360 \frac{m}{sec}$$

Modular Staging

- Use identical modules to form multiple stages
- Have to cluster modules on lower stages to make up for nonideal ΔV distributions
- Advantageous from production and development cost standpoints

Module Analysis

- All modules have the same inert mass and propellant mass
- Because δ varies with payload mass, not all modules have the same δ!
- Introduce two new parameters

$$\varepsilon \equiv \frac{m_{in}}{m_{in} + m_{pr}} \qquad \sigma \equiv \frac{m_{in}}{m_{pr}}$$

- Conversions $$\varepsilon = \frac{\delta}{1 - \lambda} \qquad \sigma = \frac{\delta}{1 - \delta - \lambda}$$

Rocket Equation for Modular Boosters

- Assuming n modules in stage 1,

$$r_1 = \frac{n(m_{in}) + m_{o2}}{n(m_{in} + m_{pr}) + m_{o2}} = \frac{n\varepsilon + \frac{m_{o2}}{m_{mod}}}{n + \frac{m_{o2}}{m_{mod}}}$$

- If all 3 stages use same modules, n_j for stage j,

$$r_1 = \frac{n_1\varepsilon + n_2 + n_3 + \rho_{pl}}{n_1 + n_2 + n_3 + \rho_{pl}}$$

where $$\rho_{pl} \equiv \frac{m_{pl}}{m_{mod}}; \; m_{mod} = m_{in} + m_{pr}$$

Example: Conestoga 1620 (EER)

- Small launch vehicle (1 flight, 1 failure)
- Payload 900 kg
- Module gross mass 11,400 kg
- Module empty mass 1,400 kg
- Exhaust velocity 2754 m/sec
- Staging pattern
  - 1st stage - 4 modules
  - 2nd stage - 2 modules
  - 3rd stage - 1 module
  - 4th stage - Star 48V (gross mass 2200 kg, empty mass 140 kg, V_e 2842 m/sec)

Conestoga 1620 Performance

- 4th stage ΔV

$$\Delta V_4 = -V_{e4} \ln \frac{m_{f4}}{m_{o4}} = -2842 \ln \frac{900 + 140}{900 + 2200} = 3104 \; \frac{\text{m}}{\text{sec}}$$

- Treat like three-stage modular vehicle; M_{pl}=3100 kg

$$\epsilon = \frac{m_{in}}{m_{mod}} = \frac{1400}{11400} = 0.1228$$

$$\rho_{pl} = \frac{m_{pl}}{m_{mod}} = \frac{3100}{11400} = 0.2719$$

$$n_1 = 4; \;\; n_2 = 2; \;\; n_3 = 1$$

Constellation 1620 Performance (cont.)

$$r_1 = \frac{n_1\epsilon + n_2 + n_3 + \rho_{pl}}{n_1 + n_2 + n_3 + \rho_{pl}} = \frac{4 \times 0.1228 + 2 + 1 + 0.2719}{4 + 2 + 1 + 0.2719} = 0.5175$$

$$r_2 = \frac{n_2\epsilon + n_3 + \rho_{pl}}{n_2 + n_3 + \rho_{pl}} = \frac{2 \times 0.1228 + 1 + 0.2719}{2 + 1 + 0.2719} = 0.4638$$

$$r_3 = \frac{n_3\epsilon + \rho_{pl}}{n_3 + \rho_{pl}} = \frac{1 \times 0.1228 + 0.2719}{1 + 0.2719} = 0.3103$$

$$V_1 = 1814 \ \frac{\text{m}}{\text{sec}}; \ V_2 = 2116 \ \frac{\text{m}}{\text{sec}}$$

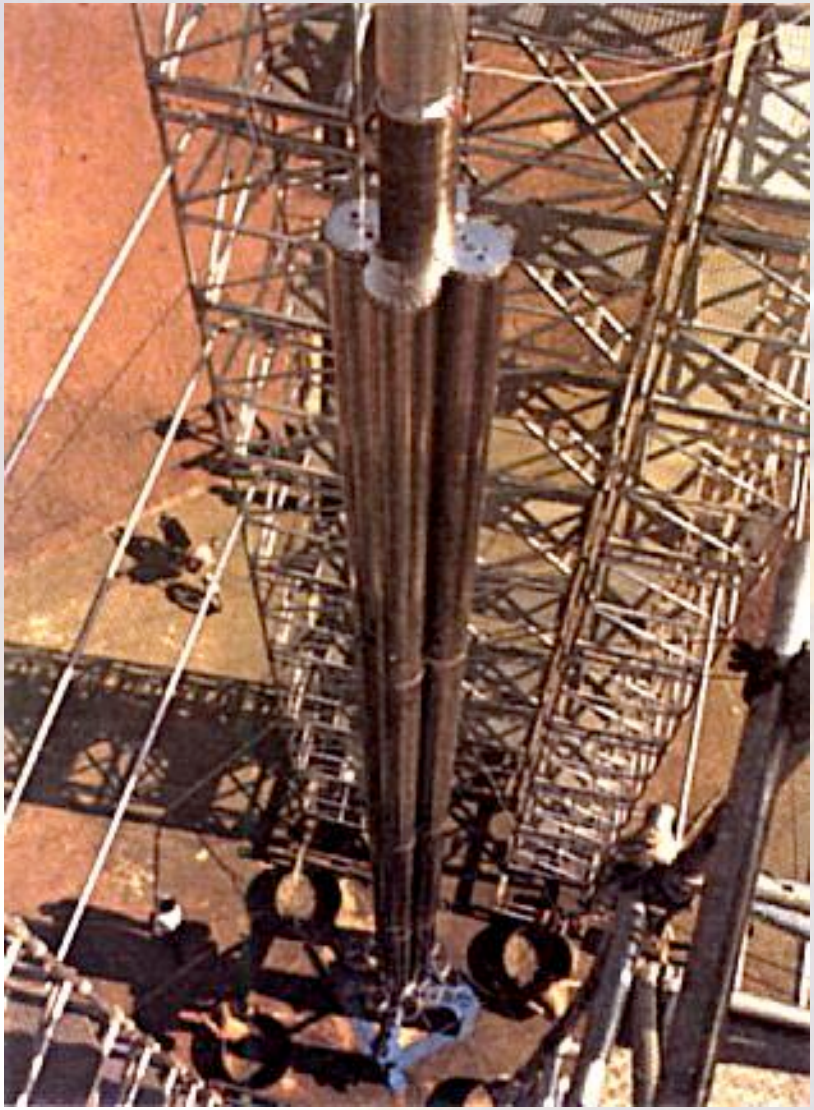
$$V_3 = 3223 \ \frac{\text{m}}{\text{sec}}; \ V_4 = 3104 \ \frac{\text{m}}{\text{sec}}$$

$$V_{total} = 10,257 \ \frac{\text{m}}{\text{sec}}$$

Discussion about Modular Vehicles

- Modularity has several advantages
  - Saves money (smaller modules cost less to develop)
  - Saves money (larger production run = lower cost/module)
  - Allows resizing launch vehicles to match payloads
- Trick is to optimize number of stages, number of modules/stage to minimize total number of modules
- Generally close to optimum by doubling number of modules at each lower stage
- Have to worry about packing factors, complexity

OTRAG - 1977-1983



Modular Example

- Let's build a launch vehicle out of seven Space Shuttle Solid Rocket Boosters
  - M_{in}=86,180 kg
  - M_{pr}=503,500 kg

$$\varepsilon \equiv \frac{m_{in}}{m_{in} + m_{pr}} = 0.1461 \qquad \sigma \equiv \frac{m_{in}}{m_{pr}} = 0.1711$$

- Look at possible approaches to sequential firing

Modular Sequencing - SRB Example

- Assume no payload
- All seven firing at once - ΔV_{tot}=5138 m/sec
- 3-3-1 sequence - ΔV_{tot}=9087 m/sec
- 4-2-1 sequence - ΔV_{tot}=9175 m/sec
- 2-2-2-1 sequence - ΔV_{tot}=9250 m/sec
- 2-1-1-1-1-1 sequence - ΔV_{tot}=9408 m/sec
- 1-1-1-1-1-1-1 sequence - ΔV_{tot}=9418 m/sec
- Sequence limited by need to balance thrust laterally