

Rocket Performance

- The rest of orbital mechanics
- The rocket equation
- Mass ratio and performance
- Structural and payload mass fractions
- Regression analysis
- Multistaging
- Optimal ΔV distribution between stages
- Trade-off ratios
- Parallel staging
- Modular staging

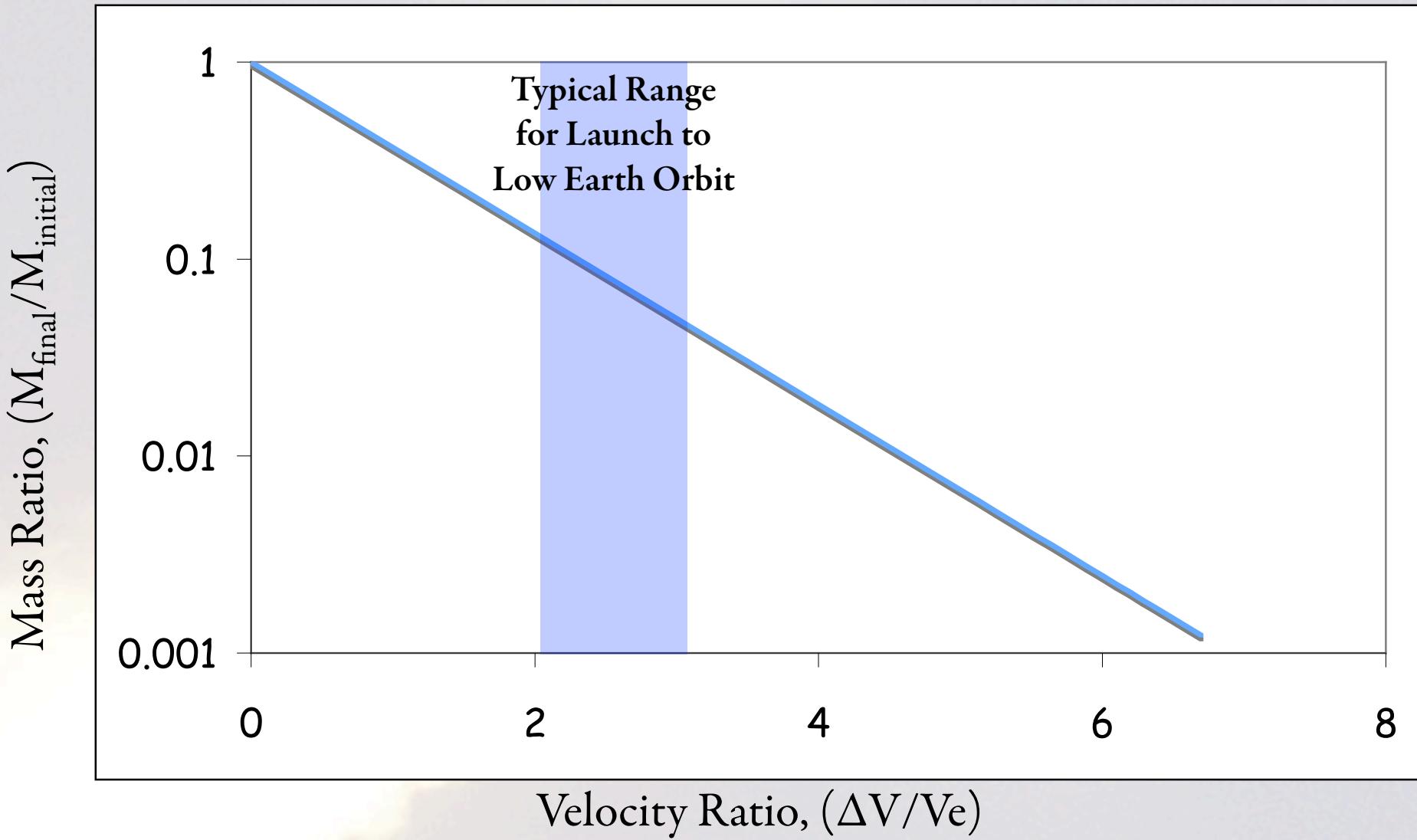
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Rocket Performance
ENAE 791 - Launch and Entry Vehicle Design

Rocket Equation (First Look)



Sources and Categories of Vehicle Mass



**Payload
Propellants
Structure
Propulsion
Avionics
Power
Mechanisms
Thermal
Etc.**



Sources and Categories of Vehicle Mass



**Payload
Propellants
Inert Mass
Structure
Propulsion
Avionics
Power
Mechanisms
Thermal
Etc.**



Basic Vehicle Parameters

- Basic mass summary

$$m_o = m_{pl} + m_{pr} + m_{in}$$

- Inert mass fraction

$$\delta \equiv \frac{m_{in}}{m_o} = \frac{m_{in}}{m_{pl} + m_{pr} + m_{in}}$$

- Payload fraction

$$\lambda \equiv \frac{m_{pl}}{m_o} = \frac{m_{pl}}{m_{pl} + m_{pr} + m_{in}}$$

- Parametric mass ratio

$$r = \lambda + \delta$$

m_o =initial mass

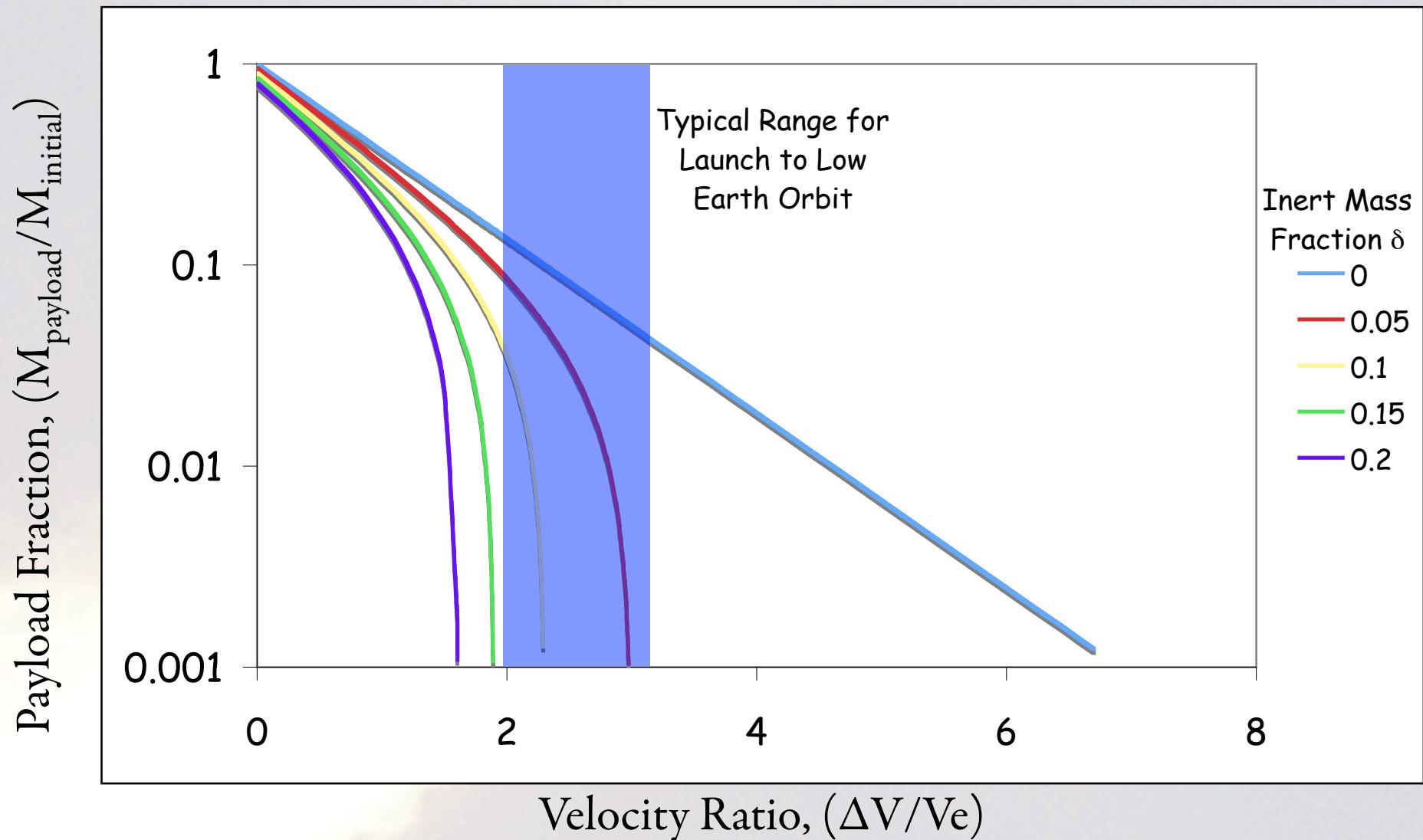
m_{pl} =payload mass

m_{pr} =propellant mass

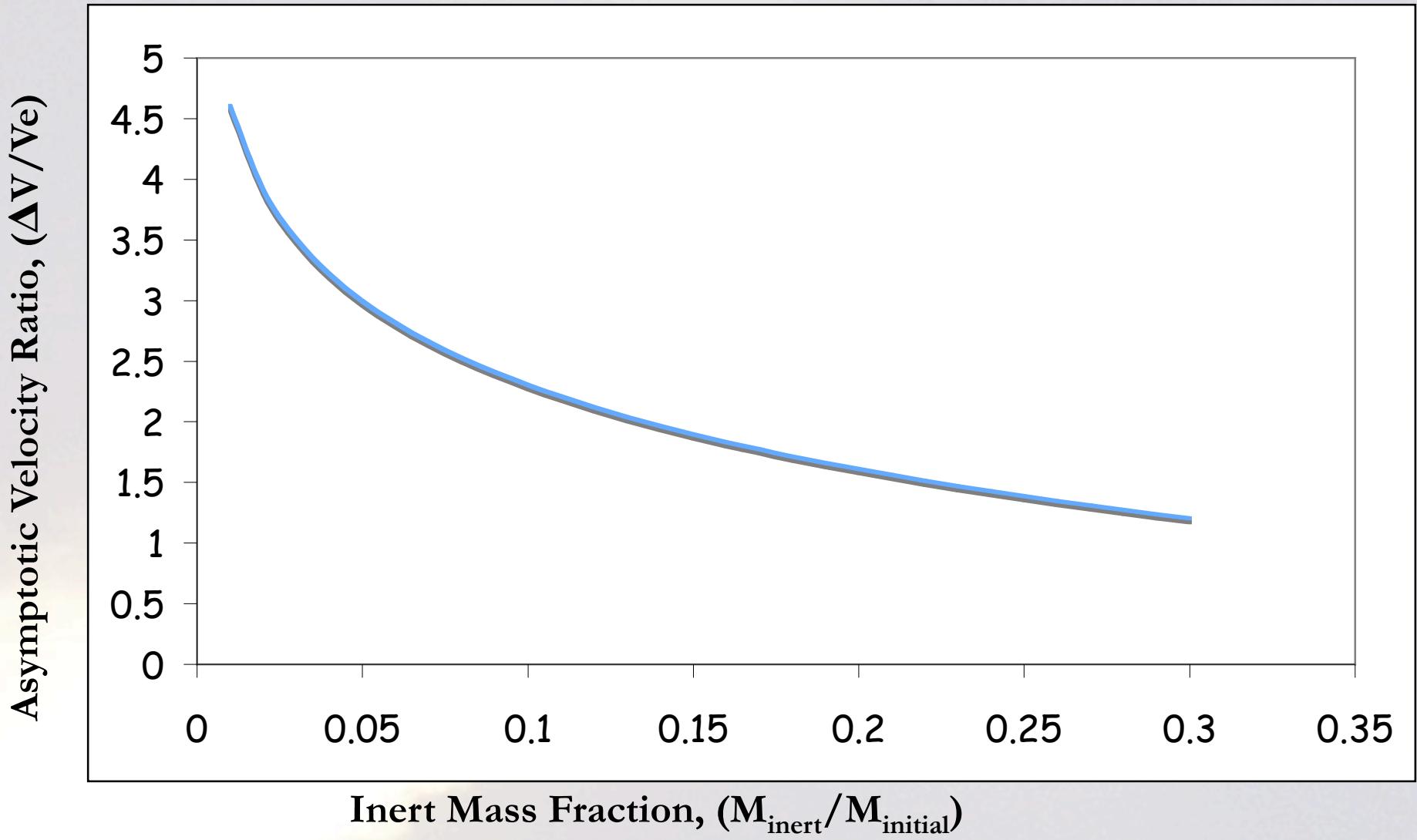
m_{in} =inert mass



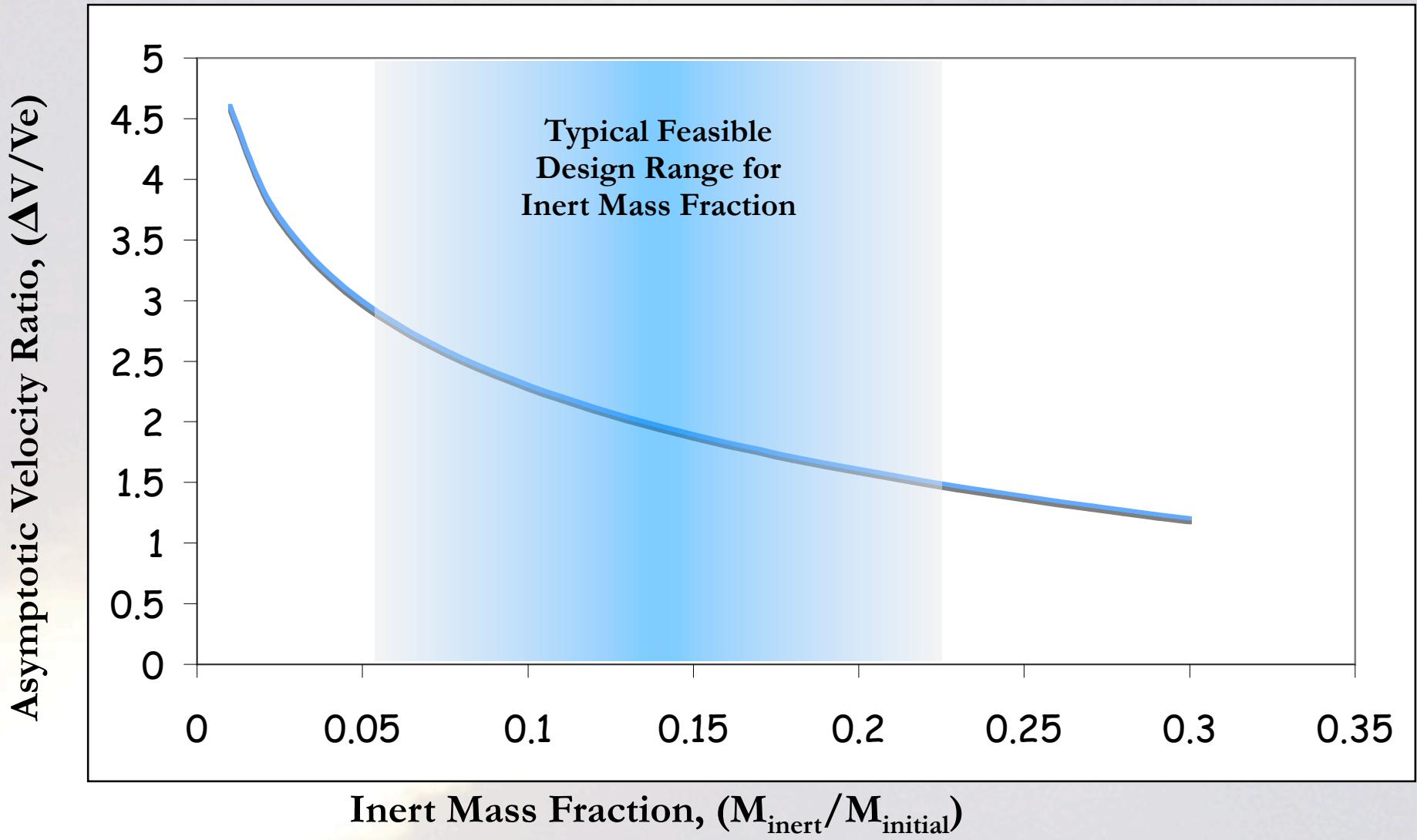
Rocket Equation (including Inert Mass)



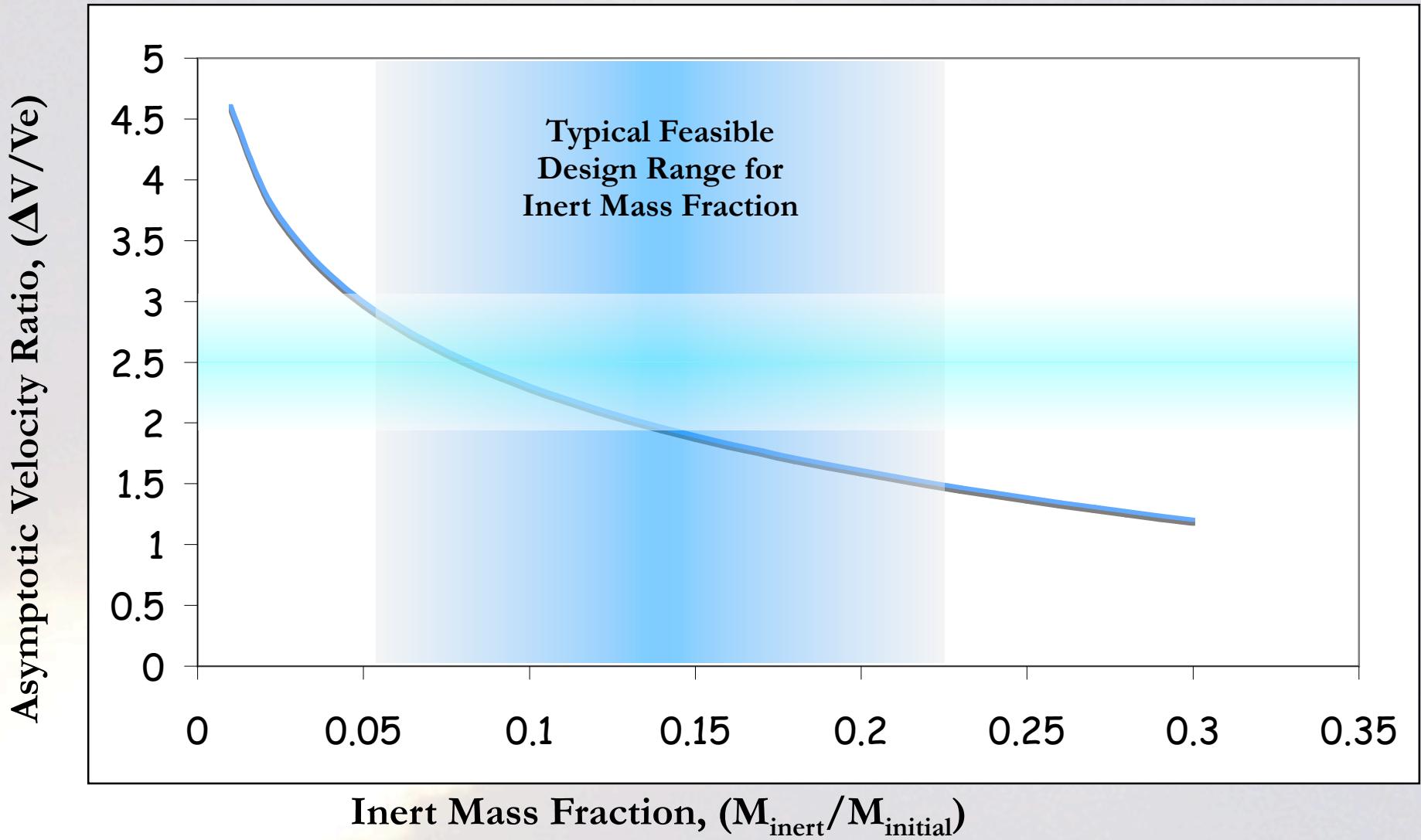
Limiting Performance Based on Inert Mass



Limiting Performance Based on Inert Mass



Limiting Performance Based on Inert Mass

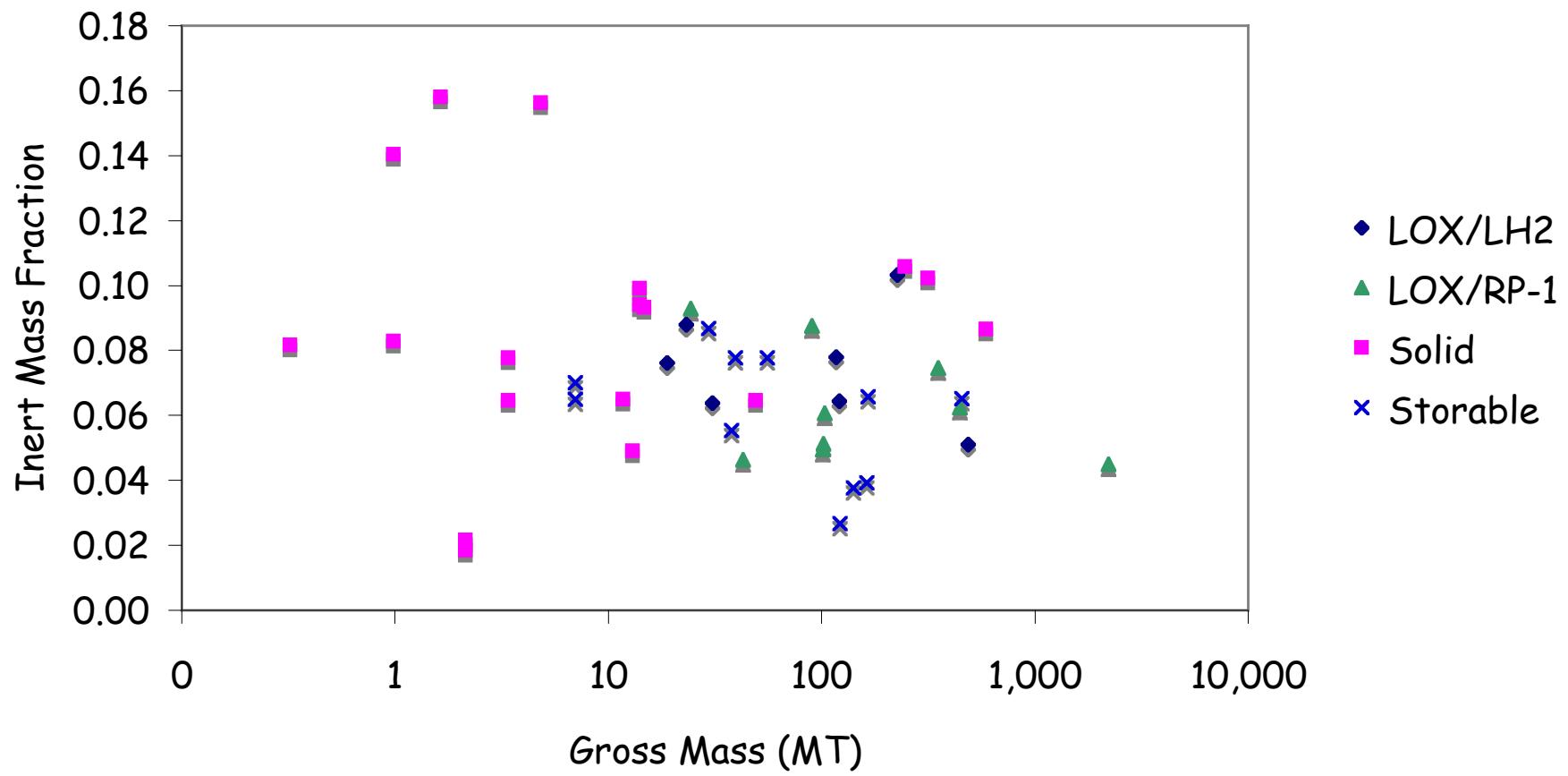


Regression Analysis of Existing Vehicles

Veh/Stage	prop mass (lbs)	gross mass (lbs)	Type	Propellants	lsp vac (sec)	isp sl (sec)	sigma	eps	delta
Delta 6925 Stage 2	13,367	15,394	Storage tank	N204-A50	319.4		0.152	0.132	0.070
Delta 7925 Stage 2	13,367	15,394	Storage tank	N204-A50	319.4		0.152	0.132	0.065
Titan II Stage 2	59,000	65,000	Storage tank	N204-A50	316.0		0.102	0.092	0.087
Titan III Stage 2	77,200	83,600	Storage tank	N204-A50	316.0		0.083	0.077	0.055
Titan IV Stage 2	77,200	87,000	Storage tank	N204-A50	316.0		0.127	0.113	0.078
Proton Stage 3	110,000	123,000	Storage tank	N204-A50	315.0		0.118	0.106	0.078
Titan II Stage 1	260,000	269,000	Storage tank	N204-A50	296.0		0.035	0.033	0.027
Titan III Stage 1	294,000	310,000	Storage tank	N204-A50	302.0		0.054	0.052	0.038
Titan IV Stage 1	340,000	359,000	Storage tank	N204-A50	302.0		0.056	0.053	0.039
Proton Stage 2	330,000	365,000	Storage tank	N204-A50	316.0		0.106	0.096	0.066
Proton Stage 1	904,000	1,004,000	Storage tank	N204-A50	316.0	285.0	0.111	0.100	0.065
average					312.2	285.0	0.100	0.089	0.061
standard deviation					8.1		0.039	0.033	0.019



Inert Mass Fraction Data for Existing LVs



Regression Analysis

- Given a set of N data points (x_i, y_i)
- Linear curve fit: $y = Ax + B$
 - find A and B to minimize sum squared error
$$\text{error} = \sum_{i=1}^N (Ax_i + B - y_i)^2$$
 - Analytical solutions exist, or use Solver in Excel
- Power law fit: $y = Bx^A$
$$\text{error} = \sum_{i=1}^N [A \log(x_i) + B - \log(y_i)]^2$$
- Polynomial, exponential, many other fits possible



Solution of Least-Squares Linear Regression

$$\frac{\partial(\text{error})}{\partial A} = 2 \sum_{i=1}^N (Ax_i + B - y_i)x_i = 0$$

$$\frac{\partial(\text{error})}{\partial B} = 2 \sum_{i=1}^N (Ax_i + B - y_i) = 0$$

$$A \sum_{i=1}^N x_i^2 + B \sum_{i=1}^N x_i - \sum_{i=1}^N x_i y_i = 0$$

$$A \sum_{i=1}^N x_i + NB - \sum_{i=1}^N y_i = 0$$

$$A = \frac{N \sum x_i y_i - \sum x_i \sum y_i}{N \sum x_i^2 - (\sum x_i)^2}$$

$$B = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{N \sum x_i^2 - (\sum x_i)^2}$$

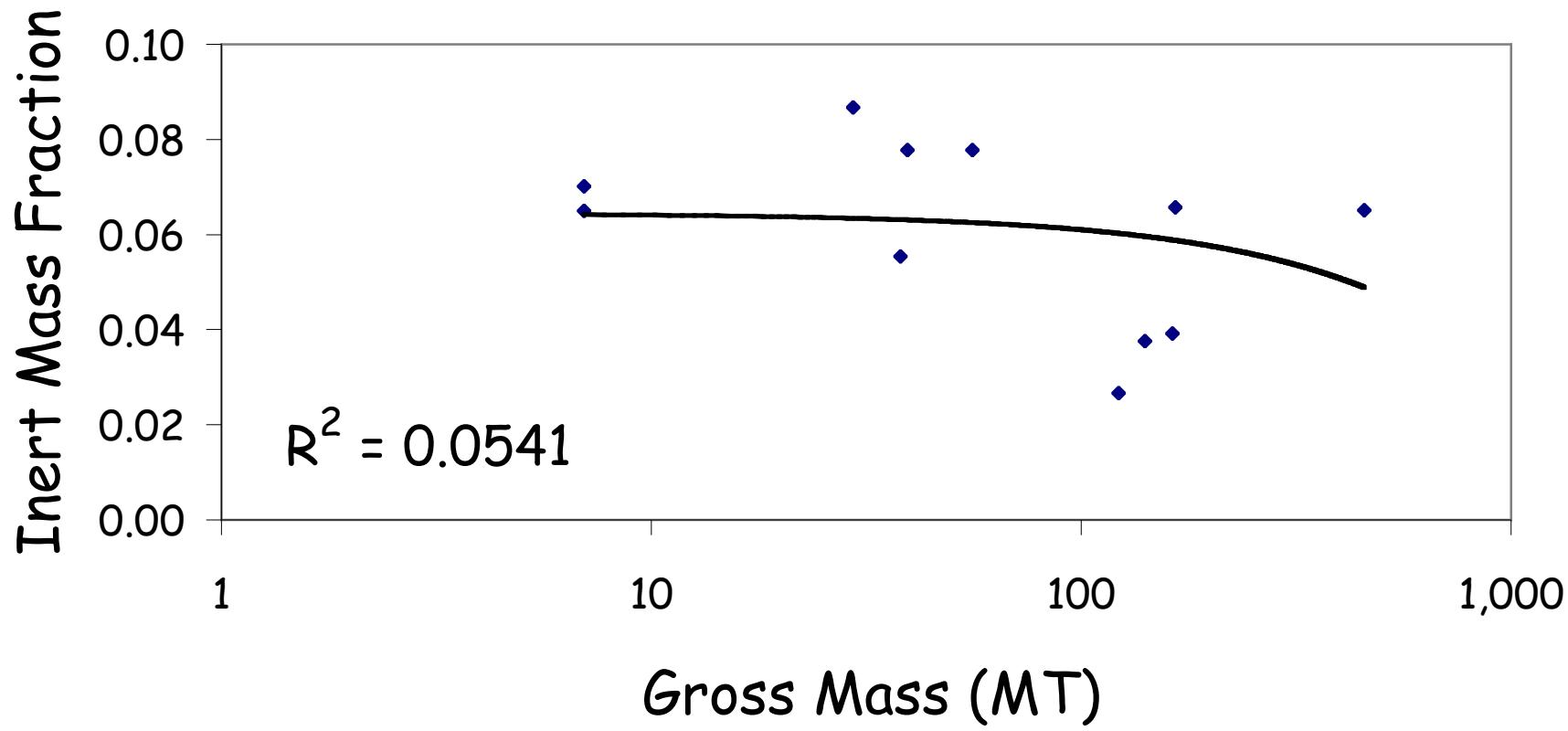


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Rocket Performance
ENAE 791 - Launch and Entry Vehicle Design

Regression Analysis - Storables

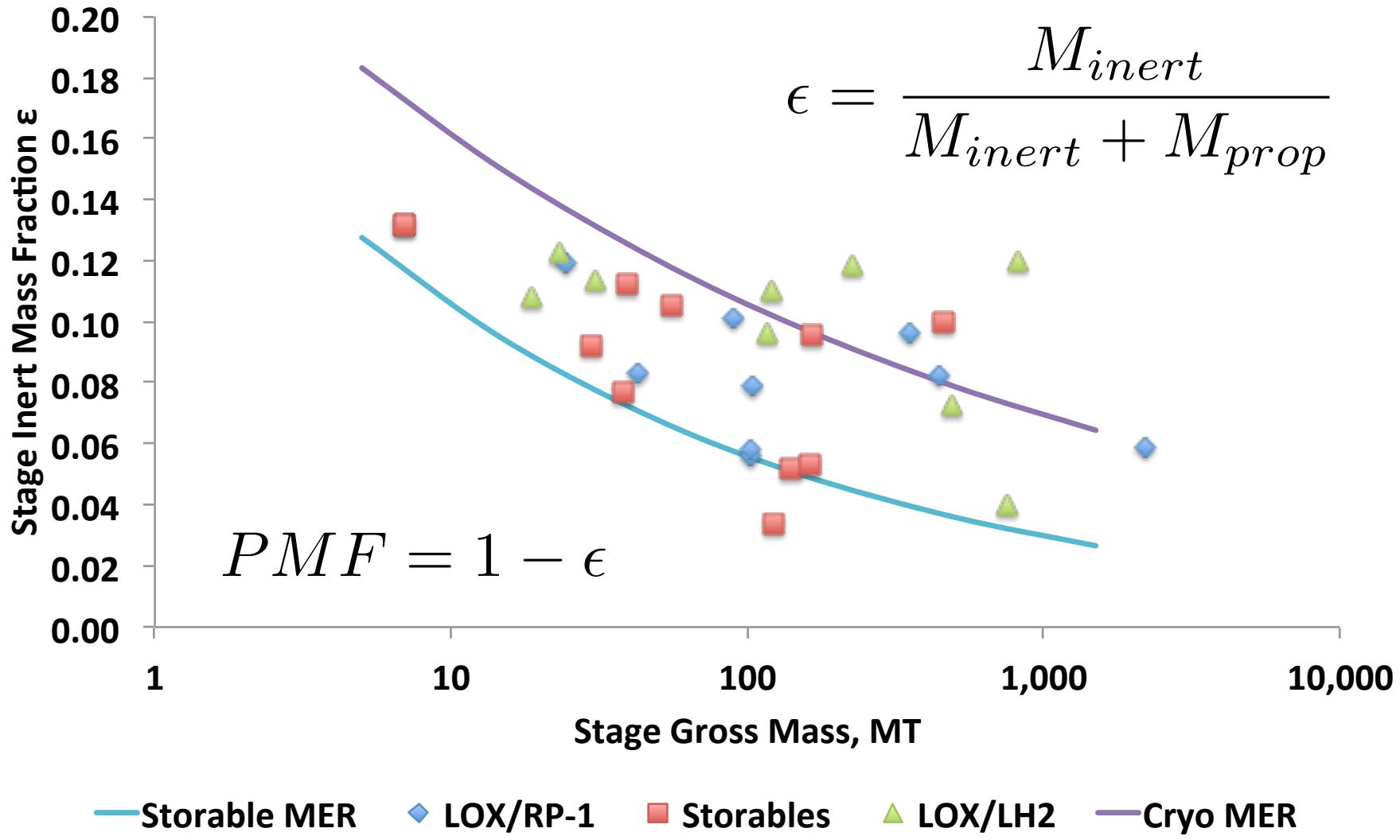


Regression Values for Design Parameters

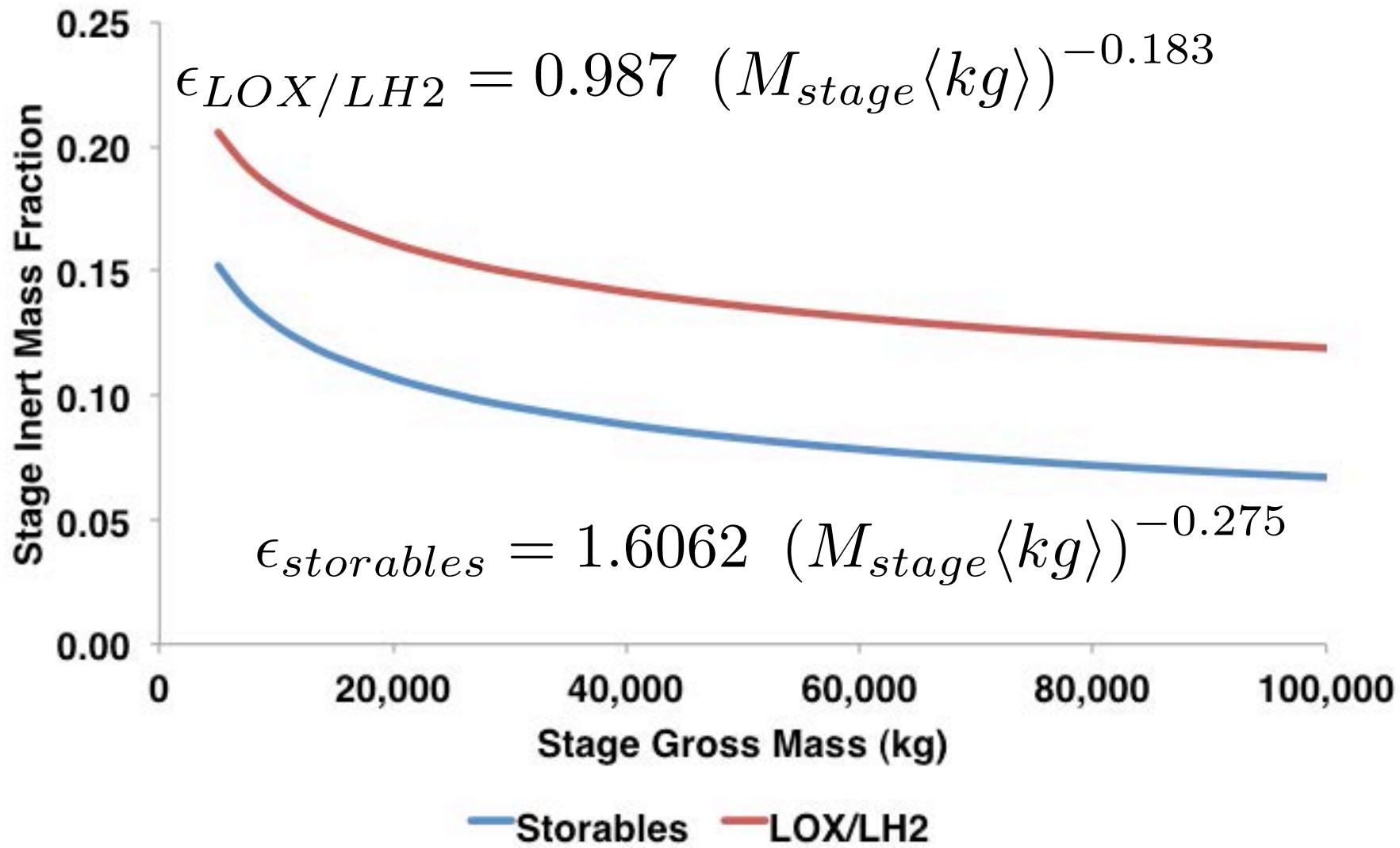
	Vacuum Ve (m/sec)	Inert Mass Fraction	Max ΔV (m/sec)
LOX/LH2	4273	0.075	11,070
LOX/RP-1	3136	0.063	8664
Storables	3058	0.061	8575
Solids	2773	0.087	6783



Economy of Scale for Stage Size



Stage Inert Mass Fraction Estimation



The Rocket Equation for Multiple Stages

- Assume two stages

$$\Delta V_1 = -V_{e1} \ln \left(\frac{m_{final1}}{m_{initial1}} \right) = -V_{e1} \ln(r_1)$$

$$\Delta V_2 = -V_{e2} \ln \left(\frac{m_{final2}}{m_{initial2}} \right) = -V_{e2} \ln(r_2)$$

- Assume $V_{e1}=V_{e2}=V_e$

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1) - V_e \ln(r_2) = -V_e \ln(r_1 r_2)$$



Continued Look at Multistaging

- There's a historical tendency to define $r_0 = r_1 r_2$

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1 r_2) = -V_e \ln(r_0)$$

- But it's important to remember that it's really

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1 r_2) = -V_e \ln\left(\frac{m_{final1}}{m_{initial1}} \frac{m_{final2}}{m_{initial2}}\right)$$

- And that r_0 has *no* physical significance, since

$$m_{final1} \neq m_{initial2} \Rightarrow r_0 \neq \frac{m_{final2}}{m_{initial1}}$$



Multistage Inert Mass Fraction

- Total inert mass fraction

$$\delta_0 = \frac{m_{in,1} + m_{in,2} + m_{in,3}}{m_0} = \frac{m_{in,1}}{m_0} + \frac{m_{in,2}}{m_0} + \frac{m_{in,3}}{m_0}$$

$$\delta_0 = \frac{m_{in,1}}{m_0} + \frac{m_{in,2}}{m_{0,2}} \frac{m_{0,2}}{m_0} + \frac{m_{in,3}}{m_{0,3}} \frac{m_{0,3}}{m_{0,2}} \frac{m_{0,2}}{m_0}$$

- Convert to dimensionless parameters

$$\delta_0 = \delta_1 + \delta_2 \lambda_1 + \delta_3 \lambda_2 \lambda_1$$

- General form of the equation

$$\delta_0 = \sum_{j=1}^{n \text{ stages}} [\delta_j \prod_{\ell=1}^{j-1} \lambda_\ell]$$



Multistage Payload Fraction

- Total payload fraction (3 stage example)

$$\lambda_0 = \frac{m_{pl}}{m_0} = \frac{m_{pl}}{m_{0,3}} \frac{m_{0,3}}{m_{0,2}} \frac{m_{0,2}}{m_0}$$

- Convert to dimensionless parameters

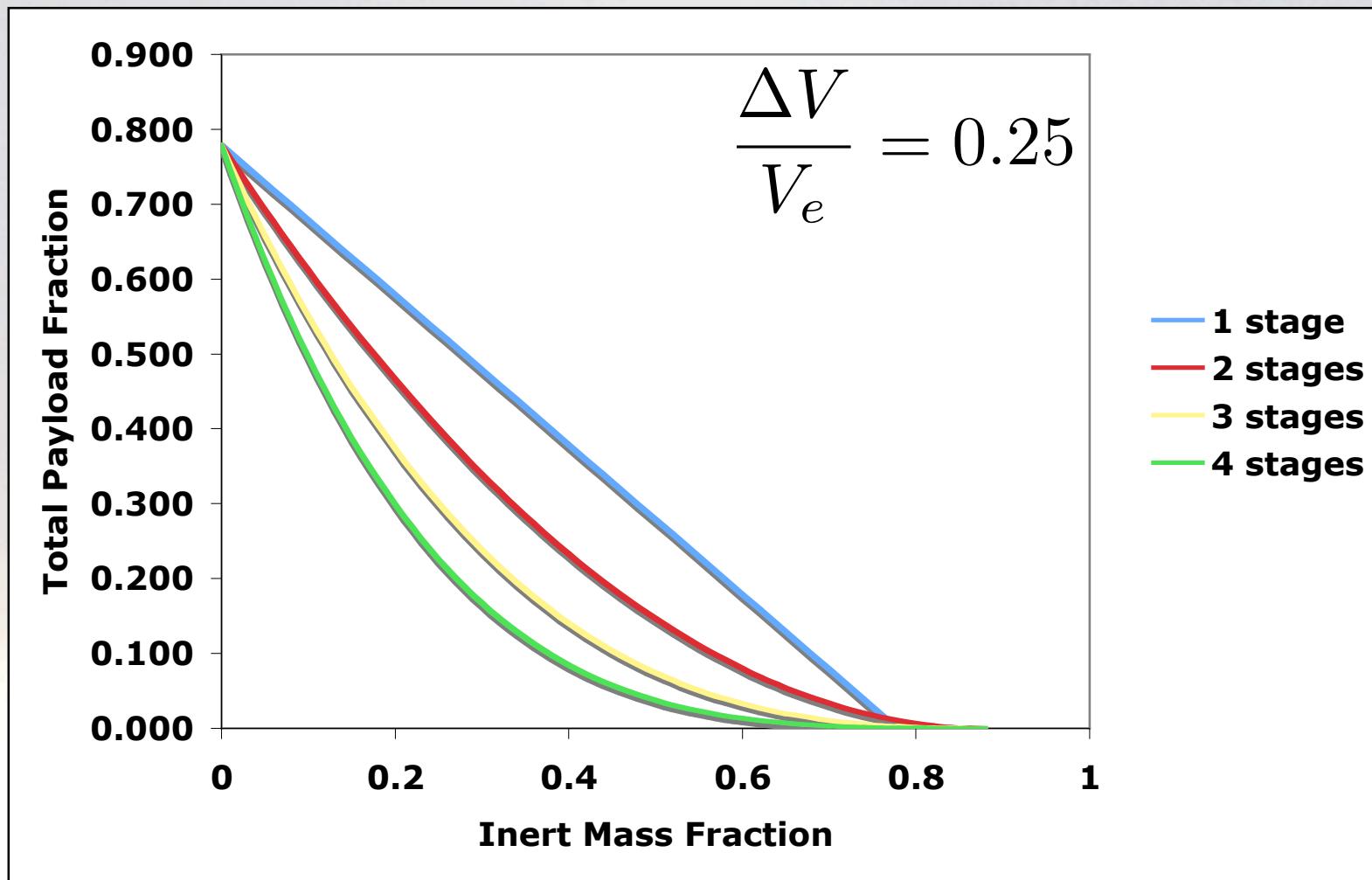
$$\lambda_0 = \lambda_3 \lambda_2 \lambda_1$$

- Generic form of the equation

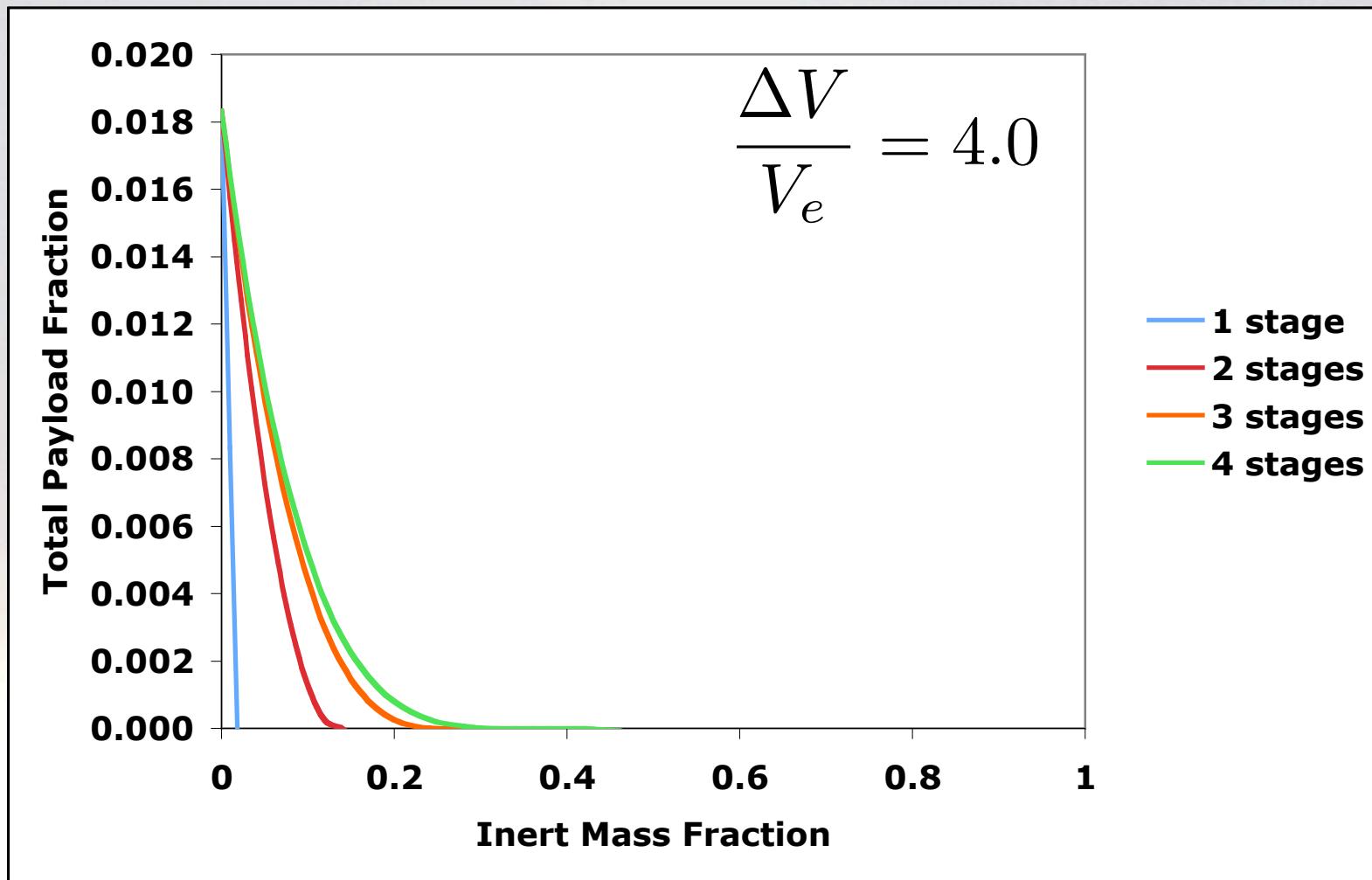
$$\lambda_0 = \prod_{j=1}^{n \text{ stages}} \lambda_j$$



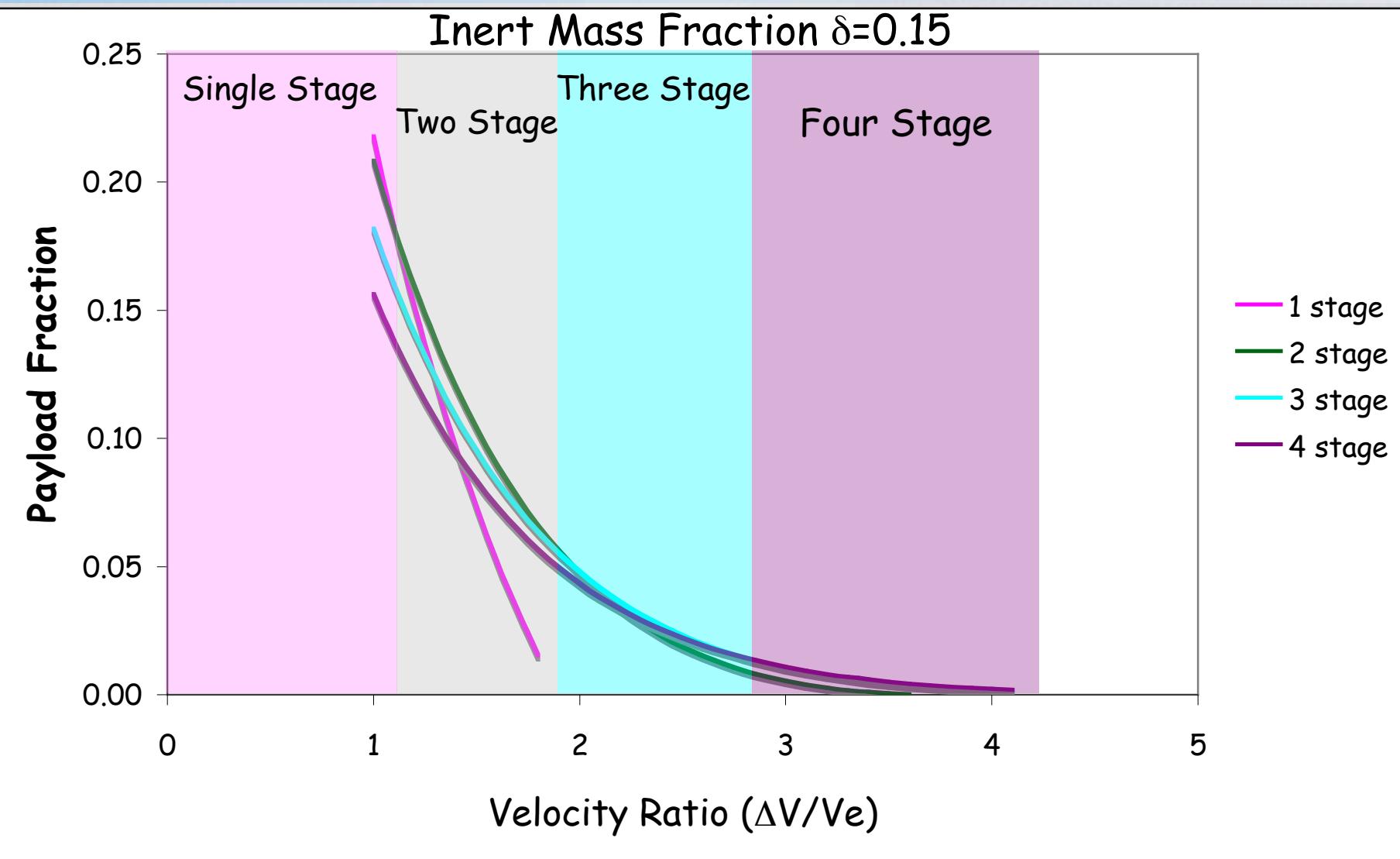
Effect of δ and $\Delta V/V_e$ on Payload



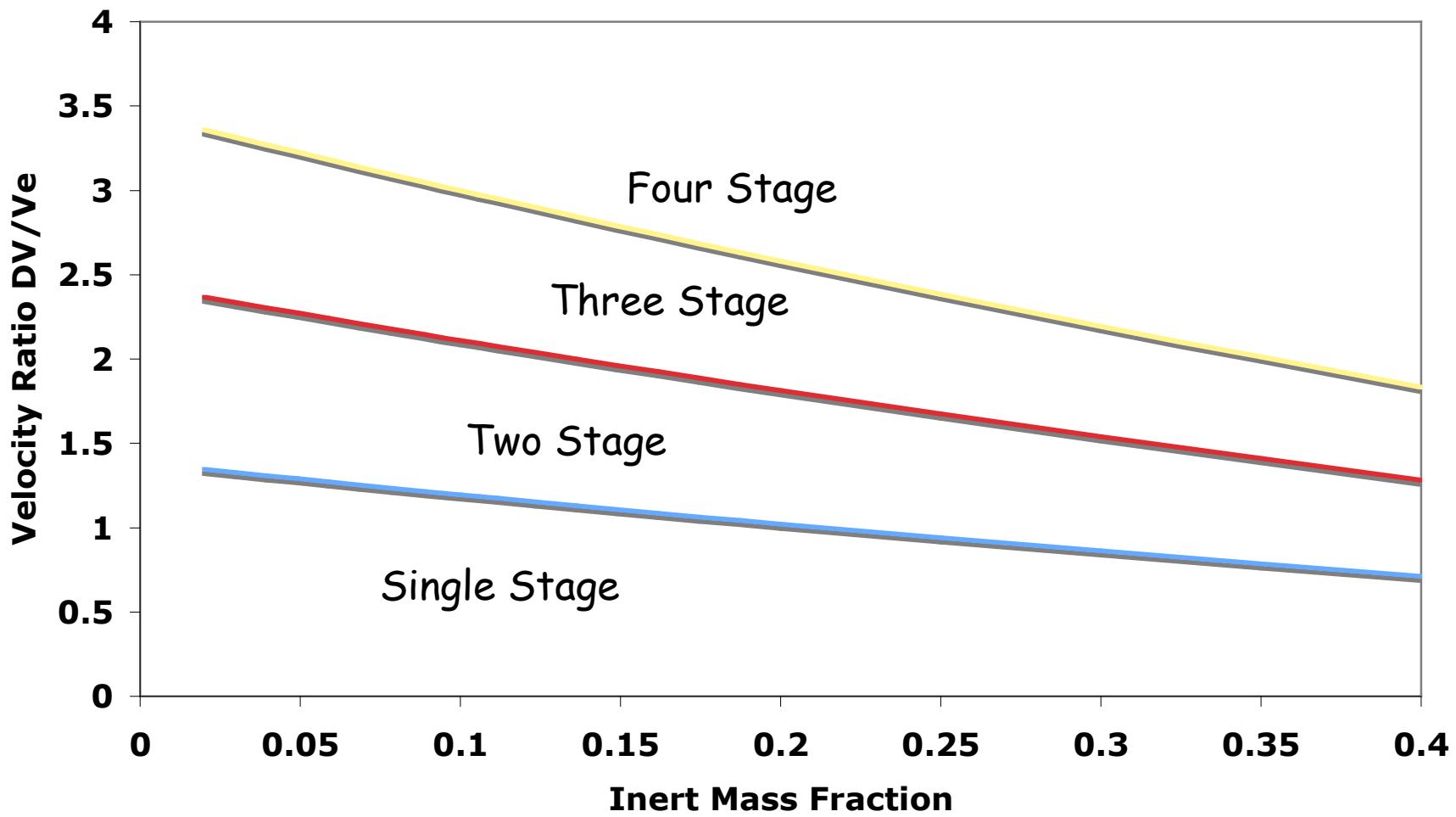
Effect of δ and $\Delta V/V_e$ on Payload



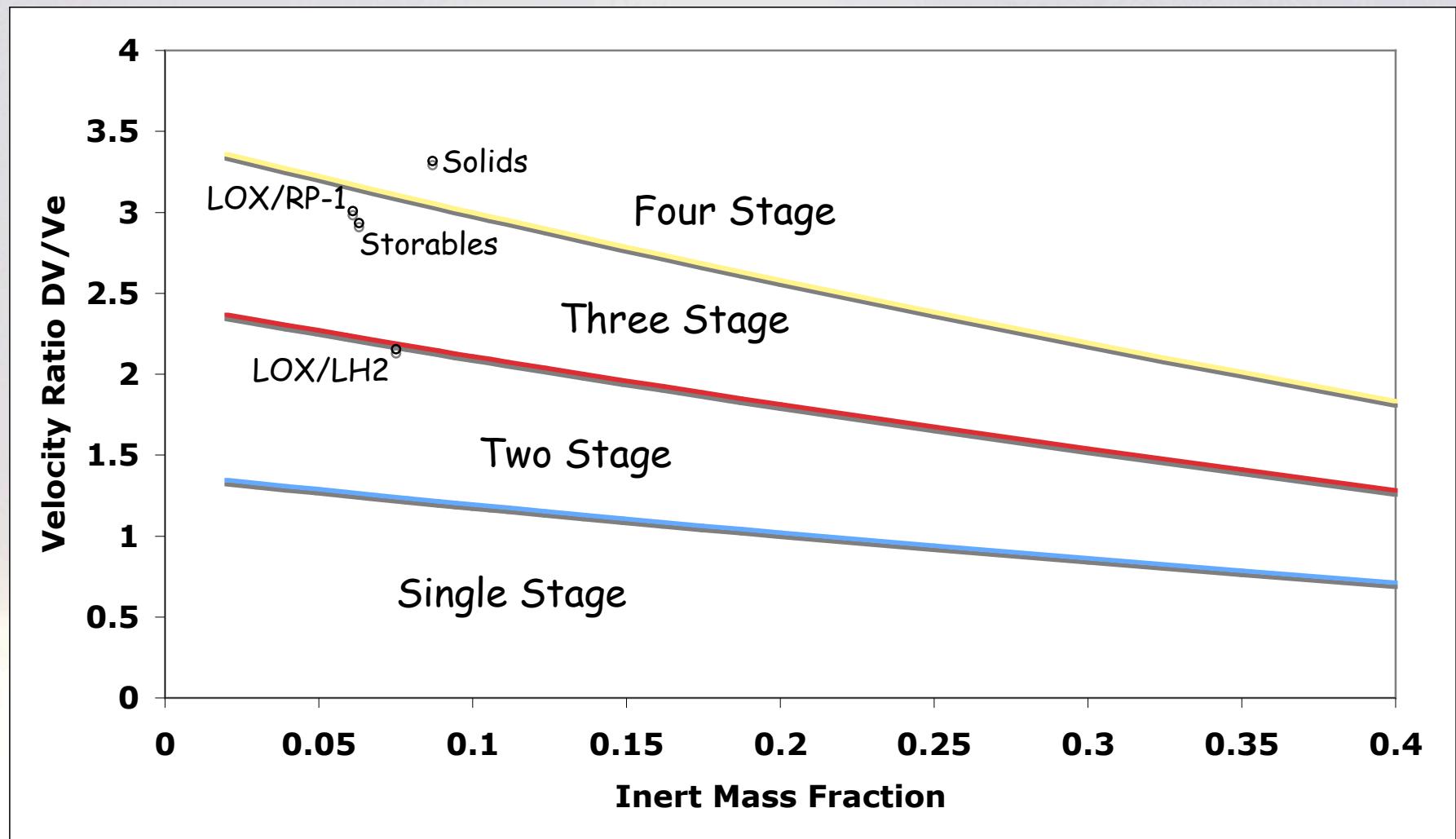
Effect of Staging



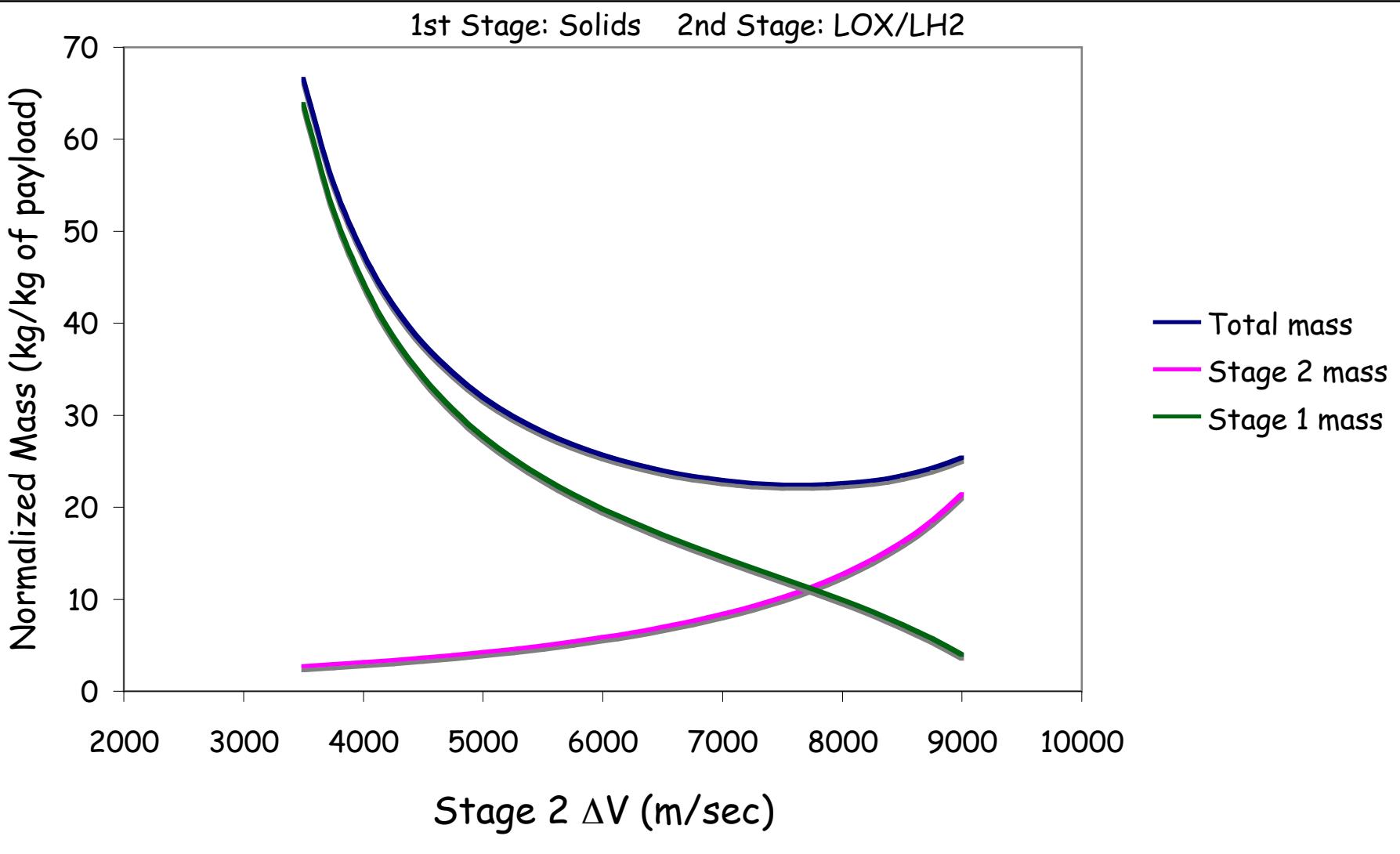
Trade Space for Number of Stages



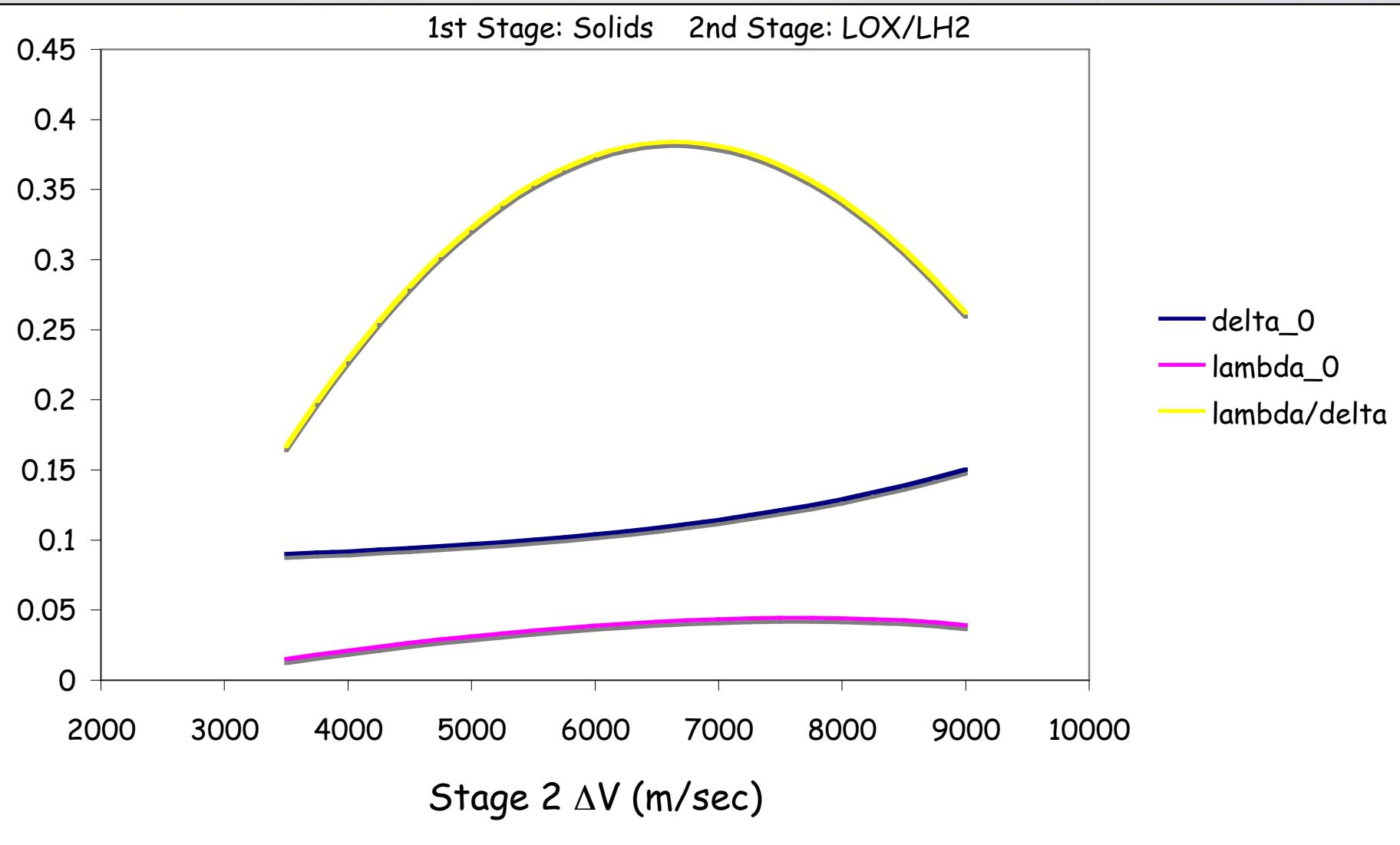
Trade Space for Number of Stages



Effect of ΔV Distribution



ΔV Distribution and Design Parameters



Lagrange Multipliers

- Given an objective function

$$y = f(x)$$

subject to constraint function

$$z = g(x)$$

- Create a new objective function

$$z = f(x) + \lambda[g(x) - z]$$

- Solve simultaneous equations

$$\frac{\partial y}{\partial x} = 0 \quad \frac{\partial y}{\partial \lambda} = 0$$



Optimum ΔV Distribution Between Stages

- Maximize payload fraction (2 stage case)

$$\lambda_0 = \lambda_1 \lambda_2 = (r_1 - \delta_1)(r_2 - \delta_2)$$

subject to constraint function

$$\Delta V_{total} = \Delta V_1 + \Delta V_2$$

- Create a new objective function

$$\lambda_o = \left(e^{\frac{-\Delta V_1}{V_{e,1}}} - \delta_1 \right) \left(e^{\frac{-\Delta V_2}{V_{e,2}}} - \delta_2 \right) + K [\Delta V_1 + \Delta V_2 - \Delta V_{total}]$$

→ Very messy for partial derivatives!



Optimum ΔV Distribution (continued)

- Use substitute objective function

$$\max (\lambda_o) \iff \max [\ln (\lambda_o)]$$

- Create a new constrained objective function

$$\ln (\lambda_o) = \ln (r_1 - \delta_1) + \ln (r_2 - \delta_2) + K [\Delta V_1 + \Delta V_2 - \Delta V_{total}]$$

- Take partials and set equal to zero

$$\frac{\partial [\ln (\lambda_o)]}{\partial r_1} = 0 \quad \frac{\partial [\ln (\lambda_o)]}{\partial r_2} = 0 \quad \frac{\partial [\ln (\lambda_o)]}{\partial K} = 0$$



Optimum ΔV Special Cases

- “Generic” partial of objective function

$$\frac{\partial [\ln(\lambda_o)]}{\partial r_i} = \frac{1}{r_i - \delta_i} + K \frac{V_{e,i}}{r_i} = 0$$

- Special case: $\delta_1 = \delta_2$ $V_{e,1} = V_{e,2}$

$$r_1 = r_2 \implies \Delta V_1 = \Delta V_2 = \frac{\Delta V_{total}}{2}$$

- Same principle holds for n stages

$$r_1 = r_2 = \dots = r_n \implies$$

$$\Delta V_1 = \Delta V_2 = \dots = \Delta V_n = \frac{\Delta V_{total}}{n}$$



Sensitivity to Inert Mass

ΔV for multistaged rocket

$$\Delta V_{tot} = \sum_{k=1}^{n \text{ stages}} \Delta V_k = \sum_{k=1}^n V_{e,k} \ln \left(\frac{m_{o,k}}{m_{f,k}} \right)$$

where

$$m_{o,k} = m_{pl} + m_{pr,k} + m_{in,k} + \sum_{j=k+1}^n (m_{pr,j} + m_{in,j})$$

$$m_{f,k} = m_{pl} + m_{in,k} + \sum_{j=k+1}^n (m_{pr,j} + m_{in,j})$$



Finding Payload Sensitivity to Inert Mass

- Given the equation linking mass to ΔV , take

$$\frac{\partial(\Delta V_{tot})}{\partial m_{pl}} dm_{pl} + \frac{\partial(\Delta V_{tot})}{\partial m_{in,j}} dm_{in,j} = 0$$

and solve to find

$$\left. \frac{dm_{pl}}{dm_{in,k}} \right|_{\partial(\Delta V_{tot})=0} = \frac{- \sum_{j=1}^k V_{e,j} \left(\frac{1}{m_{o,j}} - \frac{1}{m_{f,j}} \right)}{\sum_{\ell=1}^N V_{e,\ell} \left(\frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}} \right)}$$

- This equation shows the “trade-off ratio” - Δ payload resulting from a change in inert mass for stage k (for a vehicle with N total stages)



Trade-off Ratio Example: Gemini-Titan II

	Stage 1	Stage 2
Initial Mass (kg)	150,500	32,630
Final Mass (kg)	39,370	6099
V_e (m/sec)	2900	3097
$dm_{pl}/dm_{in,k}$	-0.1164	-1



Payload Sensitivity to Propellant Mass

- In a similar manner, solve to find

$$\left. \frac{dm_{pl}}{dm_{pr,k}} \right|_{\partial(\Delta V_{tot})=0} = \frac{- \sum_{j=1}^k V_{e,j} \left(\frac{1}{m_{o,j}} \right)}{\sum_{\ell=1}^N V_{e,\ell} \left(\frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}} \right)}$$

- This equation gives the change in payload mass as a function of additional propellant mass (all other parameters held constant)



Trade-off Ratio Example: Gemini-Titan II

	Stage 1	Stage 2
Initial Mass (kg)	150,500	32,630
Final Mass (kg)	39,370	6099
V_e (m/sec)	2900	3097
$dm_{pl}/dm_{in,k}$	-0.1164	-1
$dm_{pl}/dm_{pr,k}$	0.04124	0.2443



Payload Sensitivity to Exhaust Velocity

- Use the same technique to find the change in payload resulting from additional exhaust velocity for stage k

$$\left. \frac{dm_{pl}}{dV_{e,k}} \right|_{\partial(\Delta V_{tot})=0} = \frac{\sum_{j=1}^k \ln \left(\frac{m_{o,j}}{m_{f,j}} \right)}{\sum_{\ell=1}^N V_{e,\ell} \left(\frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}} \right)}$$

- This trade-off ratio (unlike the ones for inert and propellant masses) has units - kg/(m/sec)



Trade-off Ratio Example: Gemini-Titan II

	Stage 1	Stage 2
Initial Mass (kg)	150,500	32,630
Final Mass (kg)	39,370	6099
V_e (m/sec)	2900	3097
$dm_{pl}/dm_{in,k}$	-0.1164	-1
$dm_{pl}/dm_{pr,k}$	0.04124	0.2443
$dm_{pl}/dV_{e,k}$ (kg/m/sec)	2.870	6.459



Parallel Staging



- Multiple dissimilar engines burning simultaneously
- Frequently a result of upgrades to operational systems
- General case requires “brute force” numerical performance analysis



Parallel-Staging Rocket Equation

- Momentum at time t:

$$M = mv$$

- Momentum at time $t + \Delta t$:

(subscript “b”=boosters; “c”=core vehicle)

$$M = (m - \Delta m_b - \Delta m_c)(v + \Delta v)$$

$$+ \Delta m_b(v - V_{e,b}) + \Delta m_c(v - V_{e,c})$$

- Assume thrust (and mass flow rates) constant



Parallel-Staging Rocket Equation

- Rocket equation during booster burn

$$\Delta V = -\bar{V}_e \ln \left(\frac{m_{final}}{m_{initial}} \right) = -\bar{V}_e \ln \left(\frac{m_{in,b} + m_{in,c} + \chi m_{pr,c} + m_{0,2}}{m_{in,b} + m_{pr,b} + m_{in,c} + m_{pr,c} + m_{0,2}} \right)$$

where χ = fraction of core propellant remaining after booster burnout, and where

$$\bar{V}_e = \frac{V_{e,b} \dot{m}_b + V_{e,c} \dot{m}_c}{\dot{m}_b + \dot{m}_c} = \frac{V_{e,b} m_{pr,b} + V_{e,c} (1-\chi) m_{pr,c}}{m_{pr,b} + (1-\chi) m_{pr,c}}$$



Analyzing Parallel-Staging Performance

Parallel stages break down into pseudo-serial stages:

- Stage “0” (boosters and core)

$$\Delta V_0 = -\bar{V}_e \ln \left(\frac{m_{in,b} + m_{in,c} + \chi m_{pr,c} + m_{0,2}}{m_{in,b} + m_{pr,b} + m_{in,c} + m_{pr,c} + m_{0,2}} \right)$$

- Stage “1” (core alone)

$$\Delta V_1 = -V_{e,c} \ln \left(\frac{m_{in,c} + m_{0,2}}{m_{in,c} + \chi m_{pr,c} + m_{0,2}} \right)$$

- Subsequent stages are as before



Parallel Staging Example: Space Shuttle

- 2 x solid rocket boosters (data below for single SRB)
 - Gross mass 589,670 kg
 - Empty mass 86,183 kg
 - V_e 2636 m/sec
 - Burn time 124 sec
- External tank (space shuttle main engines)
 - Gross mass 750,975 kg
 - Empty mass 29,930 kg
 - V_e 4459 m/sec
 - Burn time 480 sec
- “Payload” (orbiter + P/L) 125,000 kg



Shuttle Parallel Staging Example

$$V_{e,b} = 2636 \frac{m}{sec}$$

$$V_{e,c} = 4459 \frac{m}{sec}$$

$$\chi = \frac{480 - 124}{480} = 0.7417$$

$$\bar{V}_e = \frac{2636(1,007,000) + 4459(721,000)(1 - .7417)}{1,007,000 + 721,000(1 - .7417)} = 2921 \frac{m}{sec}$$

$$\Delta V_0 = -2921 \ln \frac{862,000}{3,062,000} = 3702 \frac{m}{sec}$$

$$\Delta V_1 = -4459 \ln \frac{154,900}{689,700} = 6659 \frac{m}{sec}$$

$$\Delta V_{tot} = 10,360 \frac{m}{sec}$$



Modular Staging



- Use identical modules to form multiple stages
- Have to cluster modules on lower stages to make up for nonideal ΔV distributions
- Advantageous from production and development cost standpoints



Module Analysis

- All modules have the same inert mass and propellant mass
- Because δ varies with payload mass, not all modules have the same δ !
- Use module-oriented parameters

$$\varepsilon \equiv \frac{m_{in}}{m_{in} + m_{pr}} \quad \sigma \equiv \frac{m_{in}}{m_{pr}}$$

- Conversions
- $$\varepsilon = \frac{\delta}{1 - \lambda} \quad \sigma = \frac{\delta}{1 - \delta - \lambda}$$



Rocket Equation for Modular Boosters

- Assuming n modules in stage 1,

$$r_1 = \frac{n(m_{in}) + m_{o2}}{n(m_{in} + m_{pr}) + m_{o2}} = \frac{n\varepsilon + \frac{m_{o2}}{m_{mod}}}{n + \frac{m_{o2}}{m_{mod}}}$$

- If all 3 stages use same modules, n_j for stage j,

$$r_1 = \frac{n_1\varepsilon + n_2 + n_3 + \rho_{pl}}{n_1 + n_2 + n_3 + \rho_{pl}}$$

where $\rho_{pl} \equiv \frac{m_{pl}}{m_{mod}}$; $m_{mod} = m_{in} + m_{pr}$



Example: Conestoga 1620 (EER)



- Small launch vehicle (1 flight, 1 failure)
- Payload 900 kg
- Module gross mass 11,400 kg
- Module empty mass 1,400 kg
- Exhaust velocity 2754 m/sec
- Staging pattern
 - 1st stage - 4 modules
 - 2nd stage - 2 modules
 - 3rd stage - 1 module
 - 4th stage - Star 48V (gross mass 2200 kg, empty mass 140 kg, V_e 2842 m/sec)



Conestoga 1620 Performance

- 4th stage ΔV

$$\Delta V_4 = -V_{e4} \ln \frac{m_{f4}}{m_{o4}} = -2842 \ln \frac{900 + 140}{900 + 2200} = 3104 \frac{\text{m}}{\text{sec}}$$

- Treat like three-stage modular vehicle; $M_{pl}=3100 \text{ kg}$

$$\epsilon = \frac{m_{in}}{m_{mod}} = \frac{1400}{11400} = 0.1228$$

$$\rho_{pl} = \frac{m_{pl}}{m_{mod}} = \frac{3100}{11400} = 0.2719$$

$$n_1 = 4; \ n_2 = 2; \ n_3 = 1$$



Constellation 1620 Performance (cont.)

$$r_1 = \frac{n_1\epsilon + n_2 + n_3 + \rho_{pl}}{n_1 + n_2 + n_3 + \rho_{pl}} = \frac{4 \times 0.1228 + 2 + 1 + 0.2719}{4 + 2 + 1 + 0.2719} = 0.5175$$

$$r_2 = \frac{n_2\epsilon + n_3 + \rho_{pl}}{n_2 + n_3 + \rho_{pl}} = \frac{2 \times 0.1228 + 1 + 0.2719}{2 + 1 + 0.2719} = 0.4638$$

$$r_3 = \frac{n_3\epsilon + \rho_{pl}}{n_3 + \rho_{pl}} = \frac{1 \times 0.1228 + 0.2719}{1 + 0.2719} = 0.3103$$

$$V_1 = 1814 \frac{\text{m}}{\text{sec}}; V_2 = 2116 \frac{\text{m}}{\text{sec}}$$

$$V_3 = 3223 \frac{\text{m}}{\text{sec}}; V_4 = 3104 \frac{\text{m}}{\text{sec}}$$

$$V_{total} = 10,257 \frac{\text{m}}{\text{sec}}$$

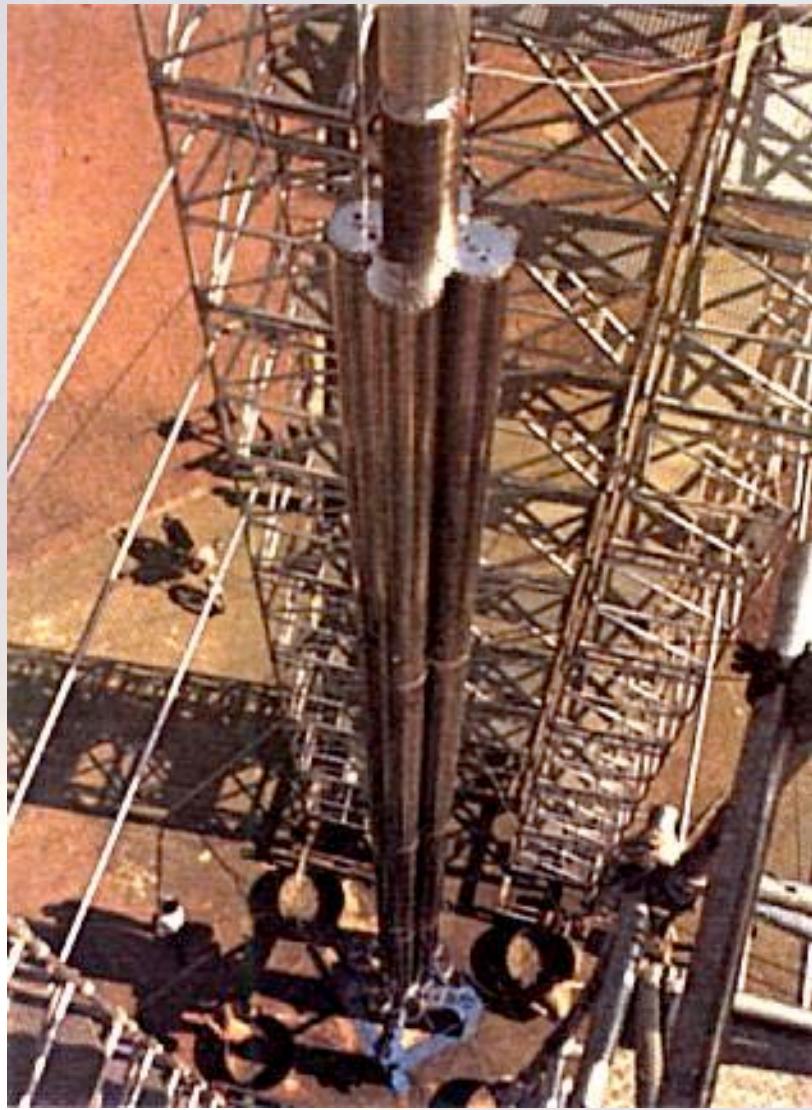


Discussion about Modular Vehicles

- Modularity has several advantages
 - Saves money (smaller modules cost less to develop)
 - Saves money (larger production run = lower cost/module)
 - Allows resizing launch vehicles to match payloads
- Trick is to optimize number of stages, number of modules/stage to minimize total number of modules
- Generally close to optimum by doubling number of modules at each lower stage
- Have to worry about packing factors, complexity



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Modular Example

- Let's build a launch vehicle out of seven Space Shuttle Solid Rocket Boosters
 - $M_{in}=86,180 \text{ kg}$
 - $M_{pr}=503,500 \text{ kg}$

$$\varepsilon \equiv \frac{m_{in}}{m_{in} + m_{pr}} = 0.1461 \quad \sigma \equiv \frac{m_{in}}{m_{pr}} = 0.1711$$

- Look at possible approaches to sequential firing



Modular Sequencing - SRB Example

- Assume no payload
- All seven firing at once - $\Delta V_{tot}=5138$ m/sec
- 3-3-1 sequence - $\Delta V_{tot}=9087$ m/sec
- 4-2-1 sequence - $\Delta V_{tot}=9175$ m/sec
- 2-2-2-1 sequence - $\Delta V_{tot}=9250$ m/sec
- 2-1-1-1-1 sequence - $\Delta V_{tot}=9408$ m/sec
- 1-1-1-1-1-1 sequence - $\Delta V_{tot}=9418$ m/sec
- Sequence limited by need to balance thrust laterally

