Ballistic/Lifting Atmospheric Entry

- Straight-line (no gravity) ballistic entry based on altitude, rather than density
- Planetary entries (at least a start)
- Basic equations of planar motion (with lift)
- Lifting equilibrium glide

Terminal Velocity

Full form of ODE -

$$\frac{d\left(v^2\right)}{d\rho} - \frac{h_s}{\beta \sin\gamma} v^2 = \frac{2 g h_s}{\rho}$$

At terminal velocity, $v = \text{constant} \equiv v_T$

$$-\frac{h_s}{\beta \sin\gamma} v_T^2 = \frac{2 g h_s}{\rho}$$

$$\boxed{v_T^2 = \sqrt{-\frac{2 g \beta \sin\gamma}{\rho}}}$$

"Cannon Ball" γ=-90° Ballistic Entry

6.75" diameter sphere, c_D=0.2, V_E=6000 m/sec

	Iron	Aluminum	Balsa Wood
Weight	40 lb	15.6 lb	14.5 oz
β (kg/m^2)	3938	1532	89
ρ_{md} (kg/m^3)	0.555	0.216	0.0125
h_{md} (m)	5600	12,300	32,500
V_{impact} (m/s)	1998	355	0*
V_{term} (m/sec)	251	156	38

*Artifact of assumption that $D \gg mg$

Atmospheric Density with Altitude

Pressure=the integral of the atmospheric density in the column above the reference area

$$P_o = \int_o^\infty \rho g dh = \rho_o g \int_o^\infty e^{-\frac{h}{h_s}} dh = -\rho_o g h_s \left[e^{-\frac{h}{h_s}} \right]_o^\infty$$

$$= -\rho_o g h_s \left[0 - 1 \right]$$

$$\boxed{P_o = \rho_o g h_s}$$

$$\text{Earth: } \rho_o = 1.226 \frac{kg}{m^3}; \; h_s = 7524m;$$

$$P_o(calc) = 90,400 \; Pa; \; P_o(act) = 101,300 \; Pa$$

Nondimensional Ballistic Coefficient

$$\frac{v}{v_e} = \exp\left(\frac{h_s \rho_o}{2\beta \sin\gamma} \frac{\rho}{\rho_o}\right) = \exp\left(\frac{P_o}{2\beta g \sin\gamma} \frac{\rho}{\rho_o}\right)$$

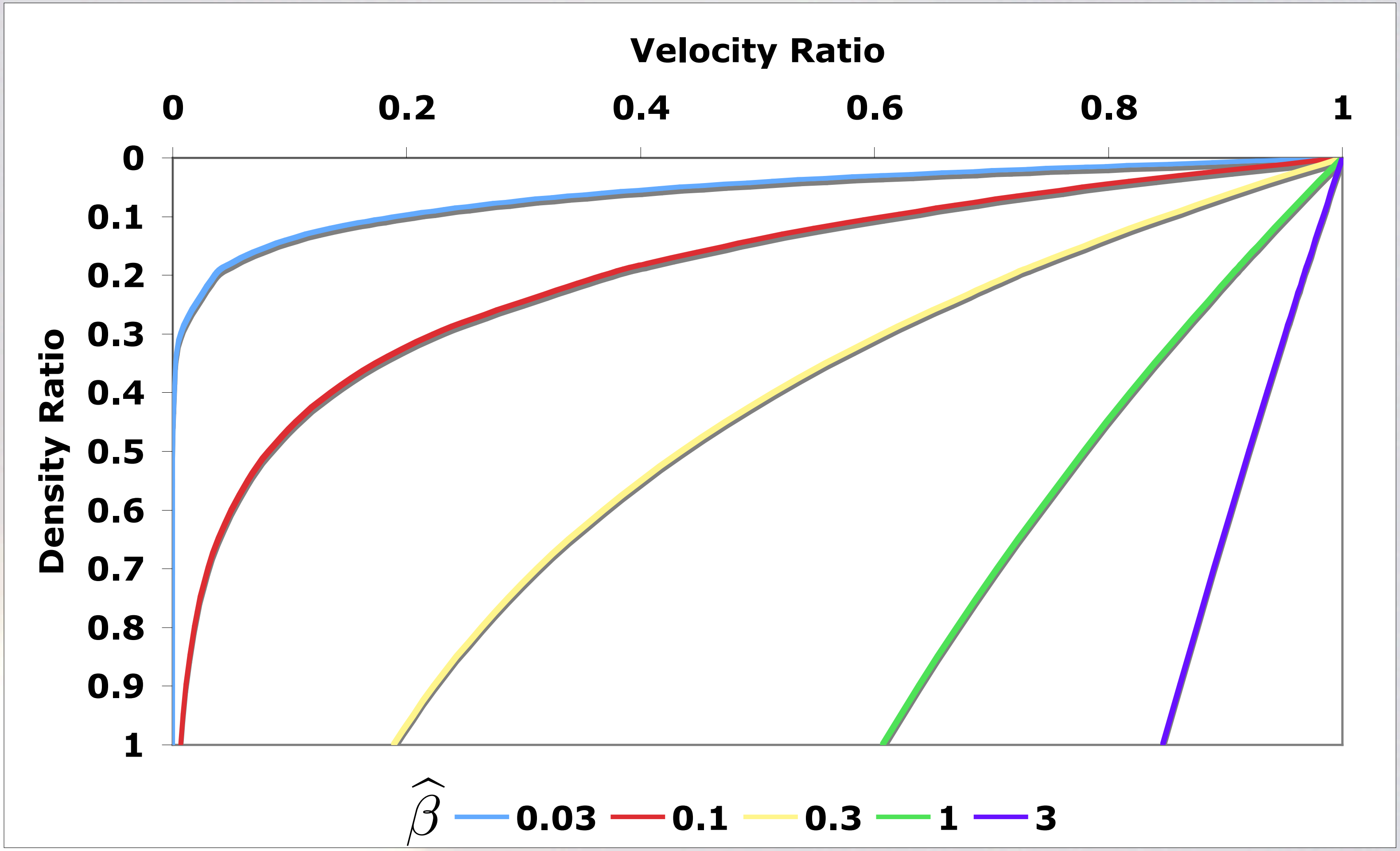
Let $\widehat{\beta} \equiv \dfrac{\beta}{\rho_o h_s} = \dfrac{\beta g}{P_o}$ (Nondimensional form of ballistic coefficient)

Note that we are using the estimated value of $P_o = \rho_o g h_s$, not the actual surface pressure.

$$\boxed{\frac{v}{v_e} = \exp\left(\frac{1}{2\widehat{\beta} \sin\gamma} \frac{\rho}{\rho_o}\right)}$$

$$\beta_{crit} = -\frac{\rho_o h_s}{\sin\gamma} \qquad \boxed{\widehat{\beta}_{crit} = -\frac{1}{\sin\gamma}}$$

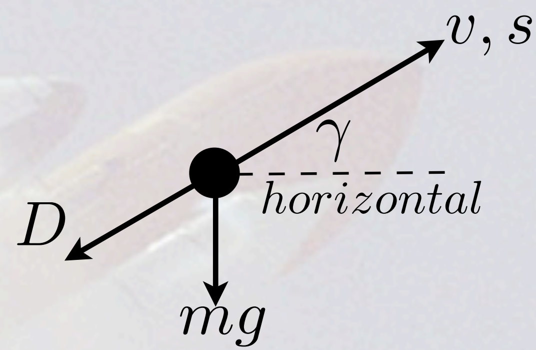
Entry Velocity Trends, γ=-90°



Ballistic Entry, Again

$s =$ distance along the flight path

$$\frac{dv}{dt} = -g\sin\gamma - \frac{D}{m}$$



Again assuming $D \gg g$,

$$\frac{dv}{dt} = -\frac{D}{m} \qquad \text{Drag } D \equiv \frac{1}{2}\rho v^2 A c_D$$

$$\frac{dv}{dt} = -\frac{\rho c_D A}{2m} v^2$$

Separating the variables,

$$\frac{dv}{v^2} = -\frac{\rho}{2\beta} dt$$

Calculating the Entry Velocity Profile

$$\frac{dh}{dt} = v\sin\gamma \Rightarrow dt = \frac{dh}{v\sin\gamma}$$

$$\frac{dv}{v^2} = -\frac{\rho}{2\beta v\sin\gamma}dh \quad \Rightarrow \frac{dv}{v} = -\frac{\rho}{2\beta\sin\gamma}dh$$

$$\frac{dv}{v} = -\frac{\rho_o}{2\beta\sin\gamma}e^{-\frac{h}{hs}}dh$$

$$\int_{v_e}^{v}\frac{dv}{v} = -\frac{\rho_o}{2\beta\sin\gamma}\int_{h_e}^{h}e^{-\frac{h}{hs}}dh$$

$$\ln\frac{v}{v_e} = \frac{\rho_o h_s}{2\beta\sin\gamma}\left[e^{-\frac{h}{hs}}\right]_{h_e}^{h} = \frac{1}{2\widehat{\beta}\sin\gamma}\left[e^{-\frac{h}{hs}} - e^{-\frac{h_e}{hs}}\right]$$

Deriving the Entry Velocity Function

Remember that $e^{-\frac{h_e}{h_s}} = \dfrac{\rho_e}{\rho_o} \approx 0$

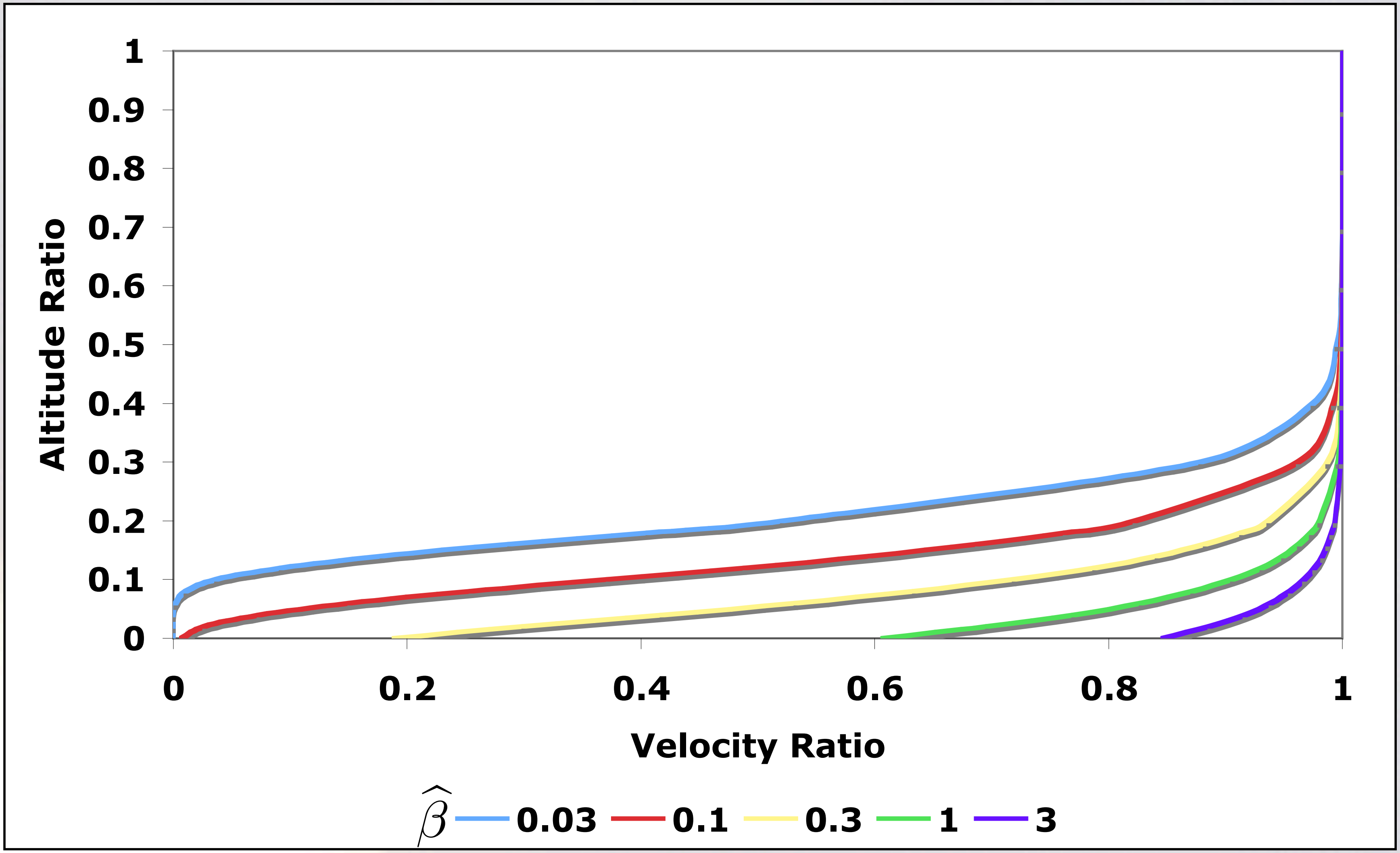
$$\frac{v}{v_e} = \exp\left[\frac{1}{2\widehat{\beta}\sin\gamma} e^{-\frac{h}{h_s}}\right]$$

We have a parametric entry equation in terms of nondimensional velocity ratios, ballistic coefficient, and altitude. To bound the nondimensional altitude variable between 0 and 1, rewrite as

$$\boxed{\frac{v}{v_e} = \exp\left[\frac{1}{2\widehat{\beta}\sin\gamma} e^{-\frac{h}{h_e}\frac{h_e}{h_s}}\right]}$$

$\dfrac{h_e}{h_s}$ and $\widehat{\beta}$ are the only variables that relate to a specific planet

Earth Entry, γ=-90°



Deceleration as a Function of Altitude

Start with

$$\frac{v}{v_e} = \exp\left[\frac{1}{2\widehat{\beta}\sin\gamma} e^{-\frac{h}{he}\frac{h_e}{h_s}}\right]$$

Let $B \equiv \dfrac{1}{2\widehat{\beta}\sin\gamma}$

$$\frac{v}{v_e} = \exp\left(Be^{-\frac{h}{h_s}}\right)$$

$$\frac{d}{dt}\left(\frac{v}{v_e}\right) = \exp\left(Be^{-\frac{h}{h_s}}\right)\frac{d}{dt}\left(Be^{-\frac{h}{h_s}}\right)$$

$$\frac{dv}{dt} = v_e \exp\left(Be^{-\frac{h}{h_s}}\right)\frac{-B}{h_s}\left(e^{-\frac{h}{h_s}}\right)\frac{dh}{dt}$$

$$\frac{dh}{dt} = v\sin\gamma \quad = v_e \sin\gamma \exp Be^{-\frac{h}{h_s}}$$

Parametric Deceleration

$$\frac{dv}{dt} = v_e \exp\left(Be^{-\frac{h}{h_s}}\right) \frac{-B}{h_s} \left(e^{-\frac{h}{h_s}}\right) v_e \sin\gamma \exp\left(Be^{-\frac{h}{h_s}}\right)$$

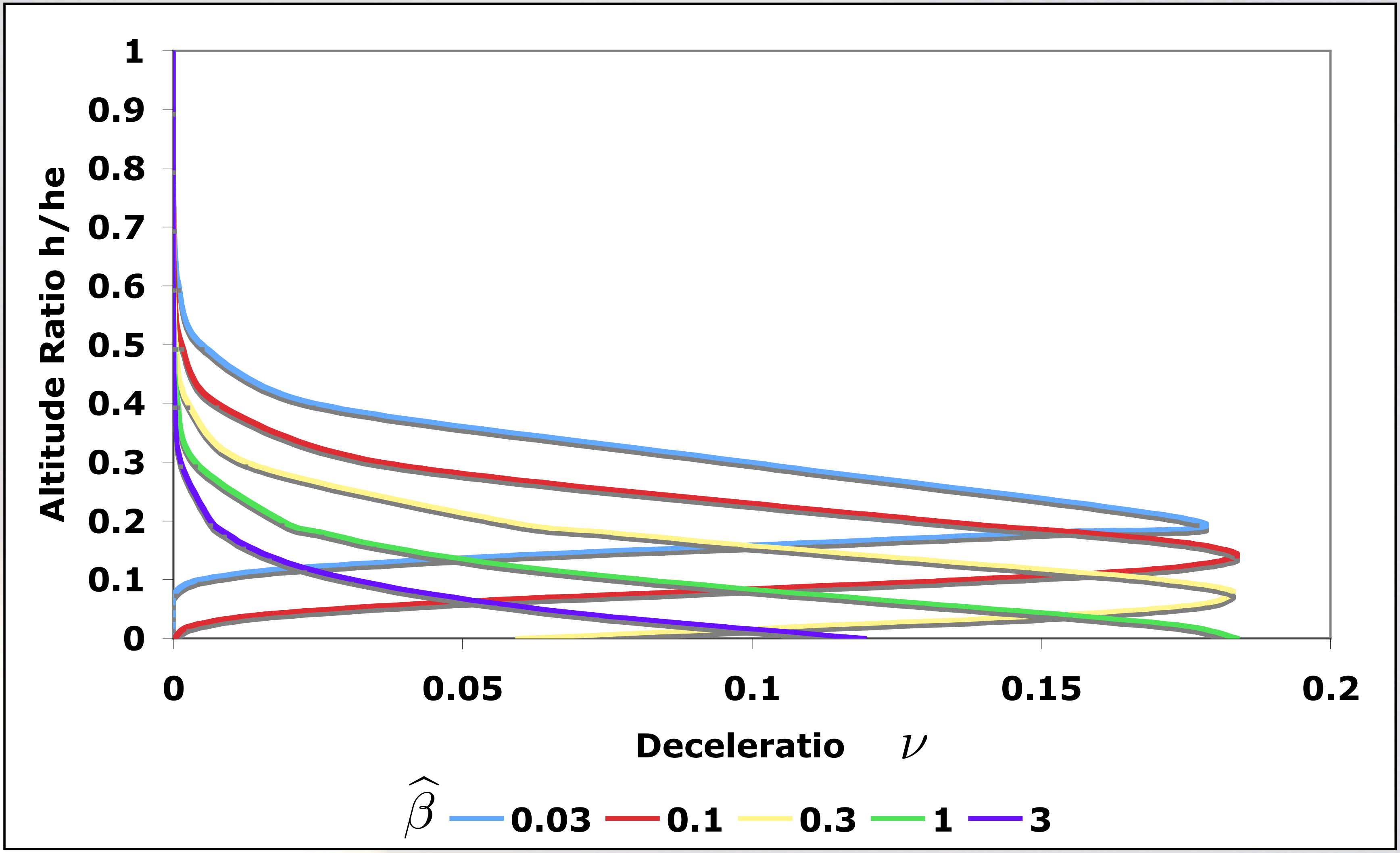
$$\frac{dv}{dt} = \frac{-Bv_e^2}{h_s} \sin\gamma \left(e^{-\frac{h}{h_s}}\right) \exp\left(2Be^{-\frac{h}{h_s}}\right)$$

$$\frac{dv}{dt} = \frac{-v_e^2}{2h_s\widehat{\beta}} \left(e^{-\frac{h}{h_s}}\right) \exp\left(\frac{1}{\widehat{\beta}\sin\gamma} e^{-\frac{h}{h_s}}\right)$$

$$\text{Let } n_{ref} \equiv \frac{v_e^2}{h_s},\ \nu \equiv \frac{dv/dt}{n_{ref}},\ \varphi \equiv \frac{h_e}{h_s}$$

$$\boxed{\nu = \frac{-1}{2\widehat{\beta}} \left(e^{-\varphi\frac{h}{h_e}}\right) \exp\left(\frac{1}{\widehat{\beta}\sin\gamma} e^{-\varphi\frac{h}{h_e}}\right)}$$

Nondimensional Deceleration, γ=-90°



Deceleration Equations

Nondimensional Form

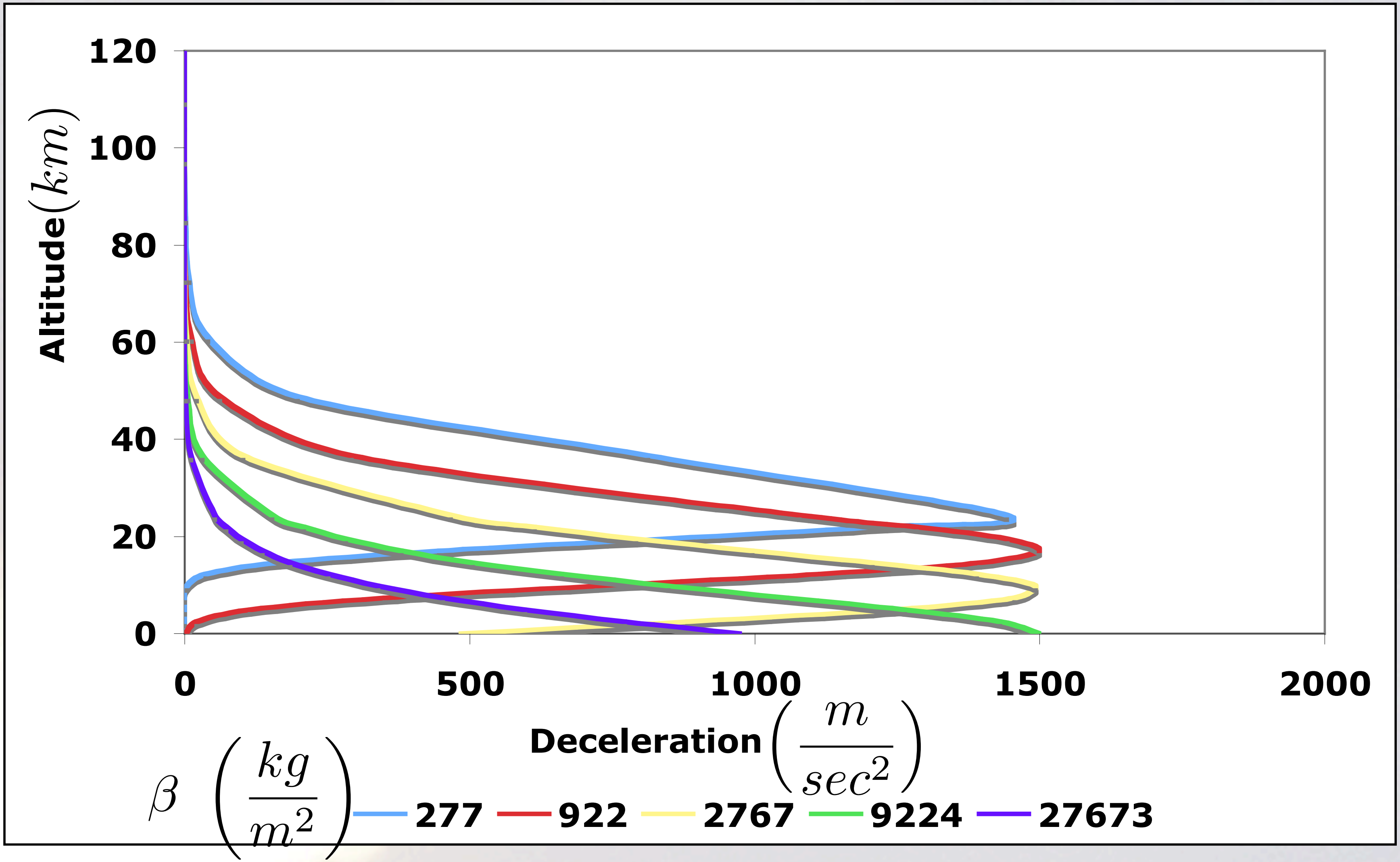
$$\boxed{\nu = \frac{-1}{2\widehat{\beta}}\left(e^{-\varphi\frac{h}{h_e}}\right)\exp\left(\frac{1}{\widehat{\beta}\sin\gamma}e^{-\varphi\frac{h}{h_e}}\right)}$$

Dimensional Form

$$\boxed{n = \frac{-\rho_o V_e^2}{2\beta}\left(e^{-\frac{h}{h_s}}\right)\exp\left(\frac{\rho_o h_s}{\beta\sin\gamma}e^{-\frac{h}{h_s}}\right)}$$

Note that these equations result in values <0 (reflecting deceleration) - graphs are absolute values of deceleration for clarity.

Dimensional Deceleration, γ=-90°



Altitude of Maximum Deceleration

Returning to shorthand notation for deceleration

$$\nu = -B\sin\gamma\left(e^{-\frac{h}{h_s}}\right)\exp\left(2Be^{-\frac{h}{h_s}}\right)$$

Let $\eta \equiv \dfrac{h}{h_s}$

$$\nu = -B\sin\gamma\left(e^{-\eta}\right)\exp\left(2Be^{-\eta}\right)$$

$$\frac{d\nu}{d\eta} = -B\sin\gamma\left[\frac{d}{d\eta}\left(e^{-\eta}\right)\exp\left(2Be^{-\eta}\right) + \left(e^{-\eta}\right)\frac{d}{d\eta}\exp\left(2Be^{-\eta}\right)\right]$$

$$\frac{d\nu}{d\eta} = -B\sin\gamma\left[-\left(e^{-\eta}\right)\exp\left(2Be^{-\eta}\right) + \left(e^{-\eta}\right)\left(-2Be^{-\eta}\right)\exp\left(2Be^{-\eta}\right)\right]$$

$$\frac{d\nu}{d\eta} = B\sin\gamma e^{-\eta}\exp\left(2Be^{-\eta}\right)\left[1 + \left(2Be^{-\eta}\right)\right] = 0$$

Altitude of Maximum Deceleration

$$1+\left(2Be^{-\eta}\right)=0 \quad \Rightarrow e^{\eta}=-2B$$

$$\eta_{n_{max}}=\ln\left(-2B\right)$$

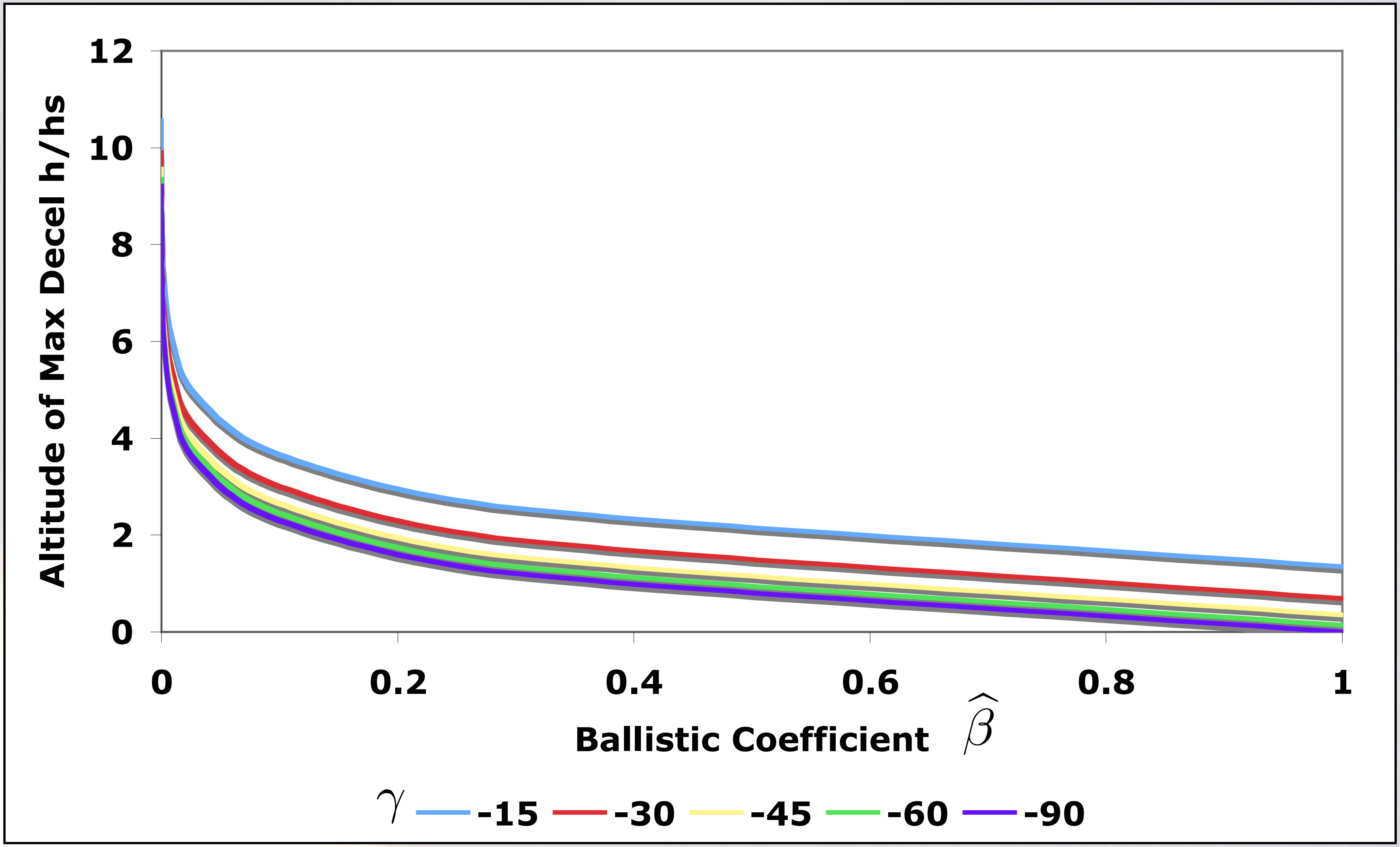
$$\boxed{\eta_{n_{max}}=\ln\left(\frac{-1}{\widehat{\beta}\sin\gamma}\right)}$$

Converting from parametric to dimensional form gives

$$\boxed{h_{n_{max}}=h_s\ln\left(\frac{-\rho_o h_s}{\beta\sin\gamma}\right)}$$

Altitude of maximum deceleration is independent of entry velocity!

Altitude of Maximum Deceleration



Magnitude of Maximum Deceleration

Start with the equation for acceleration -

$$\nu = \frac{-1}{2\widehat{\beta}} e^{-\eta} \exp\left(\frac{1}{\widehat{\beta}\sin\gamma} e^{-\eta}\right)$$

and insert the value of η at the point of maximum deceleration

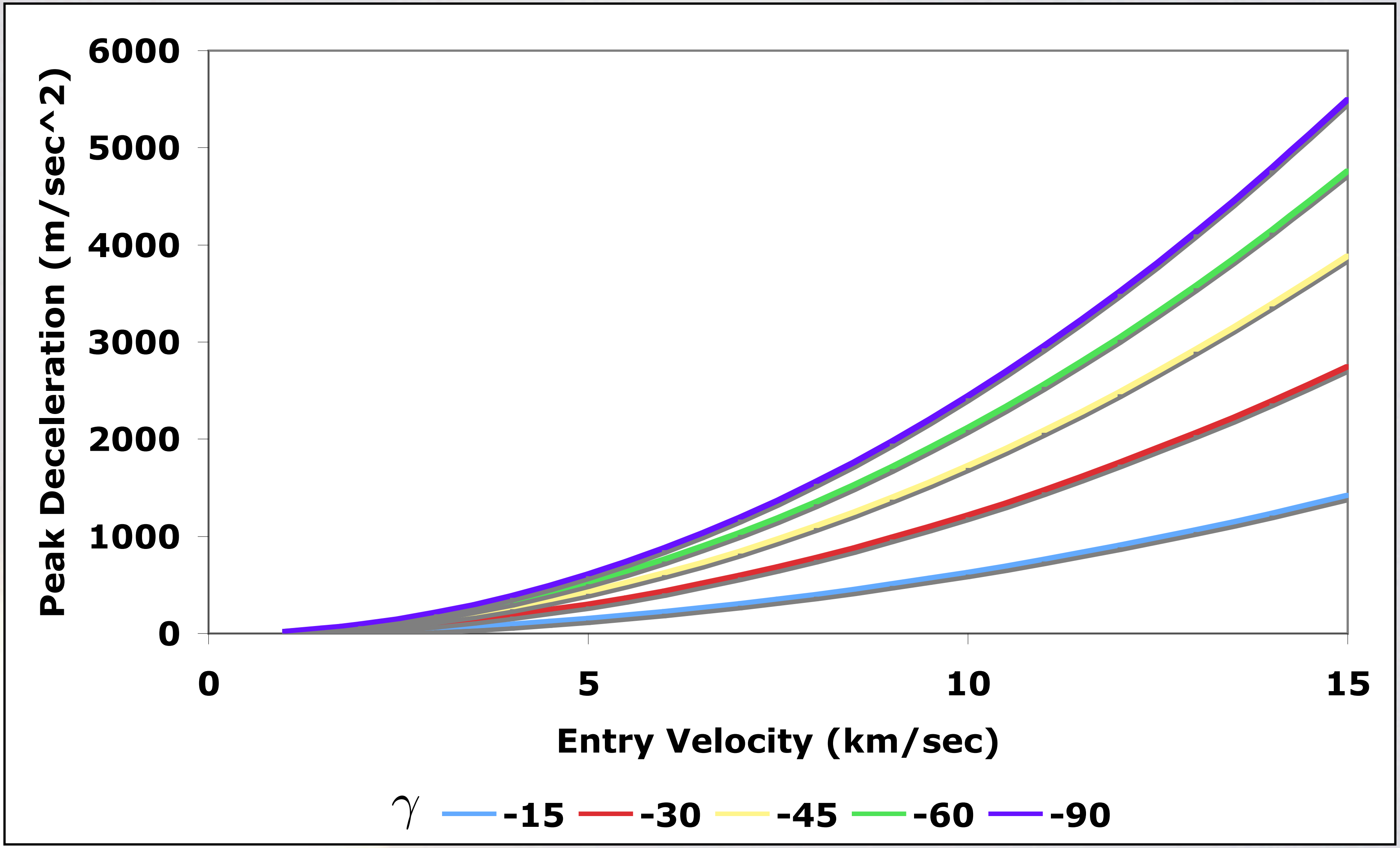
$$\eta_{n_{max}} = \ln\left(\frac{-1}{\widehat{\beta}\sin\gamma}\right) \Rightarrow e^{-\eta} = -\widehat{\beta}\sin\gamma$$

$$\nu_{n_{max}} = \frac{-1}{2\widehat{\beta}}\left(-\widehat{\beta}\sin\gamma\right)\exp\left(\frac{-\widehat{\beta}\sin\gamma}{\widehat{\beta}\sin\gamma}\right) \Rightarrow \nu_{n_{max}} = \frac{\sin\gamma}{2e}$$

$$\boxed{n_{max} = \frac{v_e^2}{h_s}\frac{\sin\gamma}{2e}}$$

Maximum deceleration is not a function of ballistic coefficient!

Peak Ballistic Deceleration for Earth Entry



Velocity at Maximum Deceleration

Start with the equation for velocity

$$\frac{v}{v_e} = \exp\left[\frac{1}{2\widehat{\beta}\sin\gamma}e^{-\eta}\right]$$

and insert the value of η at the point of maximum deceleration

$$\eta_{n_{max}} = \ln\left(\frac{-1}{\widehat{\beta}\sin\gamma}\right) \Rightarrow e^{-\eta} = -\widehat{\beta}\sin\gamma$$

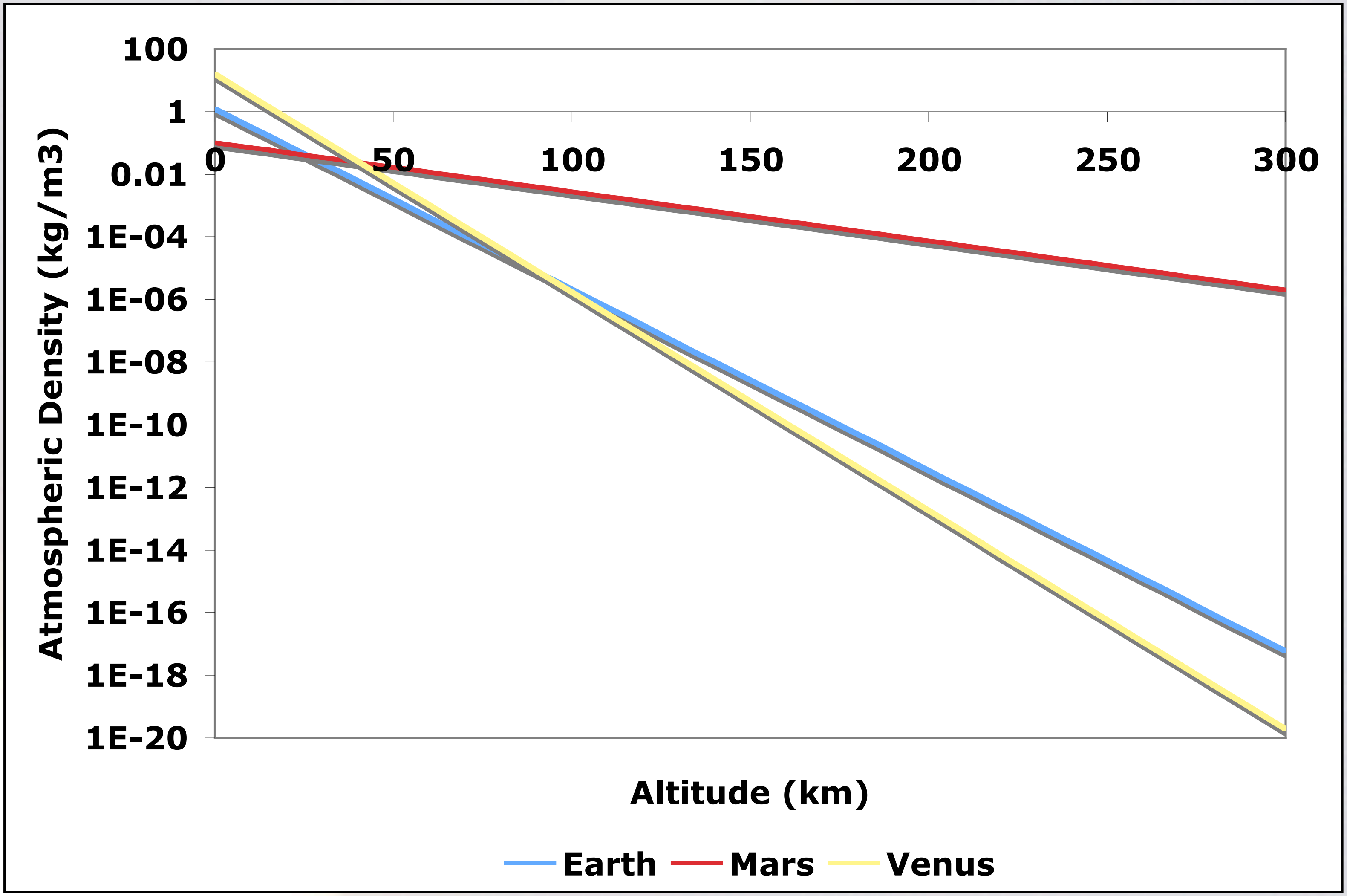
$$\frac{v}{v_e} = \exp\left[\frac{-\widehat{\beta}\sin\gamma}{2\widehat{\beta}\sin\gamma}\right] \Rightarrow \boxed{v_{n_{max}} = \frac{v_e}{\sqrt{e}} \cong 0.606 v_e}$$

Velocity at maximum deceleration is independent of everything except v_e

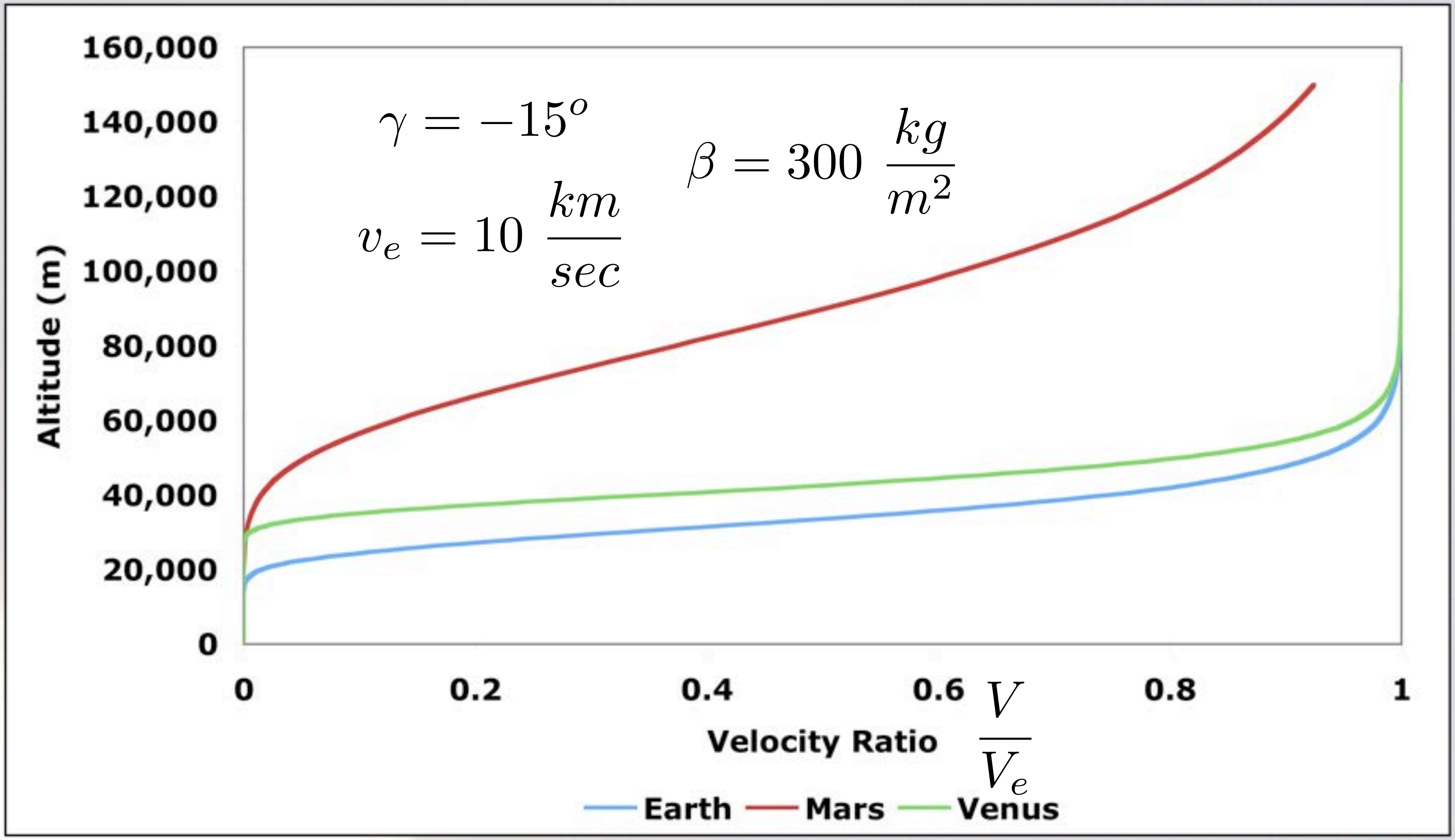
Planetary Entry - Physical Data

	Radius (km)	μ (km^3/sec^2)	ρ_o (kg/m^3)	h_s (km)	v_{esc} (km/sec)
Earth	6378	398,604	1.225	7.524	11.18
Mars	3393	42,840	0.0993	27.70	5.025
Venus	6052	325,600	16.02	6.227	10.37

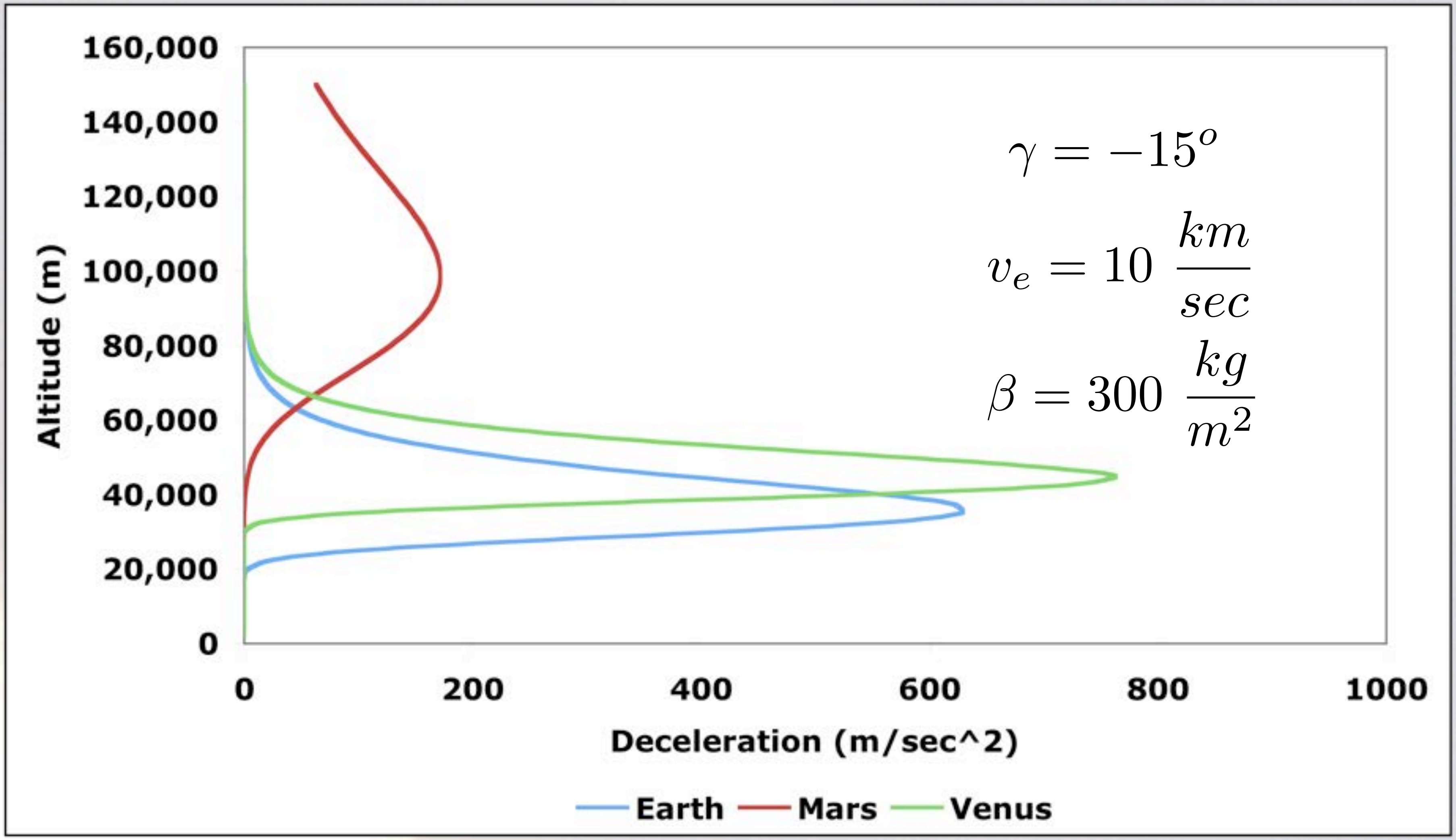
Comparison of Planetary Atmospheres



Planetary Entry Profiles



Planetary Entry Deceleration Comparison

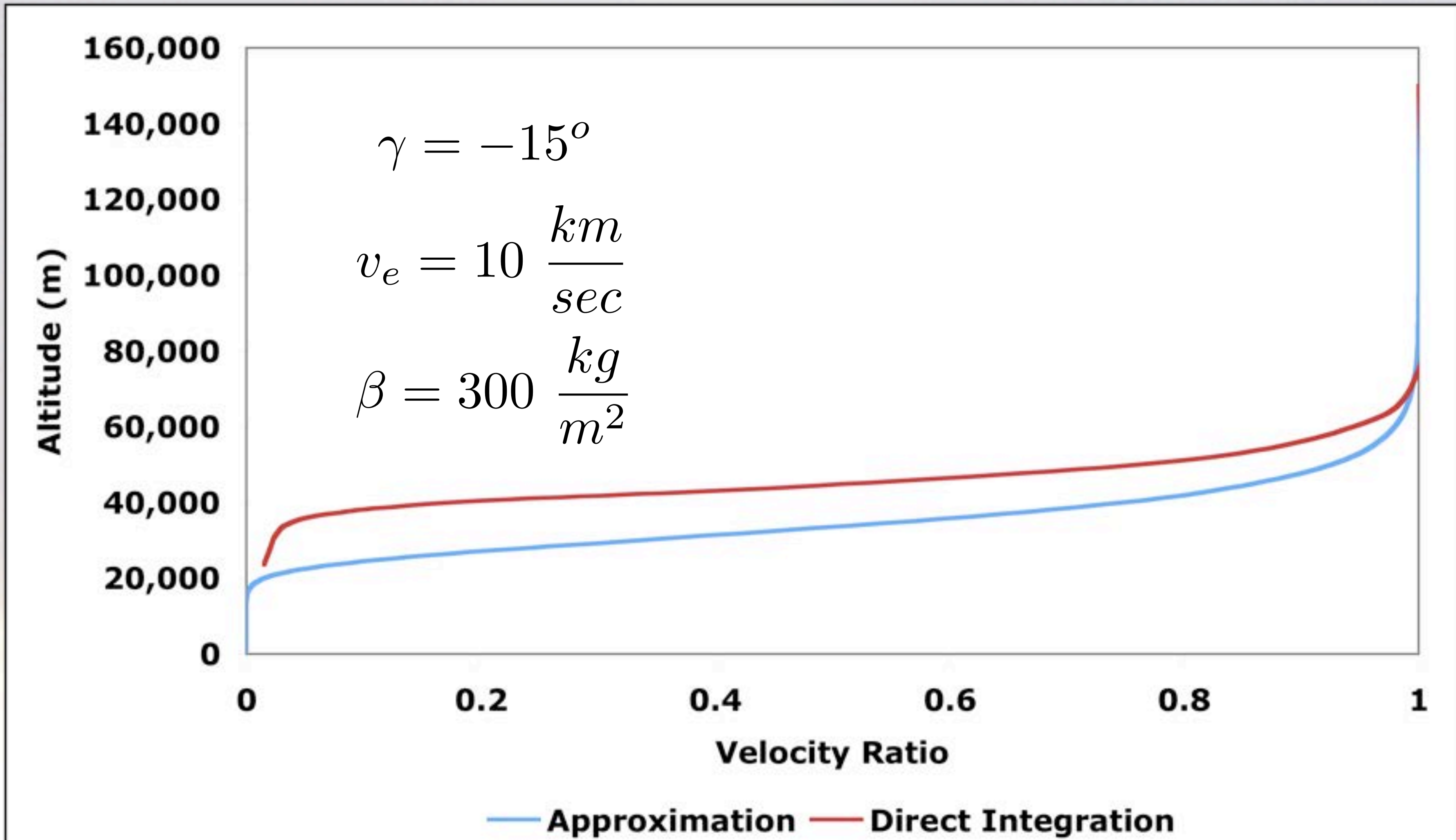


$$\gamma = -15^o$$

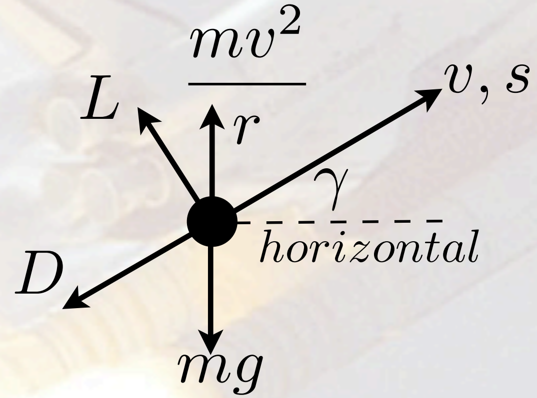
$$v_e = 10 \, \frac{km}{sec}$$

$$\beta = 300 \, \frac{kg}{m^2}$$

Check on Approximation Formulas



Lifting Entry – Free-Body Diagram



Dynamics of Lifting Entry

Along the velocity vector,

$$m\frac{dv}{dt} = m\frac{v^2}{r}\sin\gamma - mg\sin\gamma - D$$

Perpendicular to the velocity vector,

$$mv\frac{d\gamma}{dt} = L + m\frac{v^2}{r}\cos\gamma - mg\cos\gamma$$

(unbalanced lift rotates flight path angle)

Equations of Planar Lifting Entry

$$\frac{dv}{dt} = \left(\frac{v^2}{r} - g\right)\sin\gamma - \frac{D}{m}$$

$$v\frac{d\gamma}{dt} = \frac{L}{m} + \left(\frac{v^2}{r} - g\right)\cos\gamma$$

Equilibrium Glide

- Forces perpendicular to velocity vector are balanced

$$\Longrightarrow \frac{d\gamma}{dt} = 0; \;\; \gamma = \text{constant}$$

- Typically very shallow glide

$$\Longrightarrow \text{assume } \gamma \rightarrow 0; \;\; \sin(\gamma) \rightarrow 0; \;\; \cos(\gamma) \rightarrow 1$$

Equilibrium Glide Equations

$$\frac{dv}{dt} = -\frac{D}{m}$$

$$D = \frac{1}{2}\rho v^2 c_D A$$

$$\rho = \rho_o e^{-\frac{h}{h_s}}$$

$$\frac{dv}{dt} = -\frac{1}{2}\rho_o v^2 \frac{c_D A}{m} e^{-\frac{h}{h_s}}$$

$$0 = \frac{L}{m} + \left(\frac{v^2}{r} - g\right)$$

$$\frac{v^2}{r} - g = -\frac{1}{2}\rho_o v^2 \frac{c_L A}{m} e^{-\frac{h}{h_s}}$$

Dynamics Perpendicular to Velocity

$$\frac{c_L}{c_D} = \frac{L}{D}$$

L/D set by vehicle aerodynamics, flight velocity, and angle of attack (assumed constant)

$$\frac{v^2}{r} - g = -\frac{1}{2}\rho_o v^2 \frac{L}{D}\frac{c_D A}{m} e^{-\frac{h}{h_s}}$$

$$\frac{v^2}{r} = -\frac{1}{2}\rho_o v^2 \frac{L/D}{\beta} e^{-\frac{h}{h_s}} + g$$

$$v^2 = -\frac{1}{2}\rho_o r v^2 \frac{L/D}{\beta} e^{-\frac{h}{h_s}} + gr$$

More Lift-Direction Dynamics

$$\text{Let } e^{-\frac{h}{h_s}} \equiv \sigma \quad \left(= \text{density ratio} \equiv \frac{\rho}{\rho_o} \right)$$

$$v^2 + \frac{1}{2}\rho_o r v^2 \frac{L/D}{\beta}\sigma = gr$$

$$v^2 \left(1 + \frac{1}{2}\rho_o r \frac{L/D}{\beta}\sigma \right) = gr$$

$$v = \sqrt{\frac{gr}{1 + \frac{\rho_o r \sigma (L/D)}{2\beta}}}$$

Velocity during Entry

$$v_{c_o} = \sqrt{\frac{\mu}{r_o}} = \sqrt{g_o r_o} \qquad (\mu = g_o r_o^2)$$

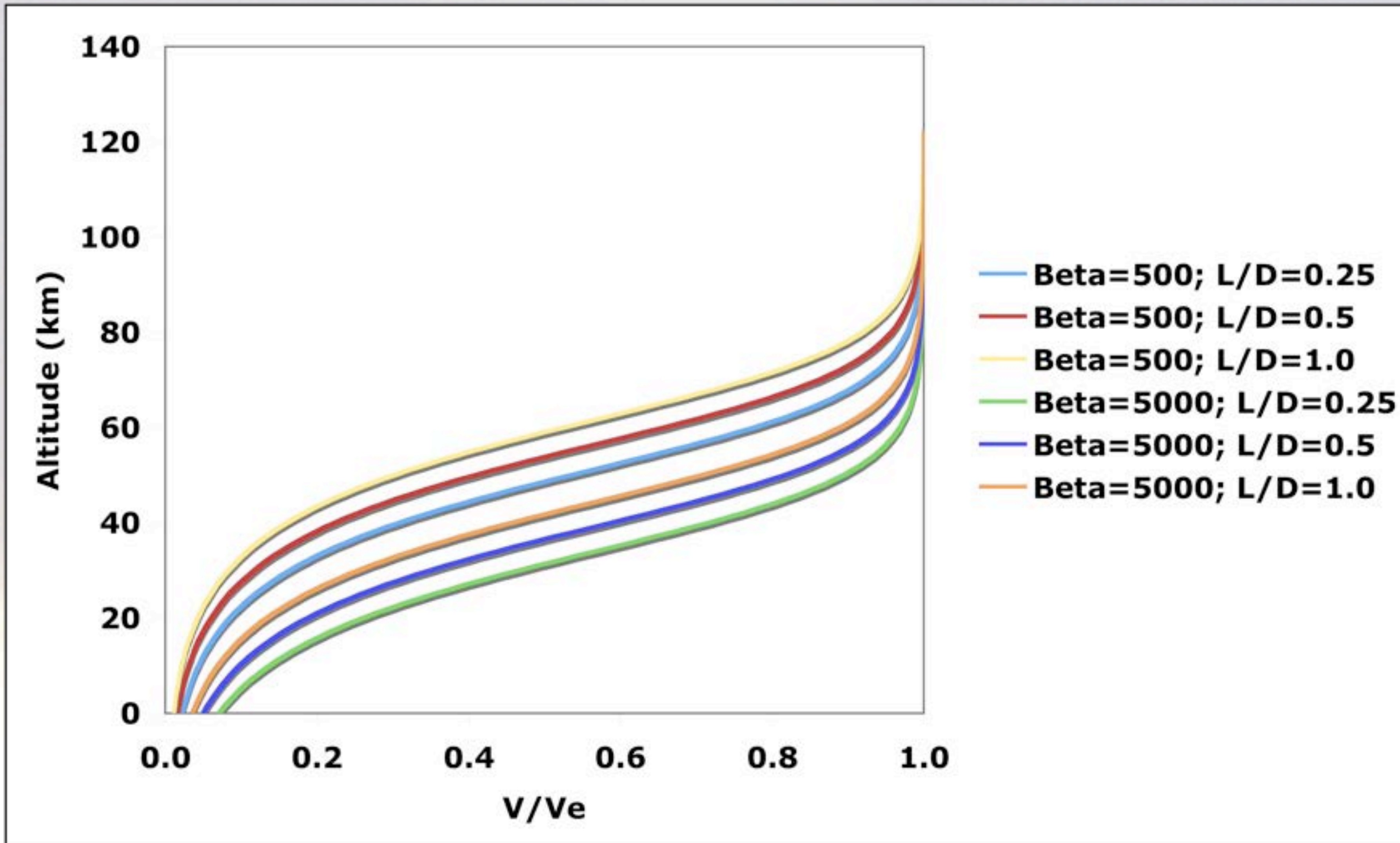
$$\frac{v}{v_{c_o}} = \sqrt{\frac{1}{1 + \frac{\rho_o r_o \sigma (L/D)}{2\beta}}}$$

$$v_{c_o} \cong v_e \quad \text{(within 1-2\% for Earth)}$$

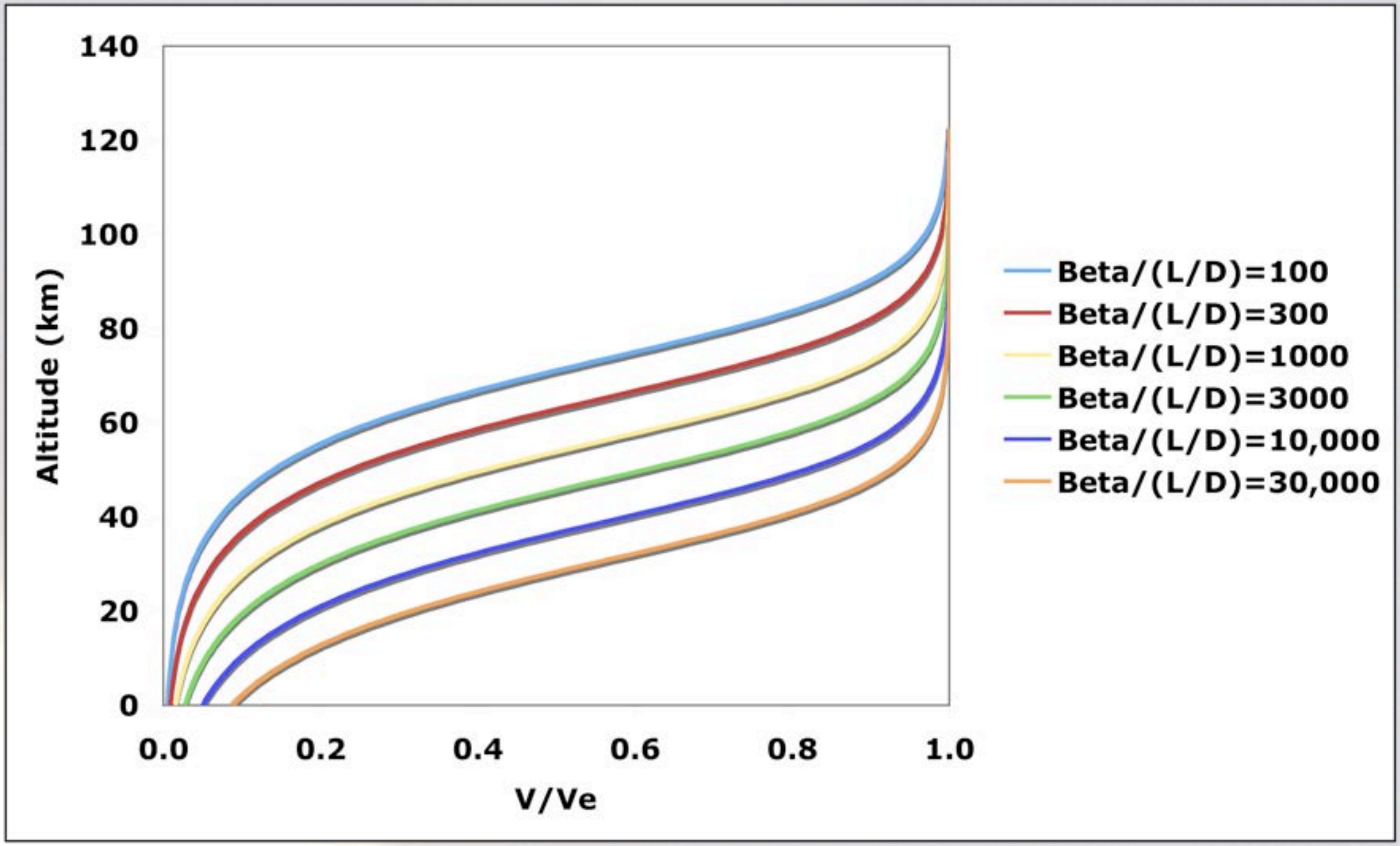
$$\boxed{\frac{v}{v_e} = \left[1 + \frac{\rho_o r_o}{2\beta} \frac{L}{D} e^{-\frac{h}{h_s}}\right]^{-\frac{1}{2}}}$$

Entry trajectory as a function of altitude, ratio $\left(\frac{\beta}{L/D}\right)$

Equilibrium Glide Velocity Trends



Trends with “Lifting Ballistic Coefficient”



Nondimensional Form of Equation

$$\frac{v}{v_e} = \left[1 + \frac{\rho_o r_o}{2\beta}\frac{L}{D}e^{-\frac{h}{h_s}}\right]^{-\frac{1}{2}}$$

$$\text{Remember } \widehat{\beta} \equiv \frac{\beta}{\rho_o h_s}$$

$$\frac{v}{v_e} = \left[1 + \frac{\rho_o h_s}{2\beta}\frac{r_o}{h_s}\frac{L}{D}e^{-\frac{h}{h_s}}\right]^{-\frac{1}{2}}$$

$$\boxed{\frac{v}{v_e} = \left[1 + \frac{1}{2\widehat{\beta}}\frac{r_o}{h_s}\frac{L}{D}e^{-\frac{h}{h_e}\frac{h_e}{h_s}}\right]^{-\frac{1}{2}}}$$

Deceleration

$$\frac{L}{m} = g - \frac{v^2}{r} = g - \frac{gv^2}{gr} = g\left[1 - \left(\frac{v}{v_{c_o}}\right)^2\right]$$

$$\frac{dv}{dt} = -\frac{D}{m} = -\frac{L}{L/D}\frac{1}{m} = -\frac{1}{L/D}\frac{L}{m}$$

$$\frac{dv}{dt} = -\frac{g}{L/D}\left[1 - \left(\frac{v}{v_{c_o}}\right)^2\right]$$

Deceleration

Let $n \equiv \frac{1}{g}\frac{dv}{dt}$ (deceleration in g's)

$$n = -\frac{1}{L/D}\left[1 - \left(\frac{v}{v_{c_o}}\right)^2\right]$$

$$n = -\frac{1}{L/D}\left[1 - \frac{1}{1 + \frac{\rho_o r_o}{2\beta}\frac{L}{D}e^{-\frac{h}{h_s}}}\right]$$

$$\left(\text{Note: } 1 - \frac{1}{1+K} = \frac{1+K}{1+K} - \frac{1}{1+K} = \frac{K}{1+K}\right)$$

Deceleration

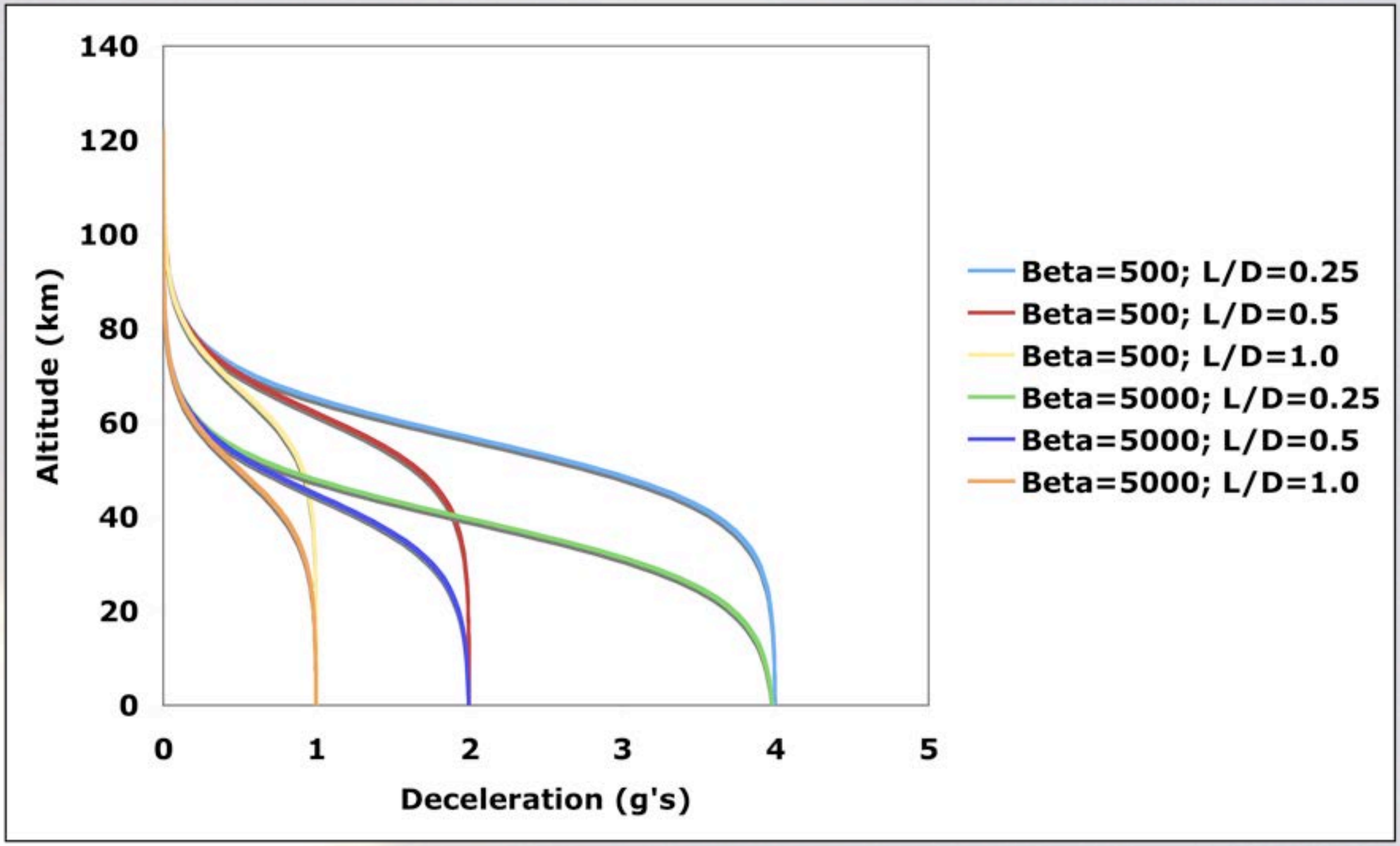
$$n = -\frac{1}{L/D}\left[\frac{\frac{\rho_o r_o}{2\beta}\frac{L}{D}e^{-\frac{h}{h_s}}}{1+\frac{\rho_o r_o}{2\beta}\frac{L}{D}e^{-\frac{h}{h_s}}}\right]$$

$$n = -\frac{\frac{\rho_o r_o}{2\beta}e^{-\frac{h}{h_s}}}{1+\frac{\rho_o r_o}{2\beta}\frac{L}{D}e^{-\frac{h}{h_s}}}$$

$$\boxed{n = \frac{-1}{\frac{2\beta}{\rho_o r_o}e^{+\frac{h}{h_s}}+\frac{L}{D}}}$$

n monotonically increases with decreasing altitude

Deceleration Trends with Altitude



Limiting Deceleration

$$n_{limit} = \frac{-1}{\frac{2\beta}{\rho_o r_o} + \frac{L}{D}}$$

$$\beta \sim \mathrm{O}(10^3);\ \rho_o \sim \mathrm{O}(1);\ r_o \sim \mathrm{O}(10^6) \Longrightarrow \frac{2\beta}{\rho_o r_o} \sim \mathrm{O}(10^{-3})$$

$$\boxed{n_{limit} \cong \frac{-1}{L/D}}$$

Lift significantly moderates peak g's on entry (L/D=0.25 --> n_{limit}=4 g's)

Time for Entry

$$\frac{dv}{dt} = -\frac{g}{L/D}\left[1 - \left(\frac{v}{v_{c_o}}\right)^2\right]$$

$$dt = \frac{-(L/D)dv}{g\left[1 - \left(\frac{v}{v_{c_o}}\right)^2\right]}$$

$$\int_0^t dt = \int_{v_e}^0 \frac{-(L/D)dv}{g\left[1 - \left(\frac{v}{v_{c_o}}\right)^2\right]}$$

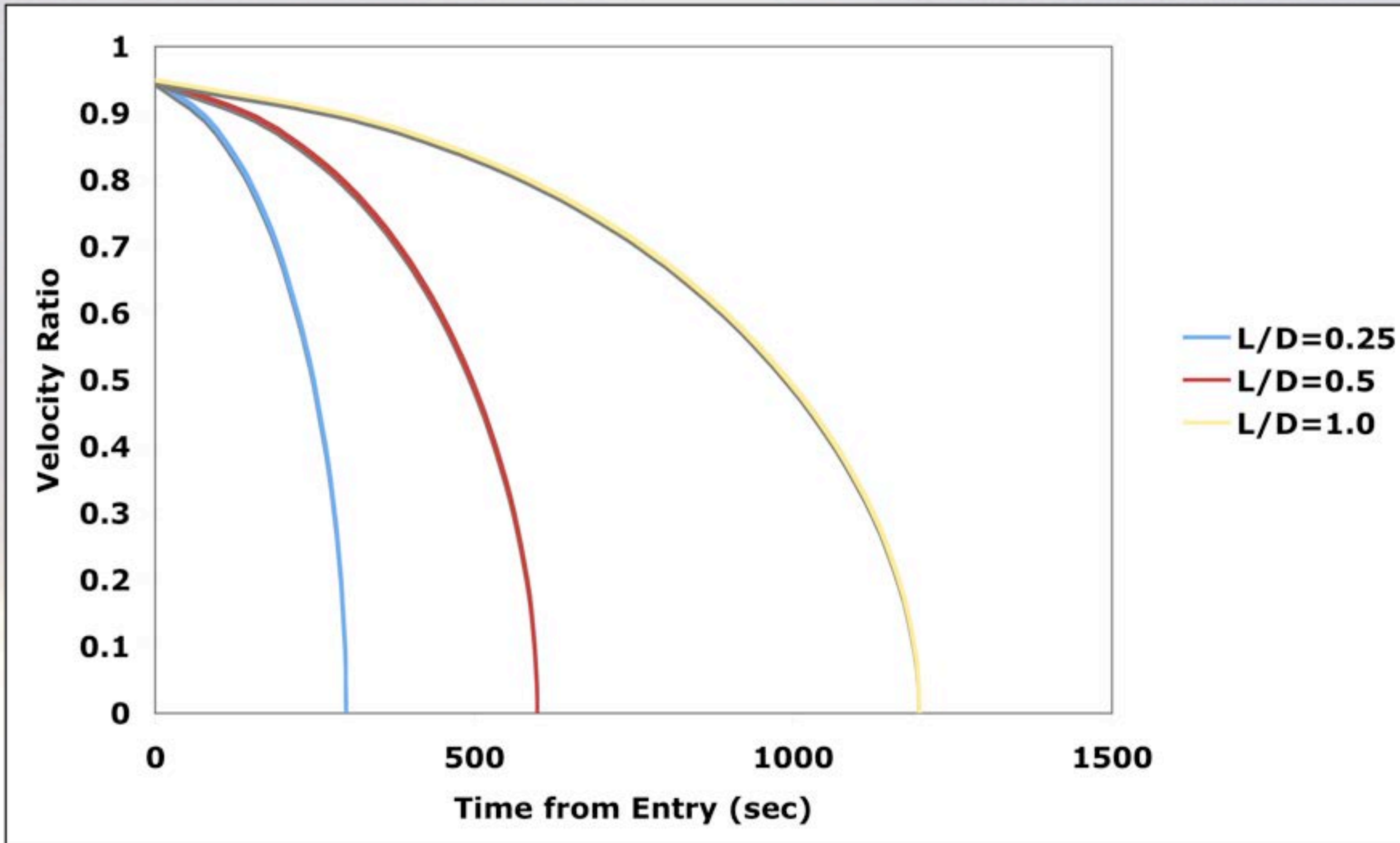
Time for Entry

$$\Delta t = \frac{1}{2}\sqrt{\frac{r_o}{g_o}}\frac{L}{D}\ln\frac{1+\left(\frac{v}{v_{co}}\right)^2}{1-\left(\frac{v}{v_{co}}\right)^2}$$

$$\text{Time for entry} \propto \frac{L}{D}$$

Not a function of β

Time From Entry Interface



Distance Along Flight Path

For shallow entry, $\frac{ds}{dt} \cong v \qquad ds = v\ dt$

$$ds = \frac{-(L/D)v\ dv}{g\left[1 - \left(\frac{v}{v_{c_o}}\right)^2\right]}$$

Let $u \equiv 1 - \left(\frac{v}{v_{c_o}}\right)^2 \qquad du = \frac{-2v}{v_{c_o}^2}dv \quad v\ dv = \frac{-v_{c_o}^2}{2}du$

$$\int ds = \frac{L/D}{2g}\int \frac{v_{c_o}^2 du}{u}$$

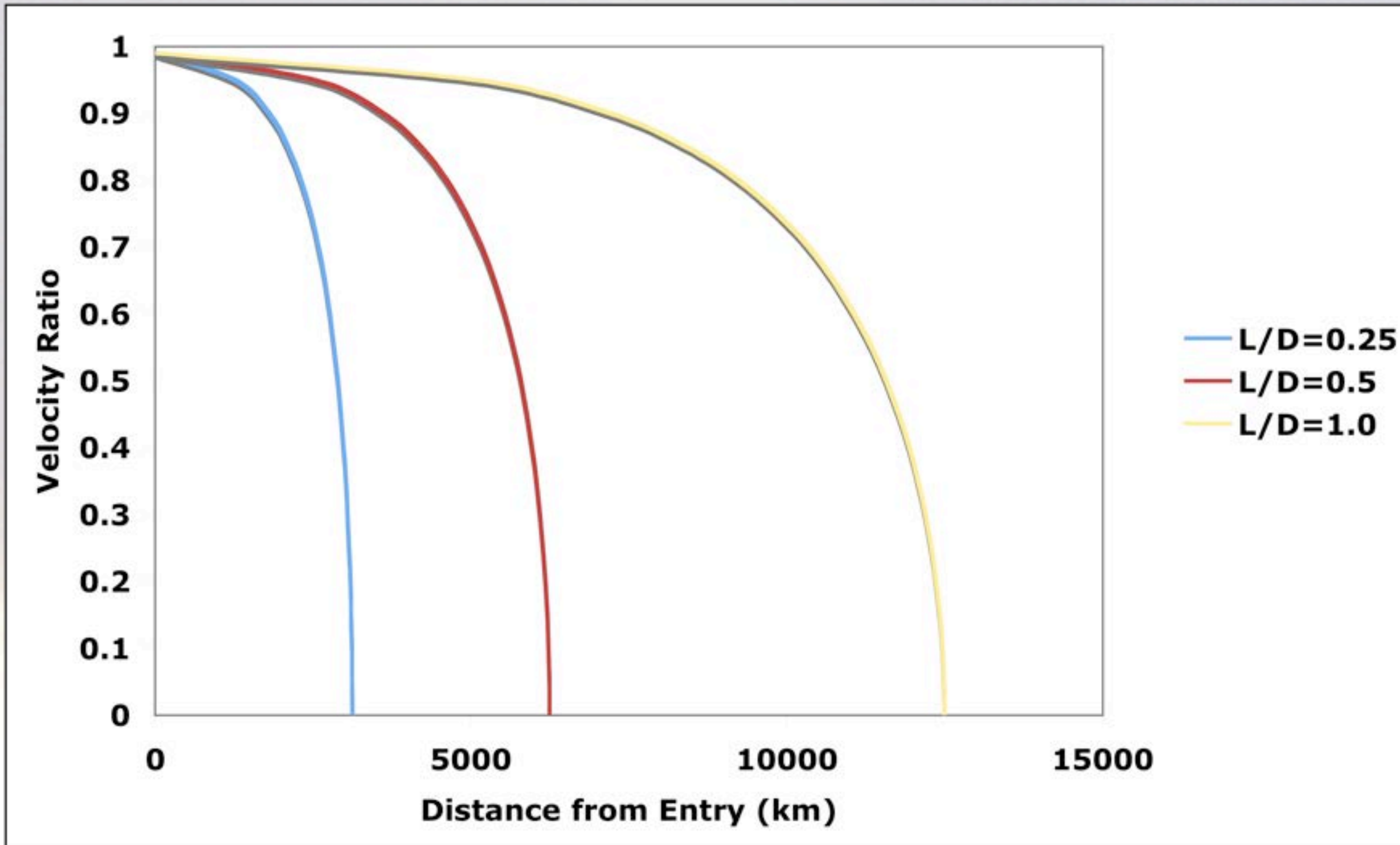
Distance Along Flight Path

$$\Delta s = \frac{(L/D)v_{c_o}^2}{2g}\ln u = \frac{(L/D)g_o r_o}{2g_o}\ln\left[1-\left(\frac{v}{v_{c_o}}\right)^2\right]$$

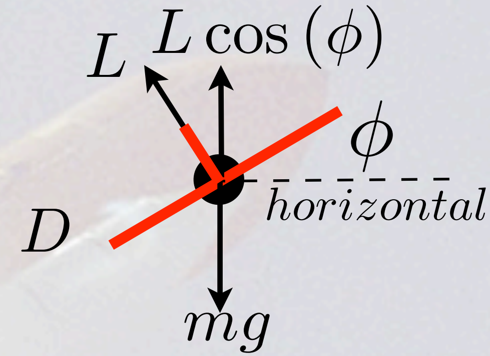
$$\boxed{\Delta s = \frac{r_o}{2}\frac{L}{D}\ln\left[1-\left(\frac{v}{v_{c_o}}\right)^2\right]}$$

Not a function of β or g

Distance from Entry



Bank Angle



$$\phi \equiv \text{Bank Angle}$$

$$\text{Level turn: } L\cos\phi = mg$$

$$\frac{L}{mg} = \frac{1}{\cos\phi} \equiv n_{bank}$$

(g's you pull in the banked turn)

Crossrange (without derivations)

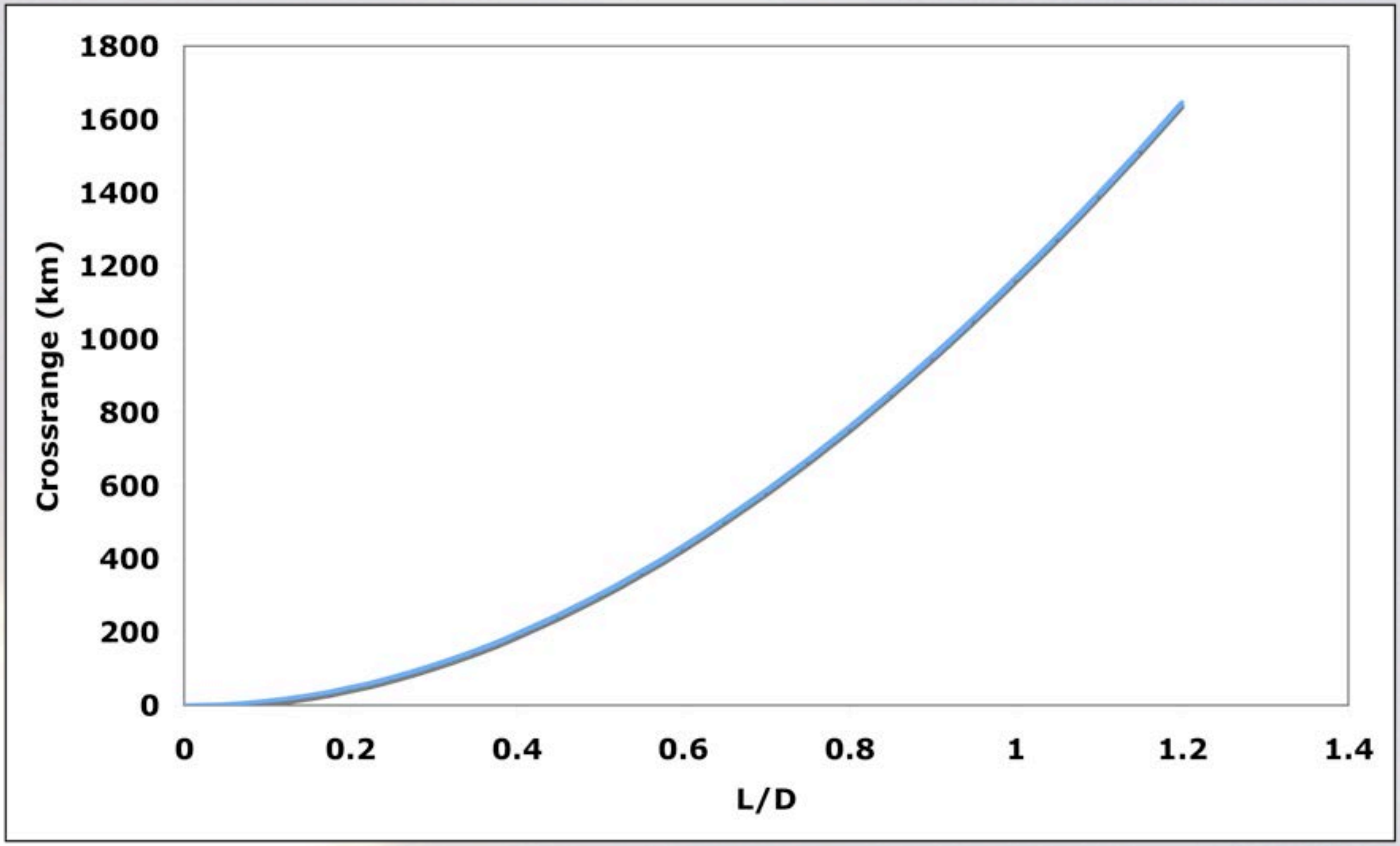
- Use of lift to deviate laterally from planar groundtrack
- Optimum bank angle to maximize crossrange

$$\phi_{opt} \cong \cot^{-1}\sqrt{1 + 0.106\left(\frac{L}{D}\right)^2}$$

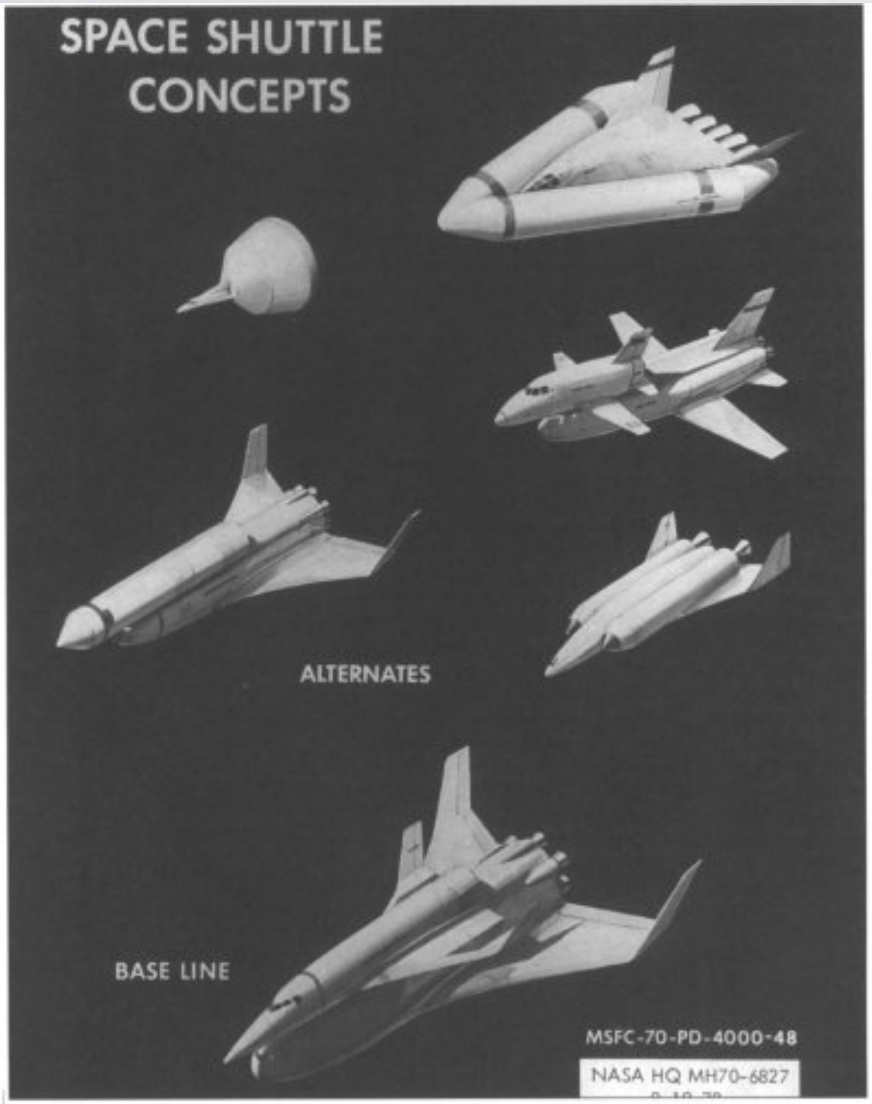
- Maximum achievable crossrange

$$y_{max} \cong \frac{r_o}{5.2}\left(\frac{L}{D}\right)^2 \frac{1}{\sqrt{1 + 0.106\left(\frac{L}{D}\right)^2}}$$

Crossrange Based on L/D



Early Shuttle Design Configurations



- Delta wing configuration
  - High hypersonic lift
  - High landing velocity
  - High crossrange
- Straight-wing configuration
  - High subsonic lift
  - Low(er) landing velocity
  - Low crossrange