Rocket Performance

- Parallel staging
- Modular staging
- Standard atmospheres
- Orbital decay due to drag
- Straight-line (no gravity) entry based on atmospheric density

Parallel Staging

- Multiple dissimilar engines burning simultaneously
- Frequently a result of upgrades to operational systems
- General case requires "brute force" numerical performance analysis

Parallel-Staging Rocket Equation

- Momentum at time t:

$$M = mv$$

- Momentum at time t+Δt:
  (subscript "b"=boosters; "c"=core vehicle)

$$M = (m - \Delta m_b - \Delta m_c)(v + \Delta v)$$
$$+\Delta m_b(v - V_{e,b}) + \Delta m_c(v - V_{e,c})$$

- Assume thrust (and mass flow rates) constant

Parallel-Staging Rocket Equation

- Rocket equation during booster burn

$$\Delta V = -\bar{V}_e \ln\left(\frac{m_{final}}{m_{initial}}\right) = -\bar{V}_e \ln\left(\frac{m_{in,b}+m_{in,c}+\chi m_{pr,c}+m_{0,2}}{m_{in,b}+m_{pr,b}+m_{in,c}+m_{pr,c}+m_{0,2}}\right)$$

where χ= fraction of core propellant remaining after booster burnout, and where

$$\bar{V}_e = \frac{V_{e,b}\dot{m}_b+V_{e,c}\dot{m}_c}{\dot{m}_b+\dot{m}_c} = \frac{V_{e,b}m_{pr,b}+V_{e,c}(1-\chi)m_{pr,c}}{m_{pr,b}+(1-\chi)m_{pr,c}}$$

Analyzing Parallel-Staging Performance

Parallel stages break down into pseudo-serial stages:

- Stage "0" (boosters and core)

$$\Delta V_0 = -\bar{V}_e \ln\left(\frac{m_{in,b}+m_{in,c}+\chi m_{pr,c}+m_{0,2}}{m_{in,b}+m_{pr,b}+m_{in,c}+m_{pr,c}+m_{0,2}}\right)$$

- Stage "1" (core alone)

$$\Delta V_1 = -V_{e,c} \ln\left(\frac{m_{in,c}+m_{0,2}}{m_{in,c}+\chi m_{pr,c}+m_{0,2}}\right)$$

- Subsequent stages are as before

Parallel Staging Example: Space Shuttle

- 2 x solid rocket boosters (data below for single SRB)
  - Gross mass 589,670 kg
  - Empty mass 86,183 kg
  - Ve 2636 m/sec
  - Burn time 124 sec
- External tank (space shuttle main engines)
  - Gross mass 750,975 kg
  - Empty mass 29,930 kg
  - Ve 4459 m/sec
  - Burn time 480 sec
- "Payload" (orbiter + P/L) 125,000 kg

Shuttle Parallel Staging Example

$$V_{e,b} = 2636\frac{m}{sec} \qquad V_{e,c} = 4459\frac{m}{sec}$$

$$\chi = \frac{480 - 124}{480} = 0.7417$$

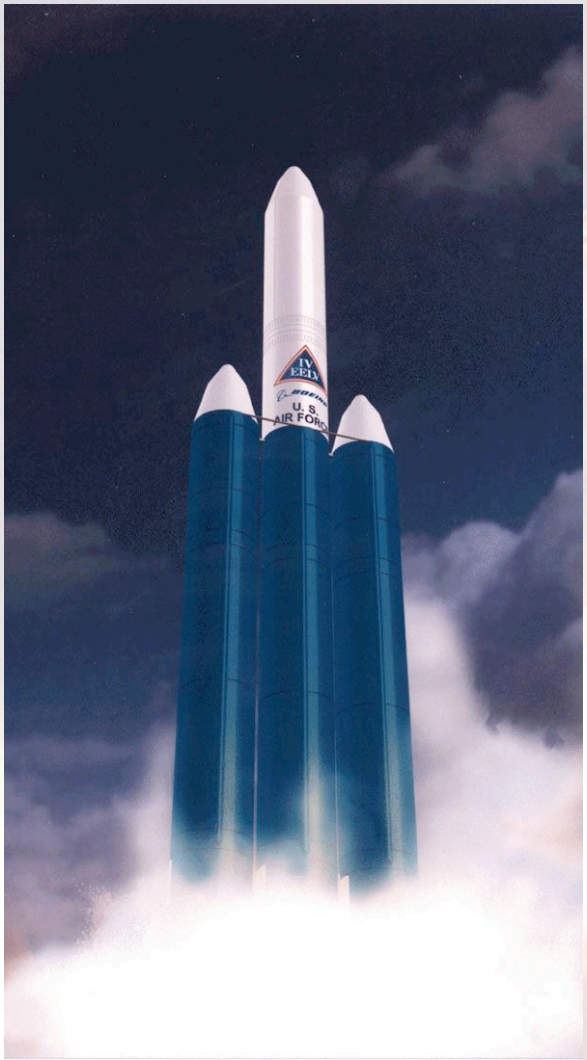
$$\bar{V}_e = \frac{2636(1,007,000) + 4459(721,000)(1 - .7417)}{1,007,000 + 721,000(1 - .7417)} = 2921\frac{m}{sec}$$

$$\Delta V_0 = -2921 \ln \frac{862,000}{3,062,000} = 3702\frac{m}{sec}$$

$$\Delta V_1 = -4459 \ln \frac{154,900}{689,700} = 6659\frac{m}{sec}$$

$$\Delta V_{tot} = 10,360\frac{m}{sec}$$

Modular Staging



- Use identical modules to form multiple stages
- Have to cluster modules on lower stages to make up for nonideal ΔV distributions
- Advantageous from production and development cost standpoints

Module Analysis

- All modules have the same inert mass and propellant mass
- Because δ varies with payload mass, not all modules have the same δ!
- Use module-oriented parameters

$$\varepsilon \equiv \frac{m_{in}}{m_{in} + m_{pr}} \qquad \sigma \equiv \frac{m_{in}}{m_{pr}}$$

- Conversions

$$\varepsilon = \frac{\delta}{1 - \lambda} \qquad \sigma = \frac{\delta}{1 - \delta - \lambda}$$

Rocket Equation for Modular Boosters

- Assuming n modules in stage 1,

$$r_1 = \frac{n(m_{in}) + m_{o2}}{n(m_{in} + m_{pr}) + m_{o2}} = \frac{n\varepsilon + \frac{m_{o2}}{m_{mod}}}{n + \frac{m_{o2}}{m_{mod}}}$$

- If all 3 stages use same modules, n_j for stage j,

$$r_1 = \frac{n_1\varepsilon + n_2 + n_3 + \rho_{pl}}{n_1 + n_2 + n_3 + \rho_{pl}}$$

where $\rho_{pl} \equiv \frac{m_{pl}}{m_{mod}}; \; m_{mod} = m_{in} + m_{pr}$

Example: Conestoga 1620 (EER)

- Small launch vehicle (1 flight, 1 failure)
- Payload 900 kg
- Module gross mass 11,400 kg
- Module empty mass 1,400 kg
- Exhaust velocity 2754 m/sec
- Staging pattern
  - 1st stage - 4 modules
  - 2nd stage - 2 modules
  - 3rd stage - 1 module
  - 4th stage - Star 48V (gross mass 2200 kg, empty mass 140 kg, V_e 2842 m/sec)

Conestoga 1620 Performance

- 4th stage ΔV

$$\Delta V_4 = -V_{e4} \ln \frac{m_{f4}}{m_{o4}} = -2842 \ln \frac{900 + 140}{900 + 2200} = 3104 \ \frac{\mathrm{m}}{\mathrm{sec}}$$

- Treat like three-stage modular vehicle; M_{pl}=3100 kg

$$\epsilon = \frac{m_{in}}{m_{mod}} = \frac{1400}{11400} = 0.1228$$

$$\rho_{pl} = \frac{m_{pl}}{m_{mod}} = \frac{3100}{11400} = 0.2719$$

$$n_1 = 4; \ n_2 = 2; \ n_3 = 1$$

Constellation 1620 Performance (cont.)

$$r_1 = \frac{n_1\epsilon + n_2 + n_3 + \rho_{pl}}{n_1 + n_2 + n_3 + \rho_{pl}} = \frac{4 \times 0.1228 + 2 + 1 + 0.2719}{4 + 2 + 1 + 0.2719} = 0.5175$$

$$r_2 = \frac{n_2\epsilon + n_3 + \rho_{pl}}{n_2 + n_3 + \rho_{pl}} = \frac{2 \times 0.1228 + 1 + 0.2719}{2 + 1 + 0.2719} = 0.4638$$

$$r_3 = \frac{n_3\epsilon + \rho_{pl}}{n_3 + \rho_{pl}} = \frac{1 \times 0.1228 + 0.2719}{1 + 0.2719} = 0.3103$$

$$V_1 = 1814\ \frac{\mathrm{m}}{\mathrm{sec}};\ V_2 = 2116\ \frac{\mathrm{m}}{\mathrm{sec}}$$

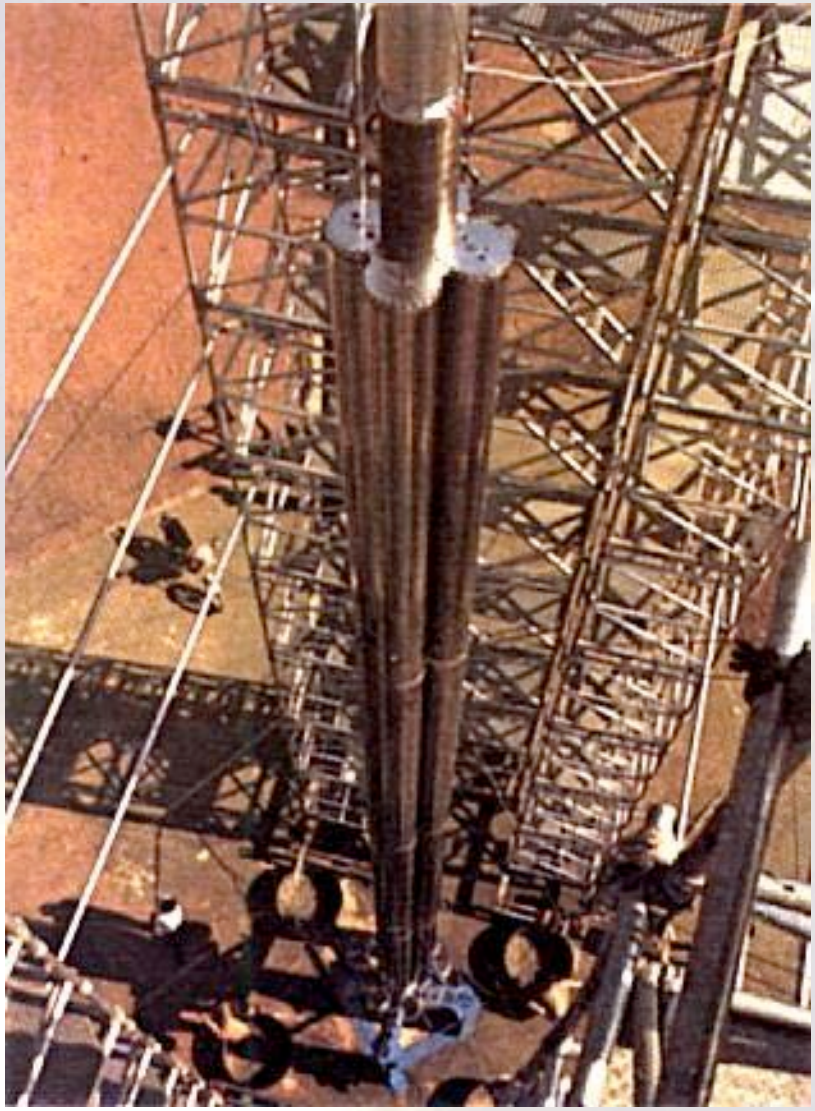
$$V_3 = 3223\ \frac{\mathrm{m}}{\mathrm{sec}};\ V_4 = 3104\ \frac{\mathrm{m}}{\mathrm{sec}}$$

$$V_{total} = 10,257\ \frac{\mathrm{m}}{\mathrm{sec}}$$

Discussion about Modular Vehicles

- Modularity has several advantages
  - Saves money (smaller modules cost less to develop)
  - Saves money (larger production run = lower cost/ module)
  - Allows resizing launch vehicles to match payloads
- Trick is to optimize number of stages, number of modules/stage to minimize total number of modules
- Generally close to optimum by doubling number of modules at each lower stage
- Have to worry about packing factors, complexity

OTRAG - 1977-1983



Modular Example

- Let's build a launch vehicle out of seven Space Shuttle Solid Rocket Boosters
  - M_{in}=86,180 kg
  - M_{pr}=503,500 kg

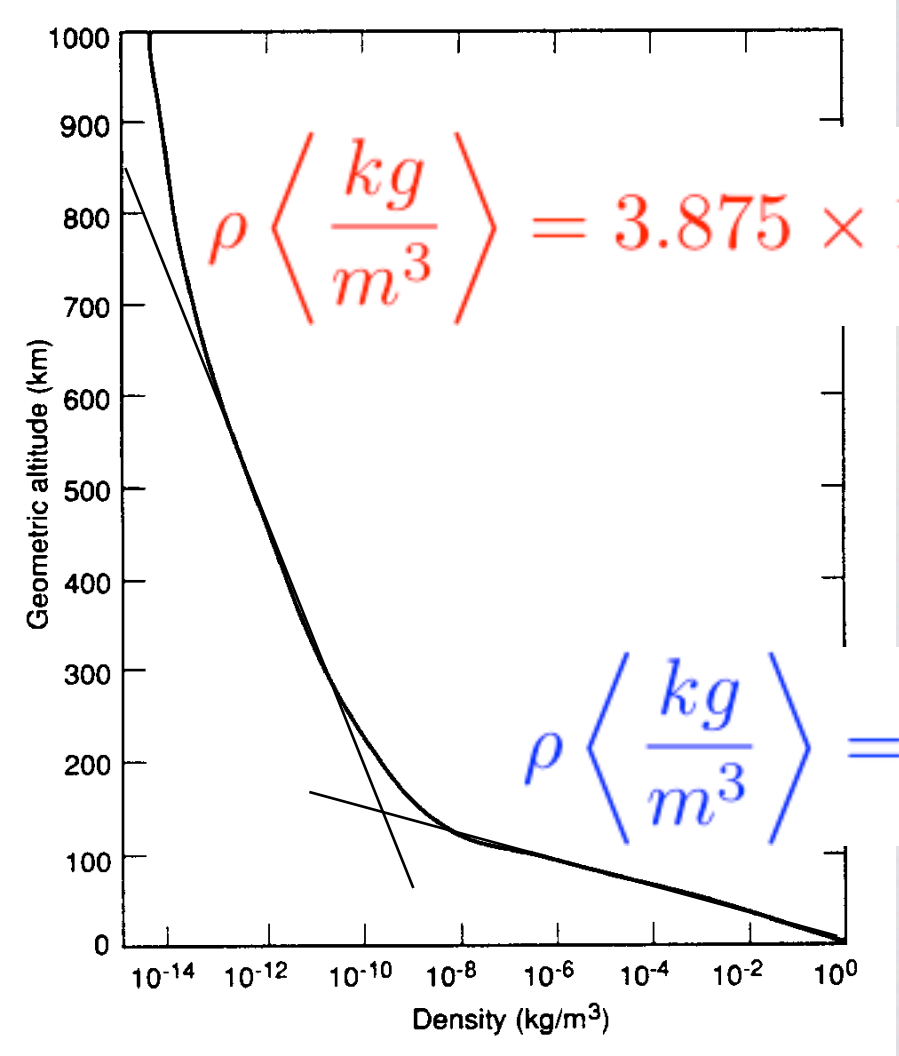
$$\varepsilon \equiv \frac{m_{in}}{m_{in} + m_{pr}} = 0.1461 \qquad \sigma \equiv \frac{m_{in}}{m_{pr}} = 0.1711$$

- Look at possible approaches to sequential firing

Modular Sequencing - SRB Example

- Assume no payload
- All seven firing at once - ΔV_{tot}=5138 m/sec
- 3-3-1 sequence - ΔV_{tot}=9087 m/sec
- 4-2-1 sequence - ΔV_{tot}=9175 m/sec
- 2-2-2-1 sequence - ΔV_{tot}=9250 m/sec
- 2-1-1-1-1-1 sequence - ΔV_{tot}=9408 m/sec
- 1-1-1-1-1-1-1 sequence - ΔV_{tot}=9418 m/sec
- Sequence limited by need to balance thrust laterally

Atmospheric Density with Altitude



$$\rho\left\langle\frac{kg}{m^3}\right\rangle = 3.875\times10^{-9}e^{-\frac{h\langle km\rangle}{59.06}}$$

$$\rho\left\langle\frac{kg}{m^3}\right\rangle = 1.226e^{-\frac{h\langle km\rangle}{7.524}}$$

Ref: V. L. Pisacane and R. C. Moore, Fundamentals of Space Systems Oxford University Press, 1994

Energy Loss Due to Atmospheric Drag

$$\text{Drag } D \equiv \frac{1}{2}\rho v^2 A c_D$$

$$\text{Drag acceleration } a_d = \frac{D}{m} = \frac{\rho v^2}{2}\frac{A c_D}{m}$$

$$\boxed{\beta \equiv \frac{m}{c_D A}}$$ <== Ballistic Coefficient

$$a_d = \frac{\rho v^2}{2\beta}$$

$$\text{orbital energy} \equiv E = -\frac{\mu}{2a}$$

$$\frac{dE}{dt} = \frac{\mu}{2a^2}\frac{da}{dt}$$

Energy Loss Due to Atmospheric Drag

Since drag is highest at perigee, the first effect of atmospheric drag is to circularize the orbit (high perigee drag lowers apogee)

$$\frac{dE_{drag}}{dt} = a_d v$$

$$v_{circ}^2 = \frac{\mu}{a} \qquad \frac{dE_{drag}}{dt} = -\frac{\rho v^2}{2\beta}\sqrt{\frac{\mu}{a}}$$

$$\frac{dE_{drag}}{dt} = -\sqrt{\frac{\mu}{a}}\frac{\rho}{2\beta}\frac{\mu}{a} = -\left(\frac{\mu}{a}\right)^{\frac{3}{2}}\frac{\rho}{2\beta}$$

Derivation of Orbital Decay Due to Drag

Set orbital energy variation equal to energy lost by drag

$$\frac{\mu}{2a^2}\frac{da}{dt} = -\frac{\rho}{2\beta}\left(\frac{\mu}{a}\right)^{\frac{3}{2}}$$

$$\frac{da}{dt} = -\frac{\rho}{\beta}\sqrt{\mu a}$$

$$\rho = \rho_o e^{-\frac{h}{h_s}} \qquad a = h + r_E \Longrightarrow \frac{da}{dt} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{\sqrt{\mu\left(h + r_E\right)}}{\beta}\rho_o e^{-\frac{h}{h_s}}$$

Derivation of Orbital Decay (2)

This is a separable differential equation...

$$\frac{1}{\sqrt{r_E + h}} e^{\frac{h}{h_s}} dh = -\frac{\sqrt{\mu}}{\beta} \rho_o dt$$

$$\int_{h_o}^{h} \frac{1}{\sqrt{r_E + h}} e^{\frac{h}{h_s}} dh = -\frac{\sqrt{\mu}}{\beta} \rho_o \int_{t_o}^{t} dt$$

Assume $\sqrt{r_E + h} \sim \sqrt{r_E}$ for $r_E \gg h$

$$\frac{1}{\sqrt{r_E}} \int_{h_o}^{h} e^{\frac{h}{h_s}} dh = -\frac{\sqrt{\mu}}{\beta} \rho_o (t - t_o)$$

Derivation of Orbital Decay (3)

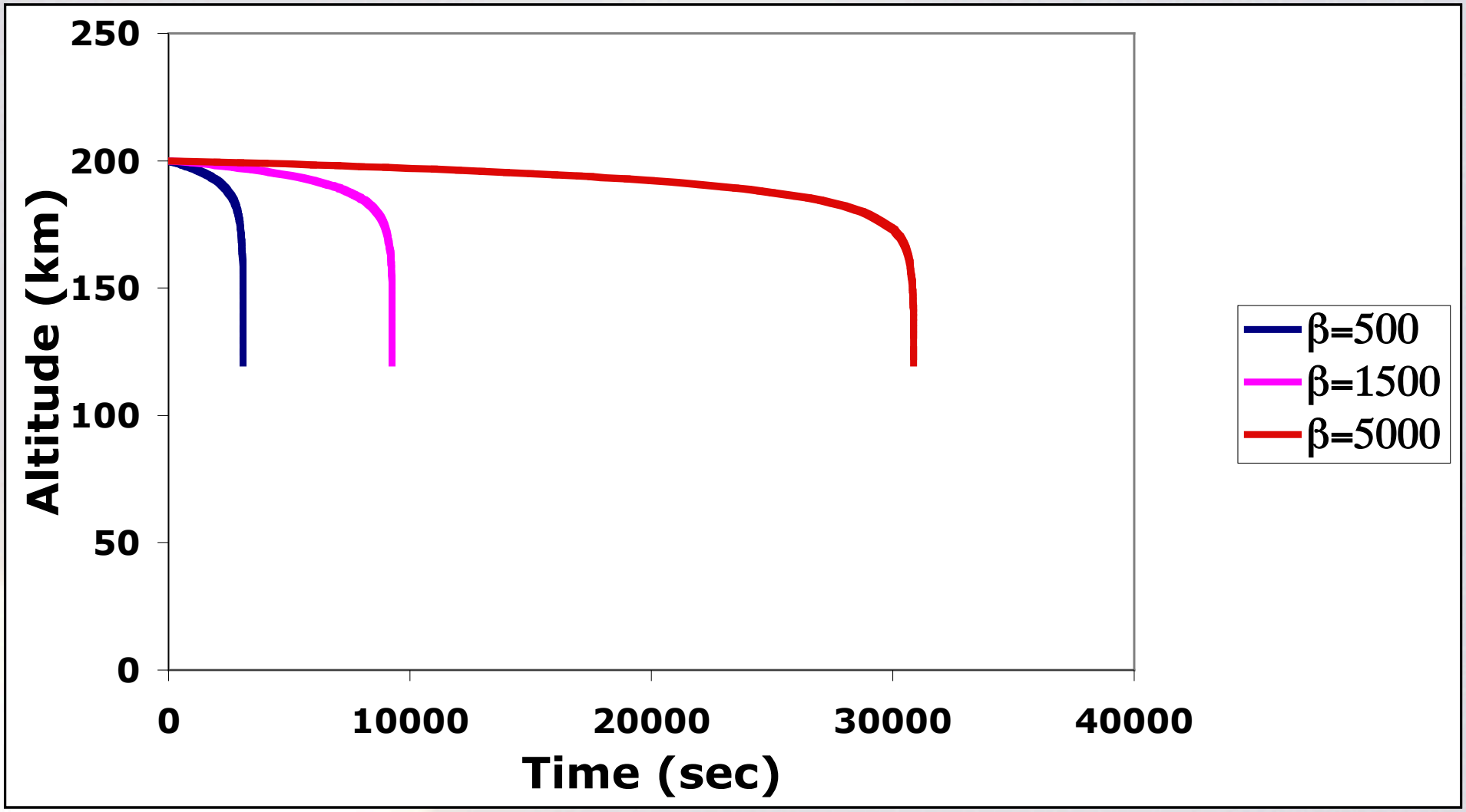
$$\frac{h_s}{\sqrt{r_E}}\left(e^{\frac{h}{h_s}} - e^{\frac{h_o}{h_s}}\right) = -\frac{\sqrt{\mu}}{\beta}\rho_o\left(t - t_o\right)$$

$$e^{\frac{h}{h_s}} - e^{\frac{h_o}{h_s}} = -\frac{\sqrt{\mu r_E}}{h_s\beta}\rho_o\left(t - t_o\right)$$

$$\boxed{h(t) = h_s \ln\left[e^{\frac{h_o}{h_s}} - \frac{\sqrt{\mu r_E}}{h_s\beta}\rho_o\left(t - t_o\right)\right]}$$

Note that some variables typically use km, and others are in meters - you have to make sure unit conversions are done properly to make this work out correctly!

Orbit Decay from Atmospheric Drag



Time Until Orbital Decay

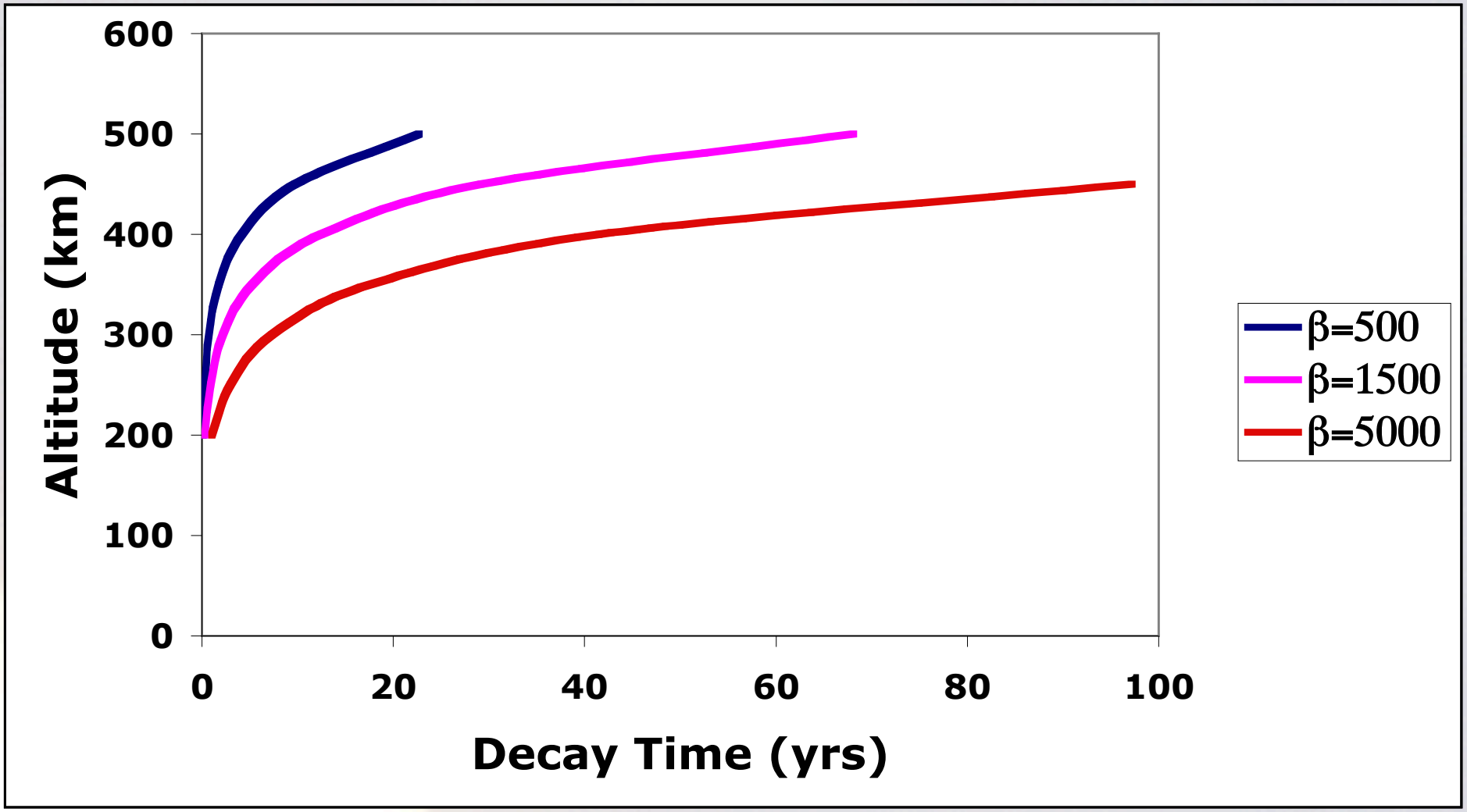
$$e^{\frac{h}{h_s}} - e^{\frac{h_o}{h_s}} = -\frac{\sqrt{\mu r_E}}{h_s \beta} \rho_o \left(t - t_o\right)$$

To find the time remaining (t_o=0) until the orbit reaches any given "critical" altitude, some algebra gives

$$t(h_{crit}) = \frac{h_s \beta}{\sqrt{\mu r_E} \rho_o} \left(e^{\frac{h_o}{h_s}} - e^{\frac{h_{crit}}{h_s}} \right)$$

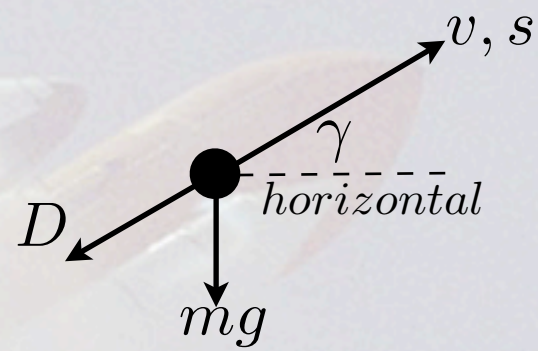
$$\boxed{t(h_{crit}) \propto \beta}$$

Decay Time to r=120 km



Ballistic Entry (no lift)

$s =$ distance along the flight path



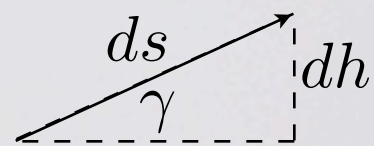
$$\frac{dv}{dt} = -g\sin\gamma - \frac{D}{m}$$

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = V\frac{dv}{ds} = \frac{1}{2}\frac{d(v^2)}{ds}$$

$$\frac{1}{2}\frac{d(v^2)}{ds} = -g\sin\gamma - \frac{D}{m}$$

$$\text{Drag } D \equiv \frac{1}{2}\rho v^2 A c_D$$

$$\frac{1}{2}\frac{d(v^2)}{ds} = -g\sin\gamma - \frac{\rho v^2}{2m}Ac_D$$



$$ds = \frac{dh}{\sin\gamma}$$

$$\frac{\sin\gamma}{2}\frac{d(v^2)}{dh} = -g\sin\gamma - \frac{\rho v^2}{2m}Ac_D$$

Ballistic Entry (2)

Exponential atmosphere $\Rightarrow \rho = \rho_o e^{-\frac{h}{h_s}}$

$$\frac{d\rho}{\rho_o} = e^{-\frac{h}{h_s}}\left(\frac{-dh}{h_s}\right) = \frac{\rho_o e^{-\frac{h}{h_s}}}{\rho_o}\left(\frac{-dh}{h_s}\right) = \frac{\rho}{\rho_o}\left(\frac{-dh}{h_s}\right)$$

$$dh = \frac{-h_s}{\rho} d\rho$$

$$\frac{\sin\gamma}{2}\frac{d(v^2)}{dh} = -g\sin\gamma - \frac{\rho v^2}{2m} A c_D$$

$$\frac{\sin\gamma}{2}\frac{d(v^2)}{d\rho}\left(\frac{-\rho}{h_s}\right) = -g\sin\gamma - \frac{\rho v^2}{2}\frac{A c_D}{m}$$

$$\frac{d(v^2)}{d\rho} = \frac{2gh_s}{\rho} + \frac{h_s v^2}{\sin\gamma}\frac{A c_D}{m}$$

Ballistic Entry (3)

Let $\beta \equiv \dfrac{m}{c_D A} \Rightarrow$ Ballistic Coefficient

$$\frac{d(v^2)}{d\rho} - \frac{h_s}{\beta \sin\gamma} v^2 = \frac{2gh_s}{\rho}$$

Assume $mg \ll D$ to get homogeneous ODE

$$\frac{d(v^2)}{d\rho} - \frac{h_s}{\beta \sin\gamma} v^2 = 0 \qquad \frac{d(v^2)}{v^2} = \frac{h_s}{\beta \sin\gamma} d\rho$$

Use (v^2) as integration variable

$$\int_{v_e}^{v} \frac{d(v^2)}{v^2} = \frac{h_s}{\beta \sin\gamma} \int_0^{\rho} d\rho \qquad v_e = \text{velocity at entry}$$

Ballistic Entry (4)

Note that the effect of ignoring gravity is that there is no force perpendicular to velocity vector ⇒ constant flight path angle γ

⇒ straight line trajectories

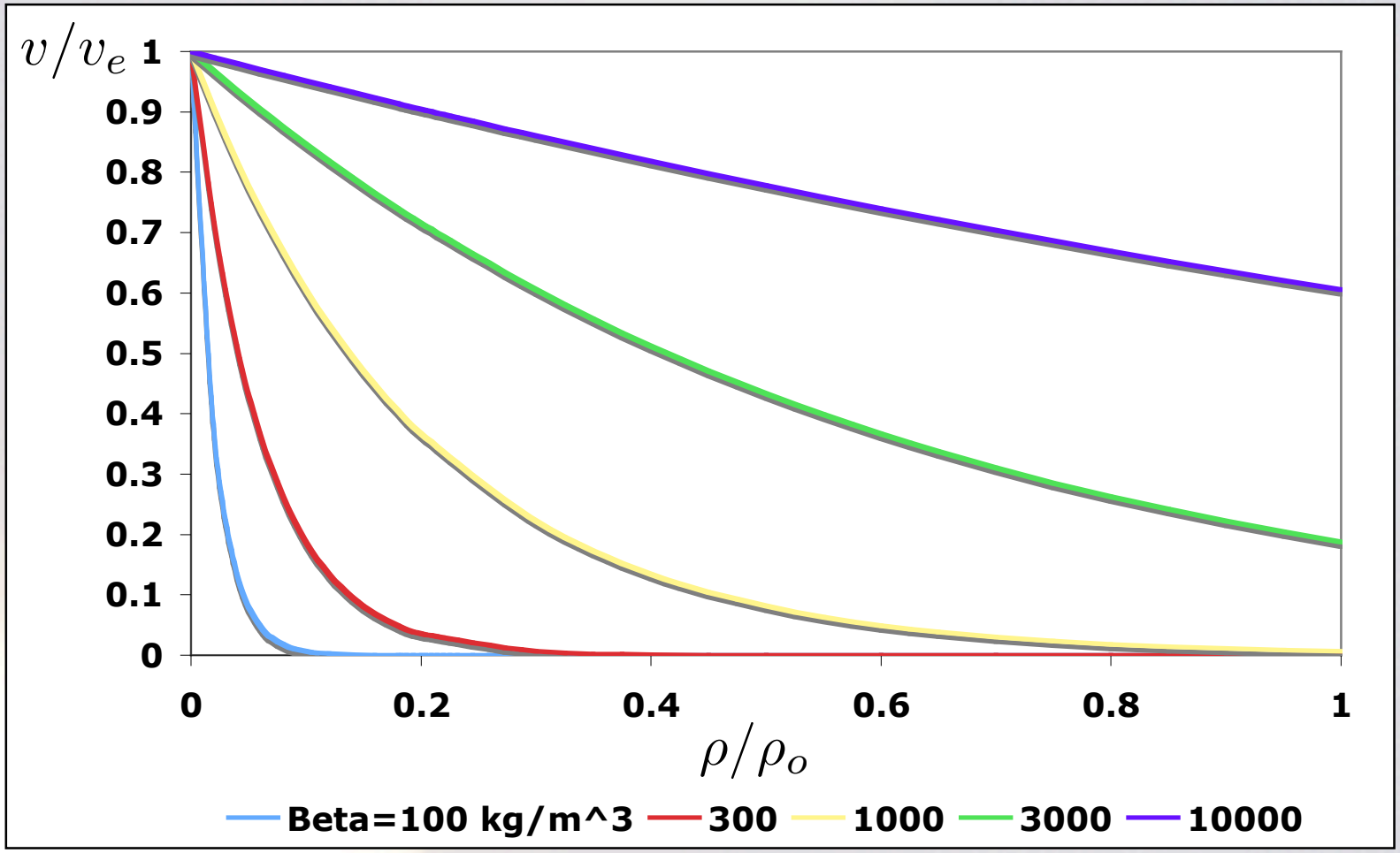
$$\ln \frac{v^2}{v_e^2} = 2 \ln \frac{v}{v_e} = \frac{h_s \rho}{\beta \sin \gamma}$$

$$\frac{v}{v_e} = \exp\left(\frac{h_s \rho}{2\beta \sin \gamma}\right)$$

$$\boxed{\frac{v}{v_e} = \exp\left(\frac{h_s \rho_o}{2\beta \sin \gamma} \frac{\rho}{\rho_o}\right)}$$

Check units: $\left(\frac{m \frac{kg}{m^3}}{\frac{kg}{m^2}}\right)$

Earth Entry, γ=-60°



What About Peak Deceleration?

$$n \equiv \boxed{\frac{dv}{dt} = -\frac{\rho v^2}{2\beta}}$$

To find n_{max}, set $\frac{d}{dt}\left(\frac{dv}{dt}\right) = \frac{d^2v}{dt^2} = 0$

$$\frac{d^2v}{dt^2} = -\frac{1}{2\beta}\left(2\rho v \boxed{\frac{dv}{dt}} + v^2\frac{d\rho}{dt}\right) = 0$$

$$\frac{d^2v}{dt^2} = -\frac{1}{2\beta}\left(-\frac{2\rho^2 v^3}{2\beta} + v^2\frac{d\rho}{dt}\right) = 0$$

$$\frac{\rho^2 v^3}{\beta} = v^2\frac{d\rho}{dt} \qquad \rho^2 v = \beta\frac{d\rho}{dt}$$

Peak Deceleration (2)

From exponential atmosphere,

$$\frac{d\rho}{dt} = -\frac{\rho_o}{h_s} e^{-\frac{h}{h_s}} \frac{dh}{dt} = -\frac{\rho}{h_s}\frac{dh}{dt}$$

From geometry, $\dfrac{dh}{dt} = v \sin\gamma$

$$\frac{d\rho}{dt} = -\frac{\rho v}{h_s}\sin\gamma \qquad \rho^2 v = \beta \frac{d\rho}{dt}$$

$$\rho^2 v = \beta\left(-\frac{\rho v}{h_s}\sin\gamma\right)$$

Remember that this refers to the conditions at max deceleration

$$\rho_{n_{max}} = -\frac{\beta}{h_s}\sin\gamma$$

Critical β for Deceleration Before Impact

At surface, $\rho = \rho_o$

$$\boxed{\beta_{crit} = -\frac{\rho_o h_s}{\sin\gamma}}$$

$\Leftarrow$ Value of β at which vehicle hits ground at point of maximum deceleration

How large is maximum deceleration?

$$\frac{dv}{dt} = \frac{\rho v^2}{2\beta} \qquad \Rightarrow \left|\frac{dv}{dt}\right|_{max} = \frac{\rho_{n_{max}} v^2}{2\beta}$$

$$\left|\frac{dv}{dt}\right|_{max} = \frac{v^2}{2\beta}\left(-\frac{\beta}{h_s}\sin\gamma\right) = -\frac{1}{2}\frac{v^2}{h_s}\sin\gamma$$

Note that this value of v is actually $v_{n_{max}}$

Peak Deceleration (3)

From page 14,

$$\frac{v}{v_e} = \exp\left(\frac{h_s \rho}{2\beta \sin\gamma}\right)$$

$$\frac{v_{n_{max}}}{v_e} = \exp\left[\frac{h_s}{2\beta \sin\gamma}\left(-\frac{\beta}{h_s}\sin\gamma\right)\right] = e^{-\frac{1}{2}}$$

$$\left|\frac{dv}{dt}\right|_{max} = -\frac{1}{2}\frac{\left(v_e e^{-\frac{1}{2}}\right)^2}{h_s}\sin\gamma = -\frac{v_e^2 \sin\gamma}{2h_s e}$$

Note that the velocity at which maximum deceleration occurs is always a fixed fraction of the entry velocity - it doesn't depend on ballistic coefficient, flight path angle, or anything else! Also, the magnitude of the maximum deceleration is not a function of ballistic coefficient - it is dependent on the entry trajectory (v_e and γ) but not spacecraft parameters (i.e., ballistic coefficient).

Terminal Velocity

Full form of ODE -

$$\frac{d\left(v^2\right)}{d\rho} - \frac{h_s}{\beta \sin\gamma} v^2 = \frac{2gh_s}{\rho}$$

At terminal velocity, $v = \text{constant} \equiv v_T$

$$-\frac{h_s}{\beta \sin\gamma} v_T^2 = \frac{2gh_s}{\rho}$$

$$\boxed{v_T^2 = \sqrt{-\frac{2g\beta \sin\gamma}{\rho}}}$$

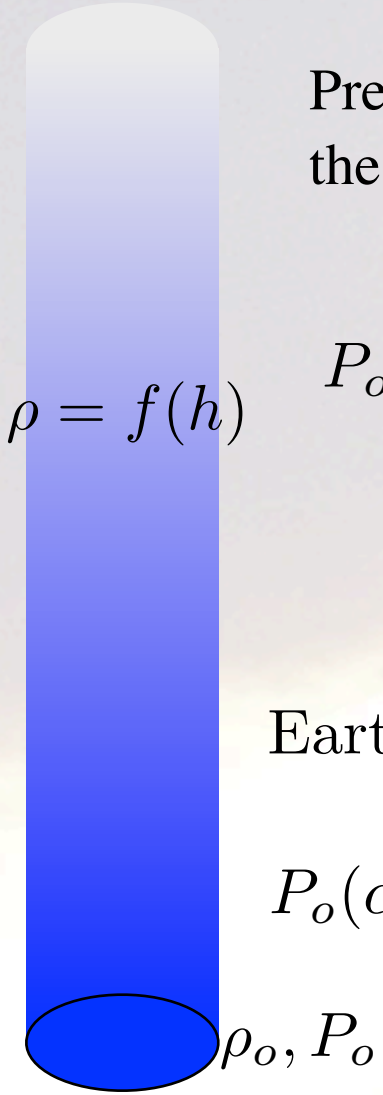
"Cannon Ball" γ=-90° Ballistic Entry

6.75" diameter sphere, c_D=0.2, V_E=6000 m/sec

	Iron	Aluminum	Balsa Wood
Weight	40 lb	15.6 lb	14.5 oz
β (kg/m^2)	3938	1532	89
ρ_{md} (kg/m^3)	0.555	0.216	0.0125
h_{md} (m)	5600	12,300	32,500
V_{impact} (m/s)	1998	355	0*
V_{term} (m/sec)	251	156	38

*Artifact of assumption that $D \gg mg$

Atmospheric Density with Altitude



Pressure=the integral of the atmospheric density in the column above the reference area

$$P_o = \int_o^\infty \rho g dh = \rho_o g \int_o^\infty e^{-\frac{h}{h_s}} dh = -\rho_o g h_s \left[e^{-\frac{h}{h_s}} \right]_o^\infty$$

$$= -\rho_o g h_s \left[0 - 1 \right]$$

$$\boxed{P_o = \rho_o g h_s}$$

Earth: $\rho_o = 1.226 \frac{kg}{m^3}$; $h_s = 7524m$;

$P_o(calc) = 90,400\ Pa$; $P_o(act) = 101,300\ Pa$

Nondimensional Ballistic Coefficient

$$\frac{v}{v_e} = \exp\left(\frac{h_s \rho_o}{2\beta \sin\gamma}\frac{\rho}{\rho_o}\right) = \exp\left(\frac{P_o}{2\beta g \sin\gamma}\frac{\rho}{\rho_o}\right)$$

Let $\widehat{\beta} \equiv \dfrac{\beta}{\rho_o h_s} = \dfrac{\beta g}{P_o}$ (Nondimensional form of ballistic coefficient)

Note that we are using the estimated value of $P_o = \rho_o g h_s$, not the actual surface pressure.

$$\boxed{\frac{v}{v_e} = \exp\left(\frac{1}{2\widehat{\beta}\sin\gamma}\frac{\rho}{\rho_o}\right)}$$

$$\beta_{crit} = -\frac{\rho_o h_s}{\sin\gamma} \qquad \boxed{\widehat{\beta}_{crit} = -\frac{1}{\sin\gamma}}$$

Entry Velocity Trends, γ=-90°