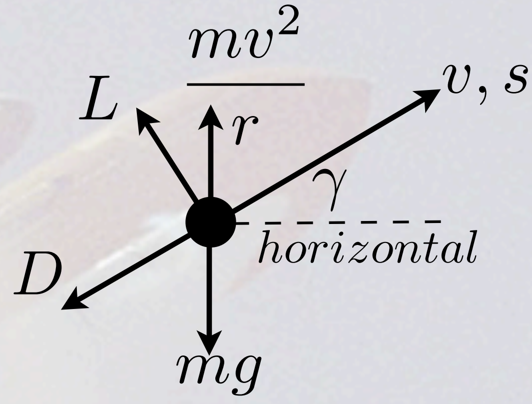
Lifting Entry (continued)

- Basic planar dynamics of motion, again
- Yet another equilibrium glide
- Hypersonic phugoid motion
- Planar state equations

Basic Equations of Motion

From last time,



$$v\frac{d\gamma}{dt} = \frac{L}{m} + \left(\frac{v^2}{r} - g\right)\cos\gamma$$

$$\frac{dv}{dt} = \left(\frac{v^2}{r} - g\right)\sin\gamma - \frac{D}{m}$$

Assume (at entry velocities close to orbital) $g \cong \frac{v^2}{r}$

$$v\frac{d\gamma}{dt} = \frac{L}{m} = \frac{\rho v^2}{2}\frac{c_L A}{m} = \frac{\rho v^2}{2\beta}\frac{L}{D} \qquad (1)$$

$$\frac{dv}{dt} = -\frac{D}{m} = -\frac{\rho v^2}{2\beta} \qquad (2)$$

Solving for Velocity

Divide (2) by (1)

$$\frac{\frac{dv}{dt}}{v\frac{d\gamma}{dt}} = \frac{-\frac{\rho v^2}{2\beta}}{\frac{\rho v^2}{2\beta}\frac{L}{D}} \implies \frac{dv}{v} = -\frac{d\gamma}{L/D} \qquad (3)$$

$$\int_{v_e}^{v} \frac{dv}{v} = -\frac{1}{L/D}\int_{\gamma_e}^{\gamma} d\gamma$$

$$\ln\frac{v}{v_e} = \frac{-(\gamma - \gamma_e)}{L/D} \implies \frac{v}{v_e} = e^{-\frac{\gamma-\gamma_e}{L/D}} \qquad (4)$$

Differential Elements

As before,

$$\frac{dh}{dt} = v \sin\gamma \tag{5}$$

$$\rho = \rho_o e^{-\frac{h}{h_s}} \tag{6}$$

Differentiating (6),

$$\frac{d\rho}{dt} = -\frac{\rho_o}{h_s} e^{-\frac{h}{h_s}} \frac{dh}{dt} = -\frac{\rho}{h_s}\frac{dh}{dt}$$

$$\frac{d\rho}{dt} = -\frac{\rho}{h_s} v \sin\gamma \tag{7}$$

Differential Elements (2)

Solve (7) for v

$$v = \frac{-h_s}{\sin\gamma}\left(\frac{1}{\rho}\frac{d\rho}{dt}\right) \tag{8}$$

Substitute (8) into (1) and rewrite as

$$\frac{d\gamma}{dt} = \frac{\rho v}{2\beta}\frac{L}{D} = \frac{\rho}{2\beta}\frac{L}{D}\left(\frac{-h_s}{\sin\gamma}\right)\frac{1}{\rho}\frac{d\rho}{dt}$$

$$\frac{d\gamma}{dt} = \frac{-h_s}{2\beta\sin\gamma}\frac{L}{D}d\rho \tag{9}$$

Solving for Flight Path Angle

$$\int_{\gamma_e}^{\gamma} \sin\gamma d\gamma = -\frac{h_s}{2\beta}\frac{L}{D}\int_0^{\rho} d\rho \tag{9}$$

$$\cos\gamma - \cos\gamma_e = \frac{h_s}{2\beta}\frac{L}{D}\rho \tag{10}$$

$$\cos\gamma = \frac{h_s}{2\beta}\frac{L}{D}\rho + \cos\gamma_e \tag{11}$$

$$\gamma = -\cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho + \cos\gamma_e\right) \tag{12}$$

Note that the negative sign was inserted because $\cos^{-1}$ is ambiguous as to direction, and the flight path angle on entry should be >0.

Flight Path Angle and Velocity Equations

$$\gamma = -\cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho_o e^{-\frac{h}{h_s}} + \cos\gamma_e\right) \qquad (13)$$

Rewrite (4) as

$$v = v_e\ exp\left(-\frac{\gamma - \gamma_e}{L/D}\right) = v_e\ exp\left(\frac{\gamma_e - \gamma}{L/D}\right)$$

and substitute into (13)

$$v = v_e\ exp\left\{\frac{1}{L/D}\left[\gamma_e + \cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho_o e^{-\frac{h}{h_s}} + \cos\gamma_e\right)\right]\right\} \qquad (14)$$

Deceleration

Along the flight path, $\left|\frac{D}{m}\right| = \frac{\rho v^2}{2\beta}$

Perpendicular to the flight path, $\left|\frac{L}{m}\right| = \frac{\rho v^2}{2\beta}\frac{L}{D}$

Total deceleration (*not* in g's)

$$n = \sqrt{\left(\frac{D}{m}\right)^2 + \left(\frac{L}{m}\right)^2} = \frac{1}{m}\sqrt{D^2 + L^2} = \frac{\rho v^2}{2\beta}\sqrt{1 + \left(\frac{L}{D}\right)^2}$$

$$\boxed{n = \frac{\rho_o v^2}{2\beta}\sqrt{1 + \left(\frac{L}{D}\right)^2} e^{-\frac{h}{h_s}}} \qquad (15)$$

Fiddling with Algebra

$$n = \frac{\rho_o}{2\beta}\sqrt{1+\left(\frac{L}{D}\right)^2} e^{-\frac{h}{h_s}} v^2$$

Substitute in (14)

$$n = \frac{\rho_o v_e^2}{2\beta}\sqrt{1+\left(\frac{L}{D}\right)^2} exp\left\{\frac{2}{L/D}\left[\gamma_e + \cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho_o e^{-\frac{h}{h_s}} + \cos\gamma_e\right)\right]\right\}$$

$$X \equiv \left[\gamma_e + \cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho_o e^{-\frac{h}{h_s}} + \cos\gamma_e\right)\right] \quad (16)$$

$$n = \frac{\rho_o v_e^2}{2\beta}\sqrt{1+\left(\frac{L}{D}\right)^2} e^{\frac{2X}{L/D}} \quad (17)$$

More Algebra

Set $\frac{dn}{dh} = 0$

$$0 = \frac{\rho_o v_e^2}{2\beta}\sqrt{1+\left(\frac{L}{D}\right)^2}\left[e^{-\frac{h}{h_s}}e^{\frac{2X}{L/D}}\frac{2}{L/D}\frac{dX}{dh}+\left(\frac{-1}{h_s}\right)e^{-\frac{h}{h_s}}e^{\frac{2X}{L/D}}\right] \quad (18)$$

Factoring out common terms,

$$\frac{1}{h_s} = \frac{2}{L/D}\frac{dX_m}{dh} \quad (19)$$

Even More Algebra

From (13),

$$\gamma = -\cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho_o e^{-\frac{h}{h_s}} + \cos\gamma_e\right)$$

$$X = \gamma_e - \gamma = -\cos^{-1}(\cos\gamma) + \gamma_e$$

$$Y \equiv \cos\gamma$$

$$X = \gamma_e - \cos^{-1} Y \qquad (20)$$

Trigonometry, for a Change

Trig identity -

$$\frac{d(\cos^{-1} u)}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \tag{21}$$

$$\frac{dX}{dh} = -\frac{d(\cos^{-1} Y)}{dh} = \frac{1}{\sqrt{1-Y^2}}\frac{dY}{dh}$$

$$\frac{dX}{dh} = \frac{1}{\sqrt{1-\cos^2\gamma}}\frac{d(\cos\gamma)}{dh}$$

Back to the Algebra

From (13),

$$\cos\gamma = \frac{h_s}{2\beta}\frac{L}{D}\rho_o e^{-\frac{h}{h_s}} + \cos\gamma_e$$

$$\frac{dX}{dh} = \frac{1}{\sqrt{1-\cos^2\gamma}}\frac{d}{dh}\left(\frac{h_s\rho_o}{2\beta}\frac{L}{D}e^{-\frac{h}{h_s}} + \cos\gamma_e\right) \quad (22)$$

$$\frac{dX}{dh} = \frac{1}{\sin\gamma}\left[\frac{h_s\rho_o}{2\beta}\frac{L}{D}\left(\frac{-1}{h_s}\right)e^{-\frac{h}{h_s}}\right]$$

$$\frac{dX}{dh} = \frac{-\rho_o}{2\beta\sin\gamma}\frac{L}{D}e^{-\frac{h}{h_s}} \quad (23)$$

Maximum Deceleration Case

If we go back to the n_{max} case, (19) gives

$$\frac{1}{h_s} = \frac{2}{L/D}\frac{dX_m}{dh}$$

$$\frac{1}{h_s} = \frac{2}{L/D}\frac{-\rho_o}{2\beta\sin\gamma_m}\frac{L}{D}e^{-\frac{h_m}{h_s}}$$

h_m, γ_m are values at n_{max}

$$\frac{1}{h_s} = \frac{-\rho_o}{\beta\sin\gamma_m}\frac{L}{D}e^{-\frac{h_m}{h_s}} \qquad (24)$$

$$\sin\gamma_m = \frac{-\rho_o h_s}{\beta}\frac{L}{D}e^{-\frac{h_m}{h_s}} \qquad (25)$$

Let's Go Back to Algebra

$$\text{Let } \Phi \equiv -\frac{\rho_o h_s}{2\beta}\frac{L}{D}$$

$$\cos\gamma_m = \Phi e^{-\frac{h_m}{h_s}} + \cos\gamma_e$$

$$\sin\gamma_m = \Phi e^{-\frac{h_m}{h_s}}$$

$$\text{Let } H \equiv e^{-\frac{h_m}{h_s}}$$

$$\left(\Phi e^{-\frac{h_m}{h_s}} + \cos\gamma_e\right)^2 + \left(\Phi e^{-\frac{h_m}{h_s}}\right)^2 = 1$$

$$(\Phi H + \cos\gamma_e)^2 + (\Phi H)^2 = 1$$

Algebra is Fun, Don't You Think?

$$2\Phi^2 H^2 + 2\Phi H \cos\gamma_e + \cos^2\gamma_e - 1 = 0$$

$$2\Phi^2 H^2 + 2\Phi H \cos\gamma_e - \sin^2\gamma_e = 0$$

$$H = \frac{-2\Phi\cos\gamma_e \pm \sqrt{4\Phi^2\cos^2\gamma_e + 4(2\Phi^2)\sin^2\gamma_e}}{4\Phi^2}$$

$$H = -\frac{1}{2\Phi}\left(\cos\gamma_e \pm \sqrt{\cos^2\gamma_e + 2\sin^2\gamma_e}\right)$$

$$H = -\frac{1}{2\Phi}\left(\cos\gamma_e \pm \sqrt{1 + \sin^2\gamma_e}\right)$$

Maximum Deceleration Equations

Skipping some painful algebra and trig,

$$h_m = h_s \ln \left\{ \frac{-\rho_o h_s}{2\beta \sin \gamma_e} \left[\sqrt{4 + \left(\frac{L}{D}\right)^2 \csc^2 \gamma_e} - \frac{L}{D} \cot \gamma_e \right] \right\}$$

$$\cos \gamma_m = \cos \gamma_e - \frac{(L/D) \sin \gamma_e}{\sqrt{4 + (L/D)^2 \csc^2 \gamma_e} - (L/D) \cot \gamma_e}$$

$$v_m = v_e e^{-\frac{\gamma_m - \gamma_e}{L/D}}$$

$$n_{max} = \frac{\rho_o v_e^2}{2\beta} \sqrt{1 + \left(\frac{L}{D}\right)^2} v_m^2$$

Phugoid Oscillations

Assume a shallow, linear entry trajectory:

$$\gamma \ll 1 \qquad D \approx 0 \qquad \dot{\gamma}_o \approx 0$$

$$\dot{h} = v \sin\gamma \cong v\gamma$$

$$\frac{v\dot{\gamma}}{g} = \frac{L}{mg} - \left(1 - \frac{v^2}{v_c^2}\right)$$

Small perturbations $\Longrightarrow \quad h = h_1 + \Delta h$

$$\gamma = \gamma_1 + \Delta\gamma \approx \Delta\gamma$$

Perturbation Analysis

$$\dot{h} = v\gamma$$

$$\dot{h} = \dot{h}_1 + \Delta\dot{h} = \cancel{\gamma_1} v_1 + \Delta\dot{h} = \Delta\dot{h}$$

$$\Delta\dot{h} = (v_1 + \Delta v_1)(\cancel{\gamma_1} + \Delta\gamma) = v_1\Delta\gamma + \Delta \cancel{v_1}\Delta\gamma$$

$$\boxed{\Delta\dot{h} = v_1\Delta\gamma}$$

$$\text{Neglecting drag} \implies v \cong \text{constant}$$

$$\frac{L}{m} = \frac{L_1 + \Delta L}{m} = \frac{v^2 A c_L}{2m}\rho_o e^{-\frac{h_1+\Delta h}{h_s}}$$

$$= \frac{v^2 A c_L}{2m}\rho_o e^{-\frac{h_1}{h_s}} e^{-\frac{\Delta h}{h_s}} = \frac{L_1}{m} e^{-\frac{\Delta h}{h_s}}$$

Perturbation Analysis

Using Taylor's series expansion,

$$\frac{L}{m} \cong \frac{L_1}{m}\left(1 - \frac{\Delta h}{h_s}\right)$$

$$\frac{\Delta L}{m} = -\frac{L_1}{m}\frac{\Delta h}{h_s}$$

$$\frac{v_1}{g}\Delta\dot{\gamma} = \frac{\Delta L}{mg} + \left[\frac{L_1}{mg} - \left(1 - \frac{v_1^2}{v_c^2}\right)\right]$$

$$v_1 \Delta\dot{\gamma} = \frac{\Delta L}{m} = \Delta\ddot{h}$$

Perturbed Lift

On an equilibrium glide,

$$\dot{\gamma} = 0 \Longrightarrow \frac{L}{m} = g - \frac{v^2}{r}$$

$$\frac{L_1}{mg} = 1 - \frac{v_1^2}{gr} = 1 - \frac{v_1^2}{v_c^2}$$

$$\Delta\ddot{h} = \frac{\Delta L}{m} = -\frac{L_1}{m}\frac{\Delta h}{h_s} = -\left(1 - \frac{v_1^2}{v_c^2}\right)\frac{g\Delta h}{h_s}$$

$$\Delta\ddot{h} + \left(1 - \frac{v_1^2}{v_c^2}\right)\frac{g}{h_s}\Delta h = 0$$

$\Longleftarrow$ Simple harmonic motion (undamped)

Phugoid Parameters

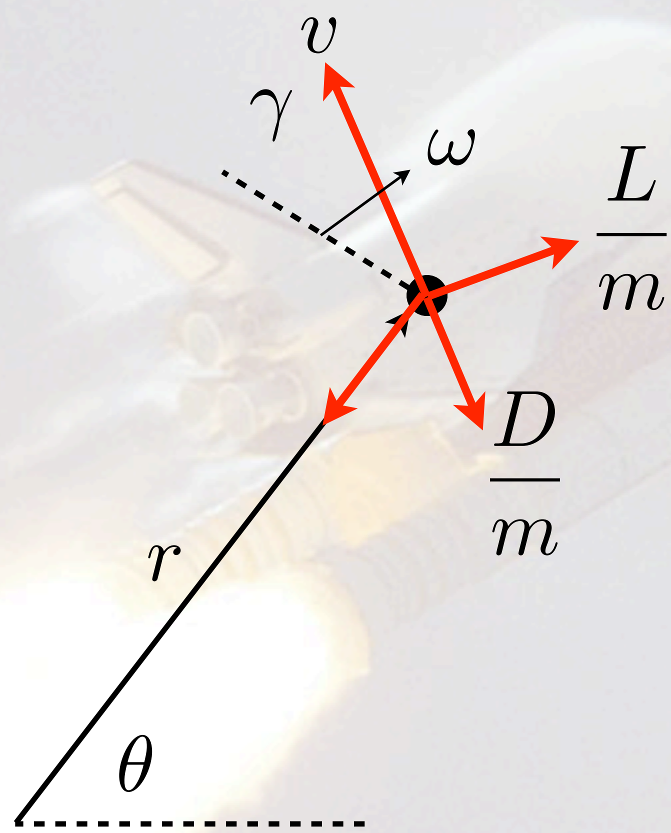
Frequency: $\omega^2 = \left(1 - \frac{v_1^2}{v_c^2}\right)\frac{g}{h_s}$

Period: $P = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\left(1 - \frac{v_1^2}{v_c^2}\right)\frac{g}{h_s}}} \cong \frac{169\ sec}{\sqrt{1 - \frac{v_1^2}{v_c^2}}}$

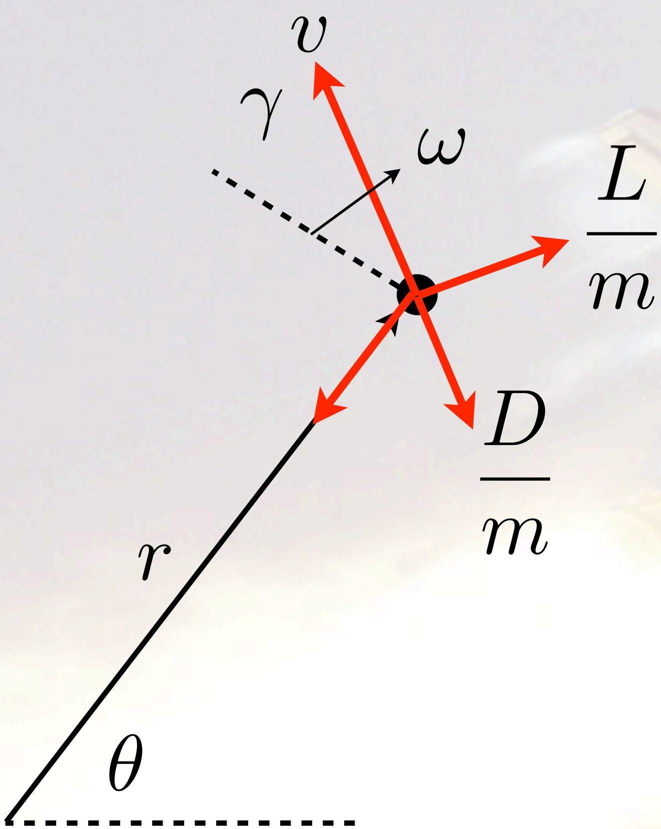
$v_c \sim 8000\ \frac{m}{sec} \Longrightarrow$

$v_1\ \left(\frac{m}{sec}\right)$	P
7750	11m21s
6000	4m15s
4000	3m15s
2000	2m55s

Free-Body Diagram with Spherical Planet



Planar State Equations



$$\omega = \dot{\gamma} - \dot{\theta}$$

Sum of transverse accelerations

$$\frac{L}{m} - g\cos\gamma = \omega v$$

Sum of parallel accelerations

$$-\frac{D}{m} - g\sin\gamma = \dot{v}$$

Planar State Equations (2)

$$\dot{r} = \dot{h} = v \sin\gamma$$

$$r\dot{\theta} = v \cos\gamma$$

$$\omega = \dot{\gamma} - \dot{\theta} = \dot{\gamma} - \frac{v}{r}\cos\gamma$$

$$\frac{L}{m} - g\cos\gamma = \left(\dot{\gamma} - \frac{v}{r}\cos\gamma\right)v$$

$$\frac{L}{m} - \left(g - \frac{v^2}{r}\right)\cos\gamma = \dot{\gamma}v$$

$$\frac{L}{m} - \left(1 - \frac{v^2}{rg}\right)g\cos\gamma = \dot{\gamma}v$$

The Canonical Planar State Equations

$$v\dot{\gamma} = \frac{L}{m} - \left(1 - \frac{v^2}{v_c^2}\right) g \cos\gamma$$

$$\dot{v} = -\frac{D}{m} - g \sin\gamma$$

$$\dot{r} = \dot{h} = v \sin\gamma$$

$$r\dot{\theta} = v \cos\gamma$$

Coupled first-order ODEs

Associated Parameters to State Eqns

$$\frac{L}{m} = \frac{1}{2}\frac{\rho v^2 A c_L}{m} = \frac{\rho v^2}{2}\frac{A c_D}{m}\frac{c_L}{c_D} = \frac{\rho v^2}{2\beta}\frac{L}{D}$$

$$\frac{D}{m} = \frac{1}{2}\frac{\rho v^2 A c_D}{m} = \frac{\rho v^2}{2\beta}$$

$$\rho = \rho_o e^{-\frac{h}{h_s}}$$

$$h = r - r_o$$

$$g = g_o \left(\frac{r_o}{r}\right)^2$$

Numerical Integration - 4th Order R-K

Given a series of equations $\dot{\bar{y}} = \bar{f}(t, \bar{x})$

$$\bar{k_1} = \Delta t \; \bar{f}\left(t_n, \bar{y_n}\right)$$

$$\bar{k_2} = \Delta t \; \bar{f}\left(t_n + \frac{\Delta t}{2}, \bar{y_n} + \frac{\bar{k_1}}{2}\right)$$

$$\bar{k_3} = \Delta t \; \bar{f}\left(t_n + \frac{\Delta t}{2}, \bar{y_n} + \frac{\bar{k_2}}{2}\right)$$

$$\bar{k_4} = \Delta t \; \bar{f}\left(t_n + \Delta t, \bar{y_n} + \bar{k_3}\right)$$

$$\bar{y_{n+1}} = \bar{y_n} + \frac{\bar{k_1}}{6} + \frac{\bar{k_2}}{3} + \frac{\bar{k_3}}{3} + \frac{\bar{k_4}}{6} + O(\Delta t^5)$$