

Lifting Entry (continued)

- Basic planar dynamics of motion, again
- Yet another equilibrium glide
- Hypersonic phugoid motion
- Planar state equations
- Top-level discussion of bank angle and cross-range

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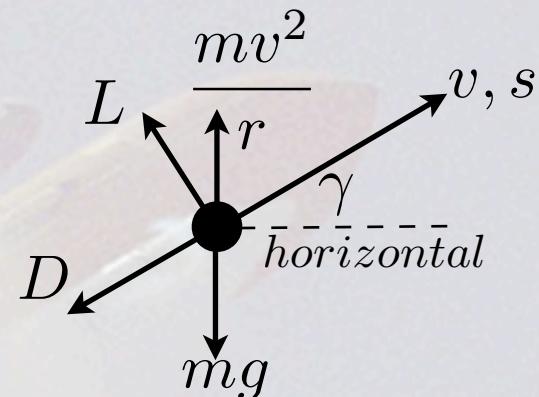
Lifting Atmospheric Entry
ENAE 791 - Launch and Entry Vehicle Design

Basic Equations of Motion

From last time,

$$v \frac{d\gamma}{dt} = \frac{L}{m} + \left(\frac{v^2}{r} - g \right) \cos \gamma$$

$$\frac{dv}{dt} = \left(\frac{v^2}{r} - g \right) \sin \gamma - \frac{D}{m}$$



Assume (at entry velocities close to orbital) $g \cong \frac{v^2}{r}$

$$v \frac{d\gamma}{dt} = \frac{L}{m} = \frac{\rho v^2}{2} \frac{c_L A}{m} = \frac{\rho v^2}{2\beta} \frac{L}{D} \quad (1)$$

$$\frac{dv}{dt} = -\frac{D}{m} = -\frac{\rho v^2}{2\beta} \quad (2)$$



Solving for Velocity

Divide (2) by (1)

$$\frac{\frac{dv}{dt}}{v \frac{d\gamma}{dt}} = \frac{-\frac{\rho v^2}{2\beta}}{\frac{\rho v^2}{2\beta} \frac{L}{D}} \implies \frac{dv}{v} = -\frac{d\gamma}{L/D} \quad (3)$$

$$\int_{v_e}^v \frac{dv}{v} = -\frac{1}{L/D} \int_{\gamma_e}^{\gamma} d\gamma$$

$$\ln \frac{v}{v_e} = \frac{-(\gamma - \gamma_e)}{L/D} \implies \frac{v}{v_e} = e^{-\frac{\gamma - \gamma_e}{L/D}} \quad (4)$$



Differential Elements

As before,

$$\frac{dh}{dt} = v \sin \gamma \quad (5)$$

$$\rho = \rho_o e^{-\frac{h}{h_s}} \quad (6)$$

Differentiating (6),

$$\begin{aligned} \frac{d\rho}{dt} &= -\frac{\rho_o}{h_s} e^{-\frac{h}{h_s}} \frac{dh}{dt} = -\frac{\rho}{h_s} \frac{dh}{dt} \\ \frac{d\rho}{dt} &= -\frac{\rho}{h_s} v \sin \gamma \end{aligned} \quad (7)$$



Differential Elements (2)

Solve (7) for v

$$v = \frac{-h_s}{\sin \gamma} \left(\frac{1}{\rho} \frac{d\rho}{dt} \right) \quad (8)$$

Substitute (8) into (1) and rewrite as

$$\begin{aligned} \frac{d\gamma}{dt} &= \frac{\rho v}{2\beta} \frac{L}{D} = \frac{\rho}{2\beta} \frac{L}{D} \left(\frac{-h_s}{\sin \gamma} \right) \frac{1}{\rho} \frac{d\rho}{dt} \\ \frac{d\gamma}{dt} &= \frac{-h_s}{2\beta \sin \gamma} \frac{L}{D} d\rho \end{aligned} \quad (9)$$



Solving for Flight Path Angle

$$\int_{\gamma_e}^{\gamma} \sin \gamma d\gamma = -\frac{h_s}{2\beta} \frac{L}{D} \int_0^{\rho} d\rho \quad (9)$$

$$\cos \gamma - \cos \gamma_e = \frac{h_s}{2\beta} \frac{L}{D} \rho \quad (10)$$

$$\cos \gamma = \frac{h_s}{2\beta} \frac{L}{D} \rho + \cos \gamma_e \quad (11)$$

$$\gamma = -\cos^{-1} \left(\frac{h_s}{2\beta} \frac{L}{D} \rho + \cos \gamma_e \right) \quad (12)$$

Note that the negative sign was inserted because \cos^{-1} is ambiguous as to direction, and the flight path angle on entry should be >0 .

Flight Path Angle and Velocity Equations

$$\gamma = -\cos^{-1} \left(\frac{h_s}{2\beta} \frac{L}{D} \rho_o e^{-\frac{h}{h_s}} + \cos \gamma_e \right) \quad (13)$$

Rewrite (4) as

$$v = v_e \exp \left(-\frac{\gamma - \gamma_e}{L/D} \right) = v_e \exp \left(\frac{\gamma_e - \gamma}{L/D} \right)$$

and substitute into (13)

$$v = v_e \exp \left\{ \frac{1}{L/D} \left[\gamma_e + \cos^{-1} \left(\frac{h_s}{2\beta} \frac{L}{D} \rho_o e^{-\frac{h}{h_s}} + \cos \gamma_e \right) \right] \right\} \quad (14)$$



Deceleration

Along the flight path, $\left| \frac{D}{m} \right| = \frac{\rho v^2}{2\beta}$

Perpendicular to the flight path, $\left| \frac{L}{m} \right| = \frac{\rho v^2}{2\beta} \frac{L}{D}$

Total deceleration (*not* in g's)

$$n = \sqrt{\left(\frac{D}{m} \right)^2 + \left(\frac{L}{m} \right)^2} = \frac{1}{m} \sqrt{D^2 + L^2} = \frac{\rho v^2}{2\beta} \sqrt{1 + \left(\frac{L}{D} \right)^2}$$

$$n = \frac{\rho_o v^2}{2\beta} \sqrt{1 + \left(\frac{L}{D} \right)^2} e^{-\frac{h}{h_s}} \quad (15)$$



Fiddling with Algebra

$$n = \frac{\rho_o}{2\beta} \sqrt{1 + \left(\frac{L}{D}\right)^2 e^{-\frac{h}{h_s}}} v^2$$

Substitute in (14)

$$n = \frac{\rho_o v_e^2}{2\beta} \sqrt{1 + \left(\frac{L}{D}\right)^2} \exp \left\{ \frac{2}{L/D} \left[\gamma_e + \cos^{-1} \left(\frac{h_s}{2\beta} \frac{L}{D} \rho_o e^{-\frac{h}{h_s}} + \cos \gamma_e \right) \right] \right\}$$

$$X \equiv \left[\gamma_e + \cos^{-1} \left(\frac{h_s}{2\beta} \frac{L}{D} \rho_o e^{-\frac{h}{h_s}} + \cos \gamma_e \right) \right] \quad (16)$$

$$n = \frac{\rho_o v_e^2}{2\beta} \sqrt{1 + \left(\frac{L}{D}\right)^2} e^{\frac{2X}{L/D}} \quad (17)$$



More Algebra

Set $\frac{dn}{dh} = 0$

$$0 = \frac{\rho_o v_e^2}{2\beta} \sqrt{1 + \left(\frac{L}{D}\right)^2} \left[e^{-\frac{h}{h_s}} e^{\frac{2X}{L/D}} \frac{2}{L/D} \frac{dX}{dh} + \left(\frac{-1}{h_s}\right) e^{-\frac{h}{h_s}} e^{\frac{2X}{L/D}} \right] \quad (18)$$

Factoring out common terms,

$$\frac{1}{h_s} = \frac{2}{L/D} \frac{dX_m}{dh} \quad (19)$$



Even More Algebra

From (13),

$$\gamma = -\cos^{-1} \left(\frac{h_s}{2\beta} \frac{L}{D} \rho_o e^{-\frac{h}{h_s}} + \cos \gamma_e \right)$$

$$X = \gamma_e - \gamma = -\cos^{-1} (\cos \gamma) + \gamma_e$$

$$Y \equiv \cos \gamma$$

$$X = \gamma_e - \cos^{-1} Y \quad (20)$$



Trigonometry, for a Change

Trig identity -

$$\frac{d(\cos^{-1} u)}{dx} = \frac{-1}{\sqrt{1 - u^2}} \frac{du}{dx} \quad (21)$$

$$\frac{dX}{dh} = -\frac{d(\cos^{-1} Y)}{dh} = \frac{1}{\sqrt{1 - Y^2}} \frac{dY}{dh}$$

$$\frac{dX}{dh} = \frac{1}{\sqrt{1 - \cos^2 \gamma}} \frac{d(\cos \gamma)}{dh}$$



Back to the Algebra

From (13),

$$\cos \gamma = \frac{h_s}{2\beta} \frac{L}{D} \rho_o e^{-\frac{h}{h_s}} + \cos \gamma_e$$

$$\frac{dX}{dh} = \frac{1}{\sqrt{1 - \cos^2 \gamma}} \frac{d}{dh} \left(\frac{h_s \rho_o}{2\beta} \frac{L}{D} e^{-\frac{h}{h_s}} + \cos \gamma_e \right) \quad (22)$$

$$\frac{dX}{dh} = \frac{1}{\sin \gamma} \left[\frac{h_s \rho_o}{2\beta} \frac{L}{D} \left(\frac{-1}{h_s} \right) e^{-\frac{h}{h_s}} \right]$$

$$\frac{dX}{dh} = \frac{-\rho_o}{2\beta \sin \gamma} \frac{L}{D} e^{-\frac{h}{h_s}} \quad (23)$$



Maximum Deceleration Case

If we go back to the n_{max} case, (19) gives

$$\frac{1}{h_s} = \frac{2}{L/D} \frac{dX_m}{dh}$$
$$\frac{1}{h_s} = \frac{2}{L/D} \frac{-\rho_o}{2\beta \sin \gamma_m} \frac{L}{D} e^{-\frac{h_m}{h_s}}$$

h_m , γ_m are values at n_{max}

$$\frac{1}{h_s} = \frac{-\rho_o}{\beta \sin \gamma_m} \frac{L}{D} e^{-\frac{h_m}{h_s}} \quad (24)$$

$$\sin \gamma_m = \frac{-\rho_o h_s}{\beta} \frac{L}{D} e^{-\frac{h_m}{h_s}} \quad (25)$$



Let's Go Back to Algebra

$$\text{Let } \Phi \equiv -\frac{\rho_o h_s}{2\beta} \frac{L}{D}$$

$$\cos \gamma_m = \Phi e^{-\frac{h_m}{h_s}} + \cos \gamma_e$$

$$\sin \gamma_m = \Phi e^{-\frac{h_m}{h_s}}$$

$$\text{Let } H \equiv e^{-\frac{h_m}{h_s}}$$

$$\left(\Phi e^{-\frac{h_m}{h_s}} + \cos \gamma_e \right)^2 + \left(\Phi e^{-\frac{h_m}{h_s}} \right)^2 = 1$$

$$(\Phi H + \cos \gamma_e)^2 + (\Phi H)^2 = 1$$



Algebra is Fun, Don't You Think?

$$2\Phi^2 H^2 + 2\Phi H \cos \gamma_e + \cos^2 \gamma_e - 1 = 0$$

$$2\Phi^2 H^2 + 2\Phi H \cos \gamma_e - \sin^2 \gamma_e = 0$$

$$H = \frac{-2\Phi \cos \gamma_e \pm \sqrt{4\Phi^2 \cos^2 \gamma_e + 4(2\Phi^2) \sin^2 \gamma_e}}{4\Phi^2}$$

$$H = -\frac{1}{2\Phi} \left(\cos \gamma_e \pm \sqrt{\cos^2 \gamma_e + 2 \sin^2 \gamma_e} \right)$$

$$H = -\frac{1}{2\Phi} \left(\cos \gamma_e \pm \sqrt{1 + \sin^2 \gamma_e} \right)$$



Maximum Deceleration Equations

Skipping some painful algebra and trig,

$$h_m = h_s \ln \left\{ \frac{-\rho_o h_s}{2\beta \sin \gamma_e} \left[\sqrt{4 + \left(\frac{L}{D} \right)^2 \csc^2 \gamma_e} - \frac{L}{D} \cot \gamma_e \right] \right\}$$

$$\cos \gamma_m = \cos \gamma_e - \frac{(L/D) \sin \gamma_e}{\sqrt{4 + (L/D)^2 \csc^2 \gamma_e} - (L/D) \cot \gamma_e}$$

$$v_m = v_e e^{-\frac{\gamma_m - \gamma_e}{L/D}}$$

$$n_{max} = \frac{\rho_o v_e^2}{2\beta} \sqrt{1 + \left(\frac{L}{D} \right)^2} v_m^2$$



Phugoid Oscillations

Assume a shallow, linear entry trajectory:

$$\gamma \ll 1 \quad D \approx 0 \quad \dot{\gamma}_o \approx 0$$

$$\dot{h} = v \sin \gamma \cong v\gamma$$

$$\frac{v\dot{\gamma}}{g} = \frac{L}{mg} - \left(1 - \frac{v^2}{v_c^2} \right)$$

Small perturbations $\implies h = h_1 + \Delta h$

$$\gamma = \gamma_1 + \Delta\gamma \approx \Delta\gamma$$



Perturbation Analysis

$$\dot{h} = v\gamma$$

$$\dot{h} = \dot{h}_1 + \Delta\dot{h} = \cancel{\gamma_1}v_1 + \Delta\dot{h} = \Delta\dot{h}$$

$$\Delta\dot{h} = (v_1 + \Delta v_1)(\cancel{\gamma_1} + \Delta\gamma) = v_1\Delta\gamma + \Delta v_1\Delta\gamma$$

$$\boxed{\Delta\dot{h} = v_1\Delta\gamma}$$

Neglecting drag $\implies v \cong \text{constant}$

$$\frac{L}{m} = \frac{L_1 + \Delta L}{m} = \frac{v^2 A c_L}{2m} \rho_o e^{-\frac{h_1 + \Delta h}{h_s}}$$

$$= \frac{v^2 A c_L}{2m} \rho_o e^{-\frac{h_1}{h_s}} e^{-\frac{\Delta h}{h_s}} = \frac{L_1}{m} e^{-\frac{\Delta h}{h_s}}$$



Perturbation Analysis

Using Taylor's series expansion,

$$\frac{L}{m} \cong \frac{L_1}{m} \left(1 - \frac{\Delta h}{h_s} \right)$$

$$\frac{\Delta L}{m} = -\frac{L_1}{m} \frac{\Delta h}{h_s}$$

$$\frac{v_1}{g} \Delta \dot{\gamma} = \frac{\Delta L}{mg} + \left[\frac{L_1}{mg} - \left(1 - \frac{v_1^2}{v_c^2} \right) \right]$$

$$v_1 \Delta \dot{\gamma} = \frac{\Delta L}{m} = \Delta \ddot{h}$$



Perturbed Lift

On an equilibrium glide,

$$\dot{\gamma} = 0 \implies \frac{L}{m} = g - \frac{v^2}{r}$$

$$\frac{L_1}{mg} = 1 - \frac{v_1^2}{gr} = 1 - \frac{v_1^2}{v_c^2}$$

$$\Delta \ddot{h} = \frac{\Delta L}{m} = -\frac{L_1}{m} \frac{\Delta h}{h_s} = -\left(1 - \frac{v_1^2}{v_c^2}\right) \frac{g \Delta h}{h_s}$$

$$\boxed{\Delta \ddot{h} + \left(1 - \frac{v_1^2}{v_c^2}\right) \frac{g}{h_s} \Delta h = 0}$$

Simple
harmonic
motion
(undamped)



Phugoid Parameters

Frequency: $\omega^2 = \left(1 - \frac{v_1^2}{v_c^2}\right) \frac{g}{h_s}$

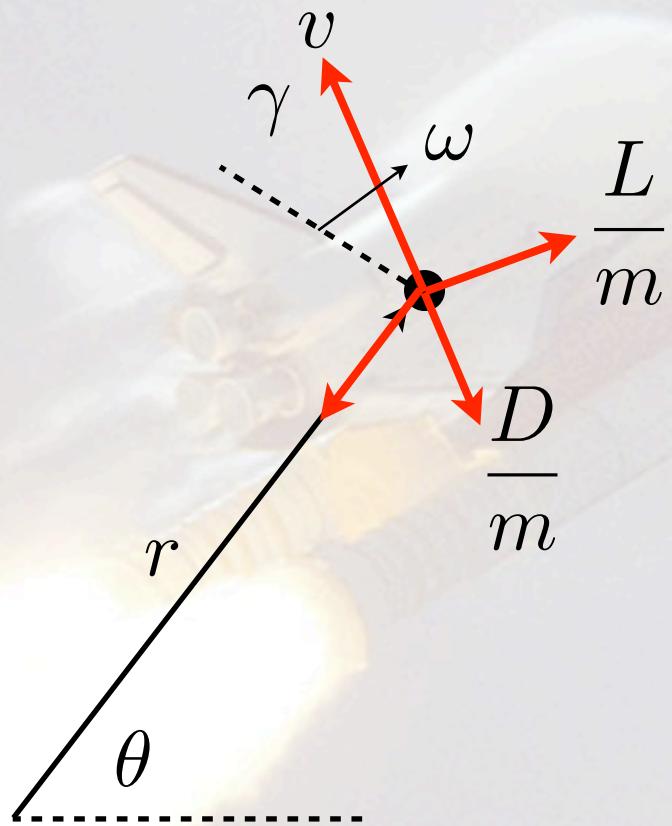
Period: $P = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\left(1 - \frac{v_1^2}{v_c^2}\right) \frac{g}{h_s}}} \approx \frac{169 \text{ sec}}{\sqrt{1 - \frac{v_1^2}{v_c^2}}}$

$v_c \sim 8000 \frac{m}{sec} \implies$

$v_1 \left(\frac{m}{sec}\right)$	P
7750	11m21s
6000	4m15s
4000	3m15s
2000	2m55s

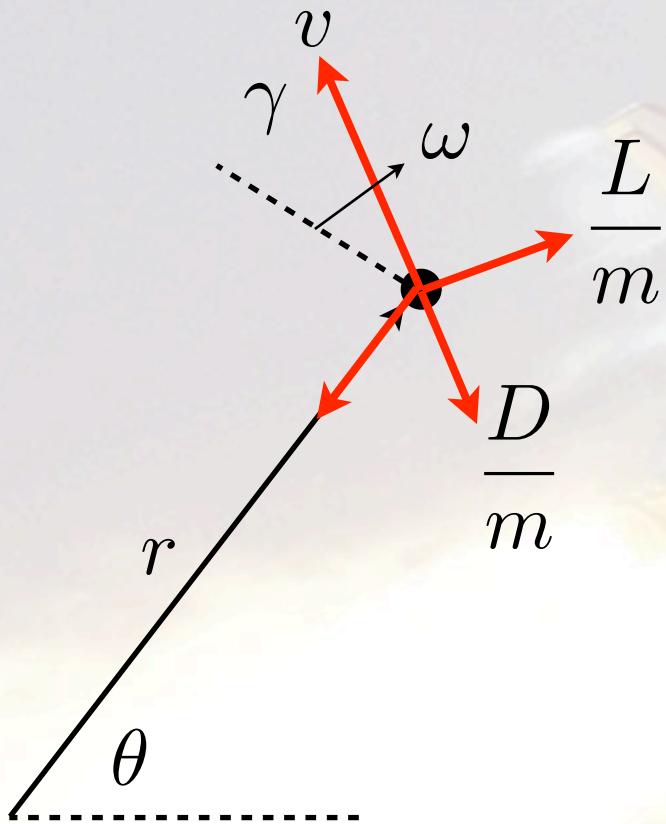


Free-Body Diagram with Spherical Planet



Planar State Equations

$$\omega = \dot{\gamma} - \dot{\theta}$$



Sum of transverse accelerations

$$\frac{L}{m} - g \cos \gamma = \omega v$$

Sum of parallel accelerations

$$-\frac{D}{m} - g \sin \gamma = \dot{v}$$



Planar State Equations (2)

$$\dot{r} = \dot{h} = v \sin \gamma$$

$$r\dot{\theta} = v \cos \gamma$$

$$\omega = \dot{\gamma} - \dot{\theta} = \dot{\gamma} - \frac{v}{r} \cos \gamma$$

$$\frac{L}{m} - g \cos \gamma = \left(\dot{\gamma} - \frac{v}{r} \cos \gamma \right) v$$

$$\frac{L}{m} - \left(g - \frac{v^2}{r} \right) \cos \gamma = \dot{\gamma} v$$

$$\frac{L}{m} - \left(1 - \frac{v^2}{rg} \right) g \cos \gamma = \dot{\gamma} v$$



The Canonical Planar State Equations

$$v\dot{\gamma} = \frac{L}{m} - \left(1 - \frac{v^2}{v_c^2}\right) g \cos \gamma$$

$$\dot{v} = -\frac{D}{m} - g \sin \gamma$$

$$\dot{r} = \dot{h} = v \sin \gamma$$

$$r\dot{\theta} = v \cos \gamma$$

Coupled first-order ODEs



Associated Parameters to State Eqns

$$\frac{L}{m} = \frac{1}{2} \frac{\rho v^2 A c_L}{m} = \frac{\rho v^2}{2} \frac{A c_D}{m} \frac{c_L}{c_D} = \frac{\rho v^2}{2\beta} \frac{L}{D}$$

$$\frac{D}{m} = \frac{1}{2} \frac{\rho v^2 A c_D}{m} = \frac{\rho v^2}{2\beta}$$

$$\rho = \rho_o e^{-\frac{h}{h_s}}$$

$$h = r - r_o$$

$$g = g_o \left(\frac{r_o}{r} \right)^2$$



Numerical Integration - 4th Order R-K

Given a series of equations $\dot{\bar{y}} = \bar{f}(t, \bar{x})$

$$\bar{k}_1 = \Delta t \bar{f}(t_n, \bar{y}_n)$$

$$\bar{k}_2 = \Delta t \bar{f}\left(t_n + \frac{\Delta t}{2}, \bar{y}_n + \frac{\bar{k}_1}{2}\right)$$

$$\bar{k}_3 = \Delta t \bar{f}\left(t_n + \frac{\Delta t}{2}, \bar{y}_n + \frac{\bar{k}_2}{2}\right)$$

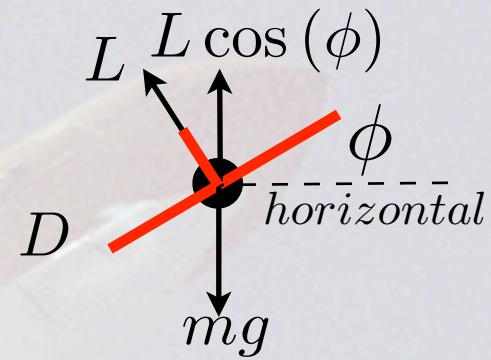
$$\bar{k}_4 = \Delta t \bar{f}(t_n + \Delta t, \bar{y}_n + \bar{k}_3)$$

$$\bar{y}_{n+1} = \bar{y}_n + \frac{\bar{k}_1}{6} + \frac{\bar{k}_2}{3} + \frac{\bar{k}_3}{3} + \frac{\bar{k}_4}{6} + O(\Delta t^5)$$



Bank Angle

$\phi \equiv \text{Bank Angle}$



Level turn: $L \cos \phi = mg$

$$\frac{L}{mg} = \frac{1}{\cos \phi} \equiv n_{bank}$$

(g's you pull in the banked turn)



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Crossrange (without derivations)

- Use of lift to deviate laterally from planar groundtrack
- Optimum bank angle to maximize crossrange

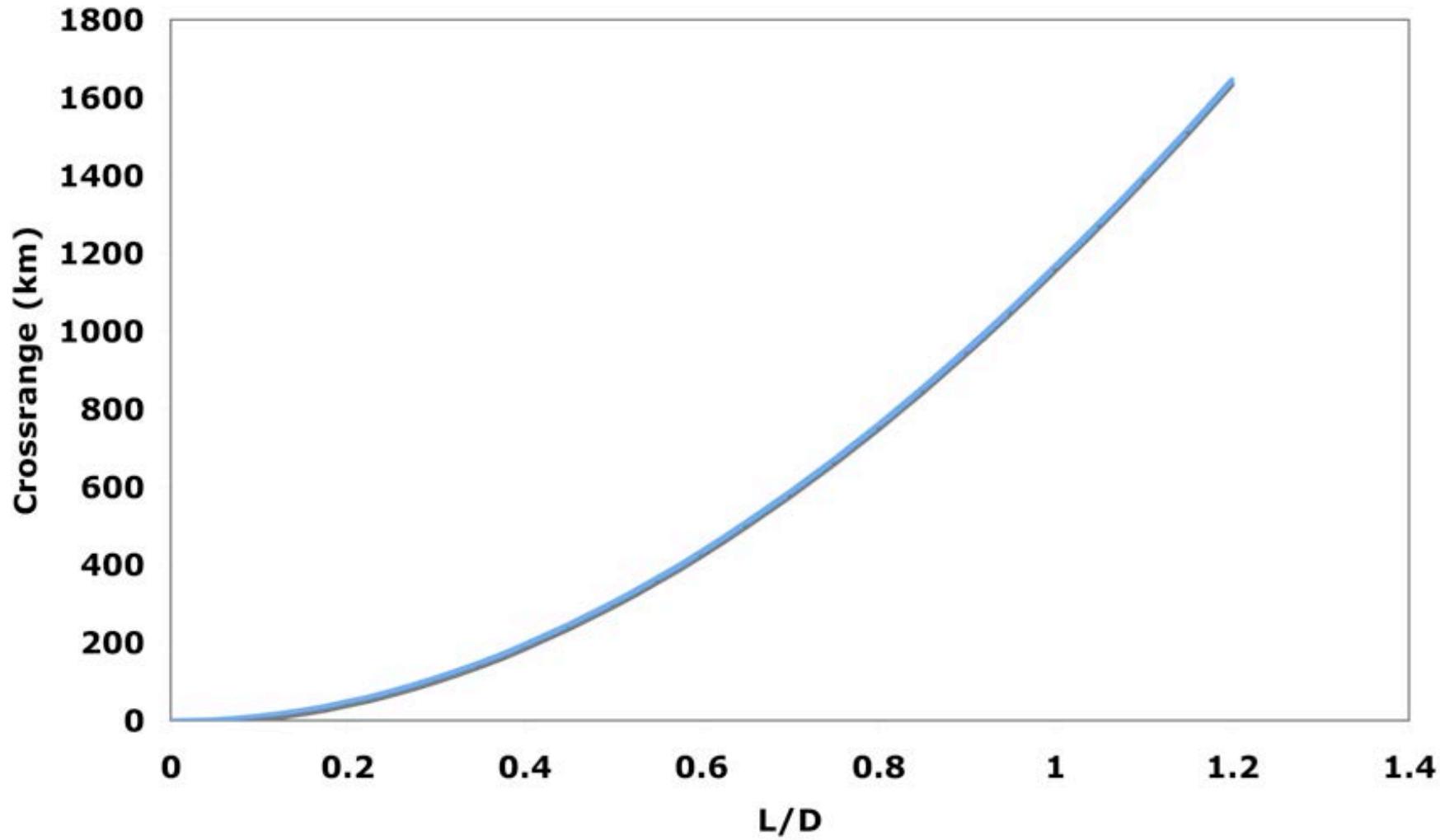
$$\phi_{opt} \cong \cot^{-1} \sqrt{1 + 0.106 \left(\frac{L}{D} \right)^2}$$

- Maximum achievable crossrange

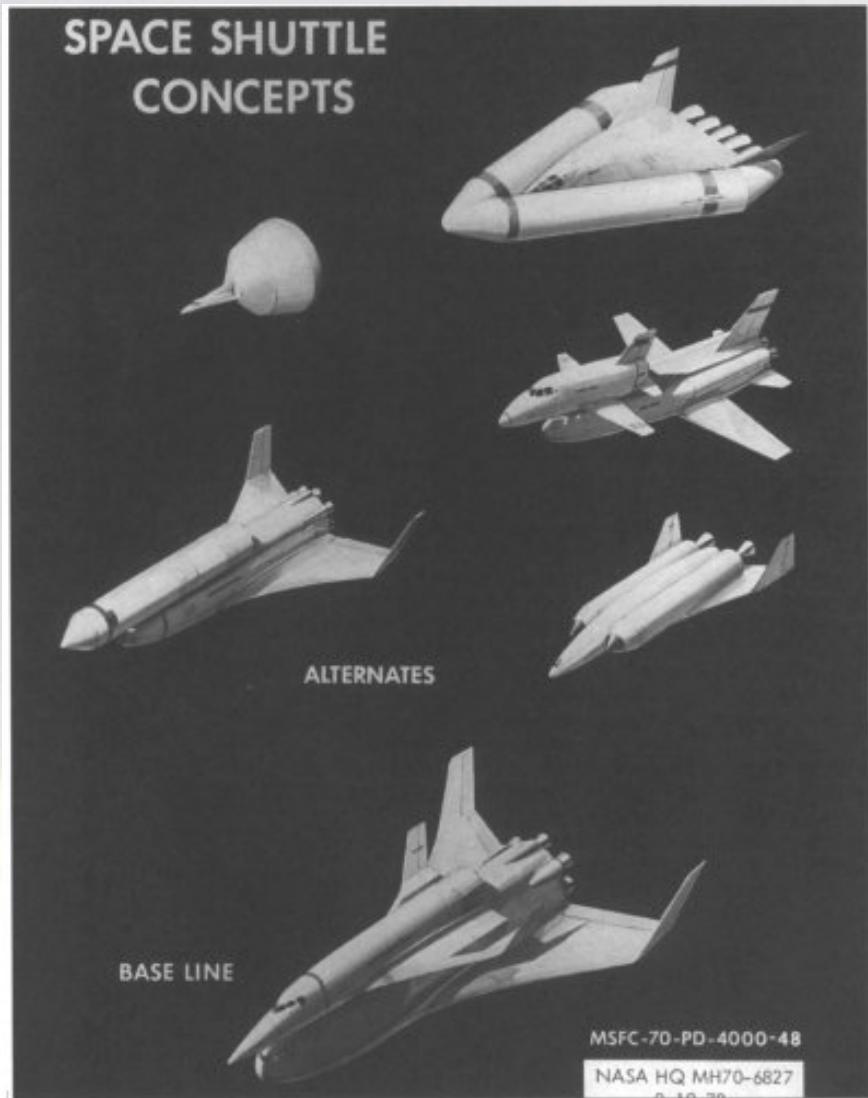
$$y_{max} \cong \frac{r_o}{5.2} \left(\frac{L}{D} \right)^2 \frac{1}{\sqrt{1 + 0.106 \left(\frac{L}{D} \right)^2}}$$



Crossrange Based on L/D



Early Shuttle Design Configurations



- Delta wing configuration
 - High hypersonic lift
 - High landing velocity
 - High crossrange
- Straight-wing configuration
 - High subsonic lift
 - Low(er) landing velocity
 - Low crossrange

