Orbital Mechanics

- Planetary launch and entry overview
- Energy and velocity in orbit
- Elliptical orbit parameters
- Orbital elements
- Coplanar orbital transfers
- Noncoplanar transfers
- Time in orbit
- Interplanetary trajectories

Orbital Mechanics: 500 years in 40 min.

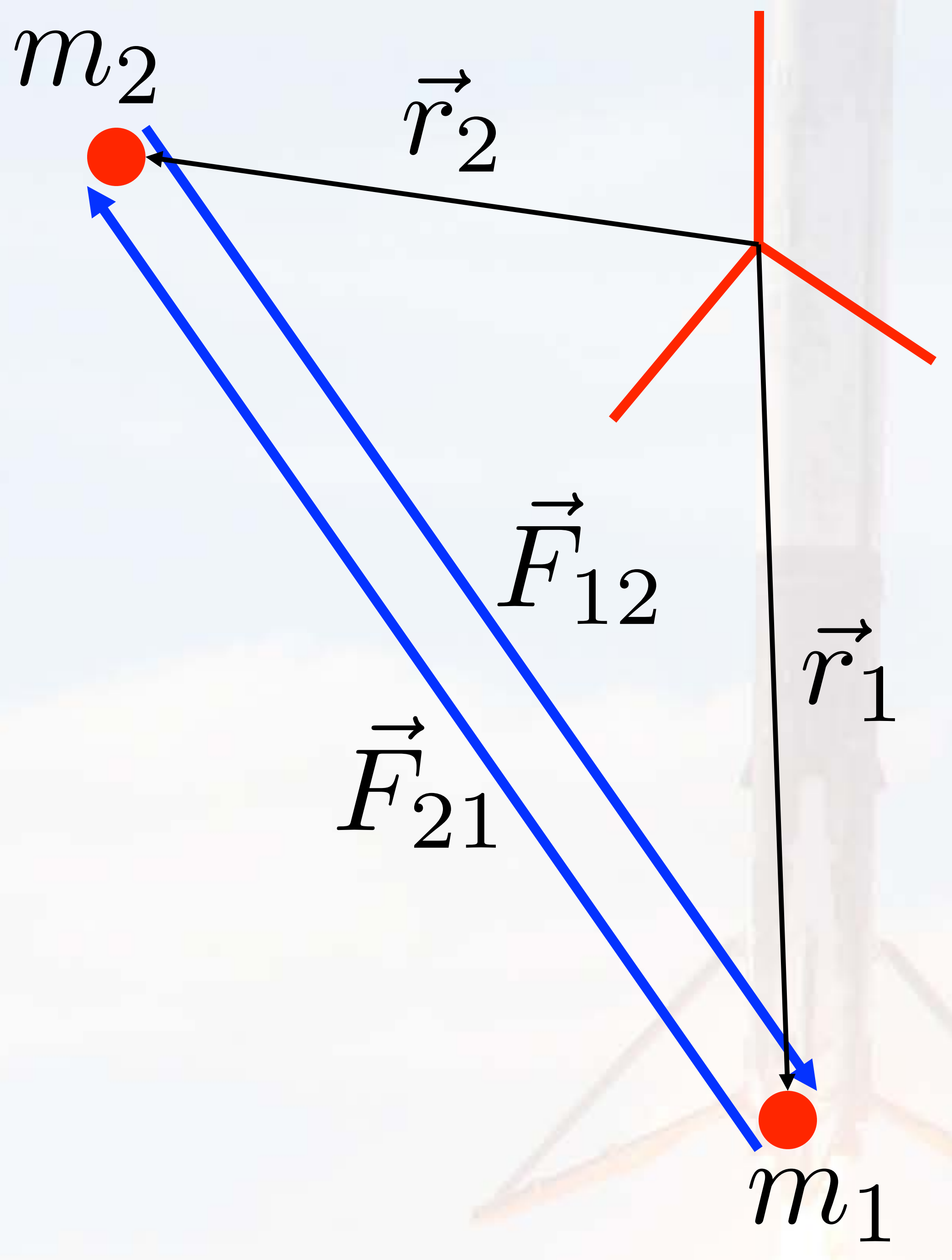
- Newton's Law of Universal Gravitation

$$F = \frac{G m_1 m_2}{r^2}$$

- Newton's First Law meets vector algebra

$$\overrightarrow{F} = m \overrightarrow{a}$$

Relative Motion Between Two Bodies



$$m_1 \frac{d^2\vec{r}_1}{dt^2} = G\frac{m_1 m_2}{\left|\vec{r}_{21}\right|^2}\frac{\vec{r}_{21}}{\left|\vec{r}_{21}\right|}$$

$$= G\frac{m_1 m_2}{\left|\vec{r}_{21}\right|^3}\vec{r}_{21} = G\frac{m_1 m_2}{\left|\vec{r}_{21}\right|^3}\left(\vec{r}_2 - \vec{r}_1\right)$$

$$m_2 \frac{d^2\vec{r}_2}{dt^2} = G\frac{m_1 m_2}{\left|\vec{r}_{12}\right|^3}\left(\vec{r}_1 - \vec{r}_2\right)$$

$\vec{F}_{12} =$ force due to body 1 on body 2

Gravitational Motion

$$\text{Let } r = |\vec{r}_{12}| = |\vec{r}_{21}| \qquad \text{Let } \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{G}{r^3}\left[m_2\left(-\vec{r}\right) - m_1\left(\vec{r}\right)\right] = \frac{-G}{r^3}\left(m_1 + m_2\right)\vec{r}$$

$$\text{Let } \mu = G(m_1 + m_2)$$

$$\frac{d^2\vec{r}}{dt^2} + \mu\frac{\vec{r}}{r^3} = \vec{0}$$

"Equation of Orbit" -
Orbital motion is simple harmonic motion

Orbital Angular Momentum

$$\vec{v} = \frac{d\vec{r}}{dt} \qquad \frac{d\vec{v}}{dt} + \mu\frac{\vec{r}}{r^3} = \vec{0}$$

$$\vec{r} \times \frac{d\vec{v}}{dt} + \frac{\mu}{r^3}\left(\vec{r} \times \vec{r}\right) = \vec{0} \qquad \vec{r} \times \frac{d\vec{v}}{dt} = \vec{0}$$

$$\frac{d}{dt}\left(\vec{r} \times \vec{v}\right) = \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}$$

$$= \vec{v} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = \vec{r} \times \frac{d\vec{v}}{dt} = \vec{0}$$

$$\frac{d}{dt}\left(\vec{r} \times \vec{v}\right) = \vec{0} \qquad \vec{r} \times \vec{v} = \vec{constant} \qquad \vec{r} \times \vec{v} = \vec{h}$$

$\vec{h}$ is angular momentum vector (constant) $\Longrightarrow$
$\vec{r}$ and $\vec{v}$ are in a constant plane

Fun and Games with Algebra

$$\frac{d\vec{v}}{dt} + \mu\frac{\vec{r}}{r^3} = \vec{0} \qquad \frac{d\vec{v}}{dt} \times \vec{h} + \frac{\mu}{r^3}\left(\vec{r} \times \vec{h}\right) = \vec{0}$$

$$\frac{d}{dt}\left(\vec{v} \times \vec{h}\right) = \frac{d\vec{v}}{dt} \times \vec{h} + \vec{v} \times \frac{d\vec{h}}{dt}$$

0

$$\frac{d}{dt}\left(\vec{v} \times \vec{h}\right) = -\frac{\mu}{r^3}\left(\vec{r} \times \vec{h}\right) = -\frac{\mu}{r^3}\left(\vec{r} \times \vec{r} \times \vec{v}\right)$$

$$\frac{d}{dt}\left(\vec{v} \times \vec{h}\right) = -\frac{\mu}{r^3}\left[\left(\vec{r} \cdot \vec{v}\right)\vec{r} - \left(\vec{r} \cdot \vec{r}\right)\vec{v}\right]$$

$$\vec{r} \cdot \vec{v} = rv\cos\gamma = r\frac{dr}{dt}$$

More Algebra, More Fun

$$\frac{d}{dt}\left(\vec{v}\times\vec{h}\right) = -\frac{\mu}{r^3}\left[r\frac{dr}{dt}\vec{r} - r^2\frac{d\vec{r}}{dt}\right]$$

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{\left(r\frac{d\vec{r}}{dt} - \vec{r}\frac{dr}{dt}\right)}{r^2} = \left(\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{\vec{r}}{r^2}\frac{dr}{dt}\right)$$

$$\frac{d}{dt}\left(\vec{v}\times\vec{h}\right) = -\mu\left(\frac{1}{r^2}\frac{dr}{dt}\vec{r} - \frac{1}{r}\frac{d\vec{r}}{dt}\right) = \mu\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

$$\frac{d}{dt}\left(\vec{v}\times\vec{h} - \mu\frac{\vec{r}}{r}\right) = \vec{0}$$

Orientation of the Orbit

$$\vec{v} \times \vec{h} - \mu\frac{\vec{r}}{r} = \vec{\text{constant}}$$

$$\vec{v} \times \vec{h} - \mu\frac{\vec{r}}{r} = \mu\vec{e}$$

$\vec{e} \equiv$ eccentricity vector, in orbital plane

$\vec{e}$ points in the direction of periapsis

$$\vec{r} \cdot \vec{v} \times \vec{h} - \vec{r} \cdot \mu\frac{\vec{r}}{r} = \mu\left(\vec{r} \cdot \vec{e}\right)$$

$$\vec{r} \times \vec{v} \cdot \vec{h} - \mu\frac{\vec{r} \cdot \vec{r}}{r} = \mu r e \cos\theta$$

$$\vec{h} \cdot \vec{h} - \mu\frac{r^2}{r} = \mu r e \cos\theta$$

Position in Orbit

$$h^2 - \mu r = \mu r e \cos\theta$$

$$r = \frac{h^2/\mu}{1 + e\cos\theta}$$

θ = true anomaly: angular travel from perigee passage

$$\text{at } \theta = \pm\frac{\pi}{2};\ \cos\theta = 0;\ r = p \equiv h^2/\mu$$

Relating Velocity and Orbital Elements

$$\mu\vec{e} = \vec{v} \times \vec{h} - \mu\frac{\vec{r}}{r}$$

$$\mu\vec{e} \cdot \mu\vec{e} = \vec{v} \times \vec{h} \cdot \vec{v} \times \vec{h} - 2\mu\left(\vec{v} \times \vec{h}\right) \cdot \frac{\vec{r}}{r} + \mu^2\left(\frac{\vec{r}}{r} \cdot \frac{\vec{r}}{r}\right)$$

$$\mu^2 e^2 = v^2 h^2 - 2\mu\frac{h^2}{r} + \mu^2$$

$$e^2 = \frac{v^2}{\mu}p - 2\frac{p}{r} + 1$$

Vis-Viva Equation

$$p \equiv a(1-e^2) = \frac{1-e^2}{\frac{2}{r}-\frac{v^2}{\mu}}$$

$$a = \left(\frac{2}{r}-\frac{v^2}{\mu}\right)^{-1}$$

$$v^2 = \mu\left(\frac{2}{r}-\frac{1}{a}\right)$$

<--Vis-Viva Equation

$$\frac{v^2}{2}-\frac{\mu}{r} = -\frac{\mu}{2a}$$

Energy in Orbit

- Kinetic Energy

$$K.E. = \frac{1}{2} m v^2 \Rightarrow \frac{K.E.}{m} = \frac{v^2}{2}$$

- Potential Energy

$$P.E. = -\frac{m\mu}{r} \Rightarrow \frac{P.E.}{m} = -\frac{\mu}{r}$$

- Total Energy

$$Const. = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

<--Vis-Viva Equation

Implications of Vis-Viva

- Circular orbit (r=a)

$$v_{circular} = \sqrt{\frac{\mu}{r}}$$

- Parabolic escape orbit (a tends to infinity)

$$v_{escape} = \sqrt{\frac{2\mu}{r}}$$

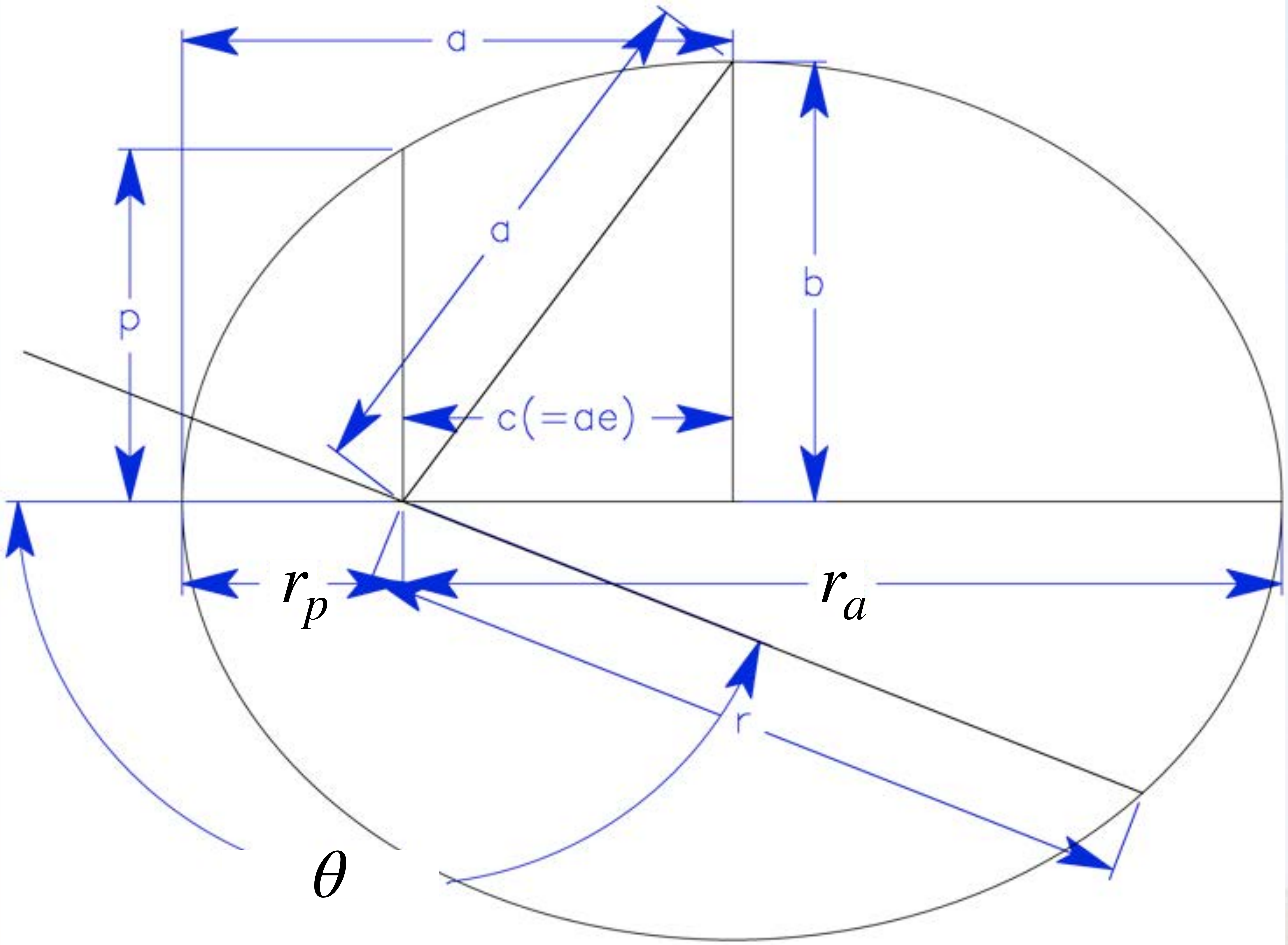
- Relationship between circular and parabolic orbits

$$v_{escape} = \sqrt{2}v_{circular}$$

Some Useful Constants

- Gravitation constant $\mu = GM$
  - Earth: 398,604 km^3/sec^2
  - Moon: 4667.9 km^3/sec^2
  - Mars: 42,970 km^3/sec^2
  - Sun: 1.327×10^{11} km^3/sec^2
- Planetary radii
  - r_{Earth} = 6378 km
  - r_{Moon} = 1738 km
  - r_{Mars} = 3393 km

Classical Parameters of Elliptical Orbits



Basic Orbital Parameters

- Semi-latus rectum (or parameter)

$$p = a(1 - e^2)$$

- Radial distance as function of orbital position

$$r = \frac{p}{1 + e\cos\theta}$$

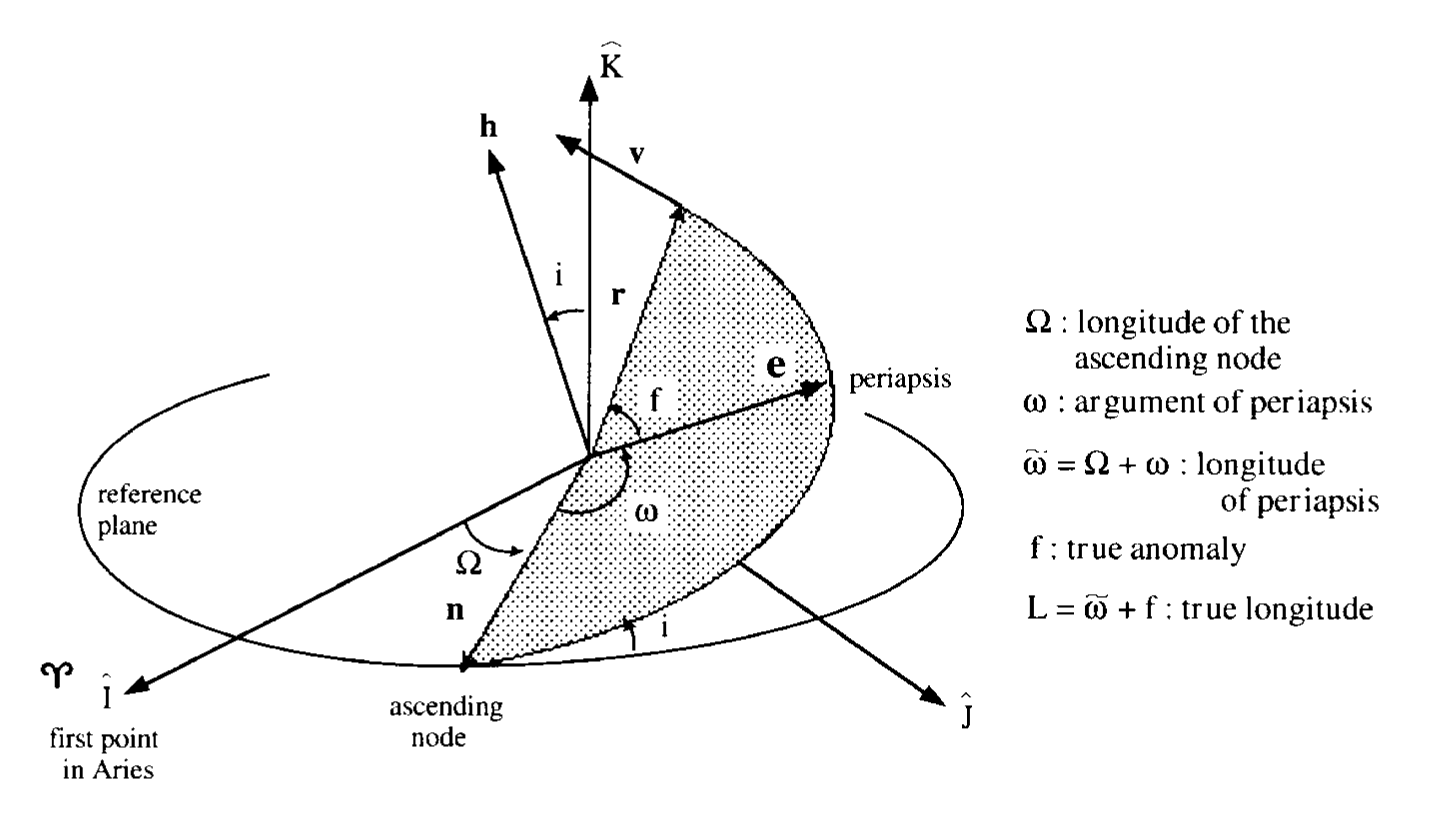
- Periapse and apoapse distances

$$r_p = a(1 - e) \qquad r_a = a(1 + e)$$

- Angular momentum

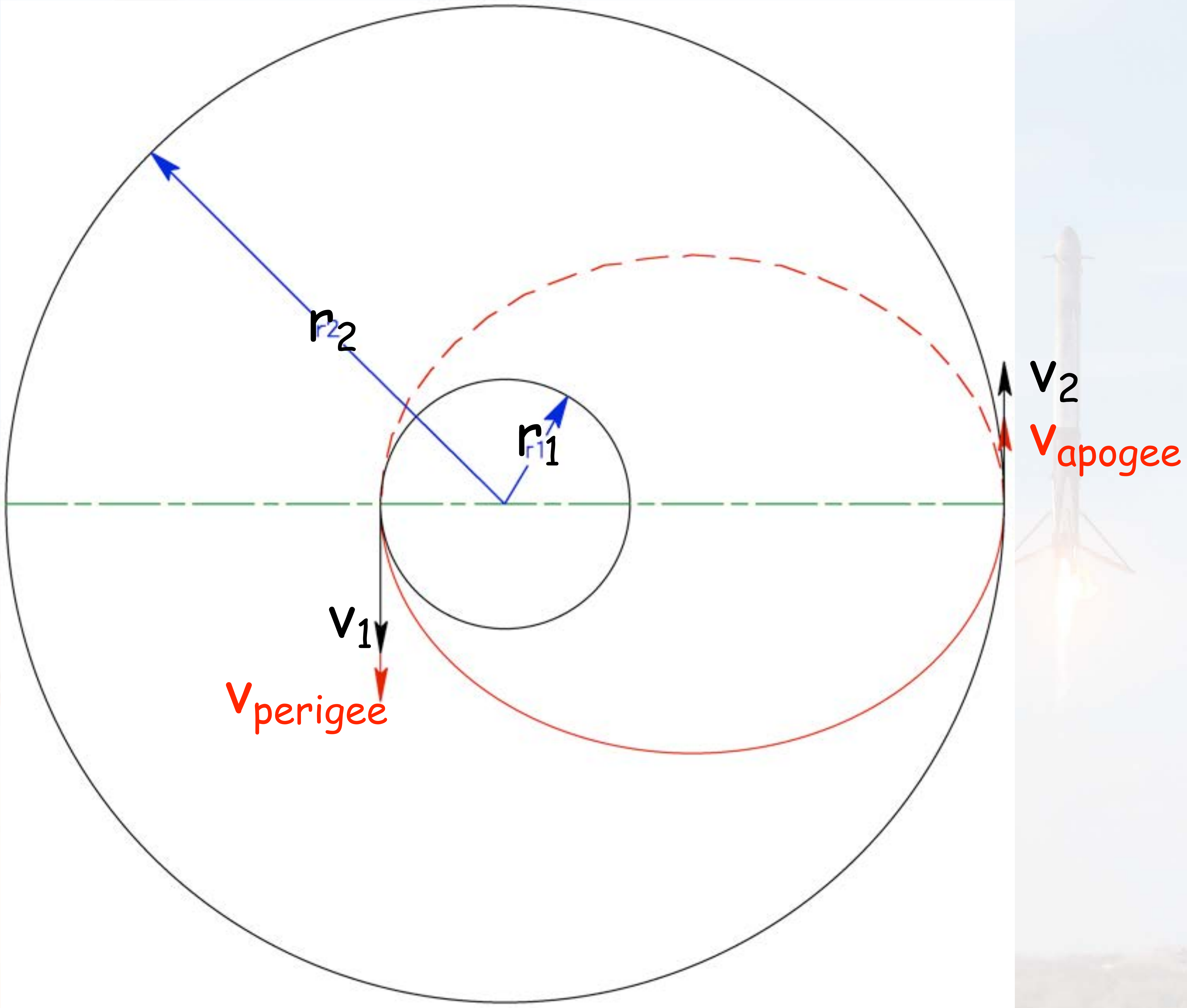
$$\vec{h} = \vec{r} \times \vec{v} \qquad h = \sqrt{\mu p}$$

The Classical Orbital Elements



Ref: J. E. Prussing and B. A. Conway, Orbital Mechanics Oxford University Press, 1993

The Hohmann Transfer



First Maneuver Velocities

- Initial vehicle velocity

$$v_1 = \sqrt{\frac{\mu}{r_1}}$$

- Needed final velocity

$$v_{perigee} = \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{2r_2}{r_1 + r_2}}$$

- Required ΔV

$$\Delta v_1 = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right)$$

Second Maneuver Velocities

- Initial vehicle velocity

$$v_{apogee} = \sqrt{\frac{\mu}{r_2}}\sqrt{\frac{2r_1}{r_1+r_2}}$$

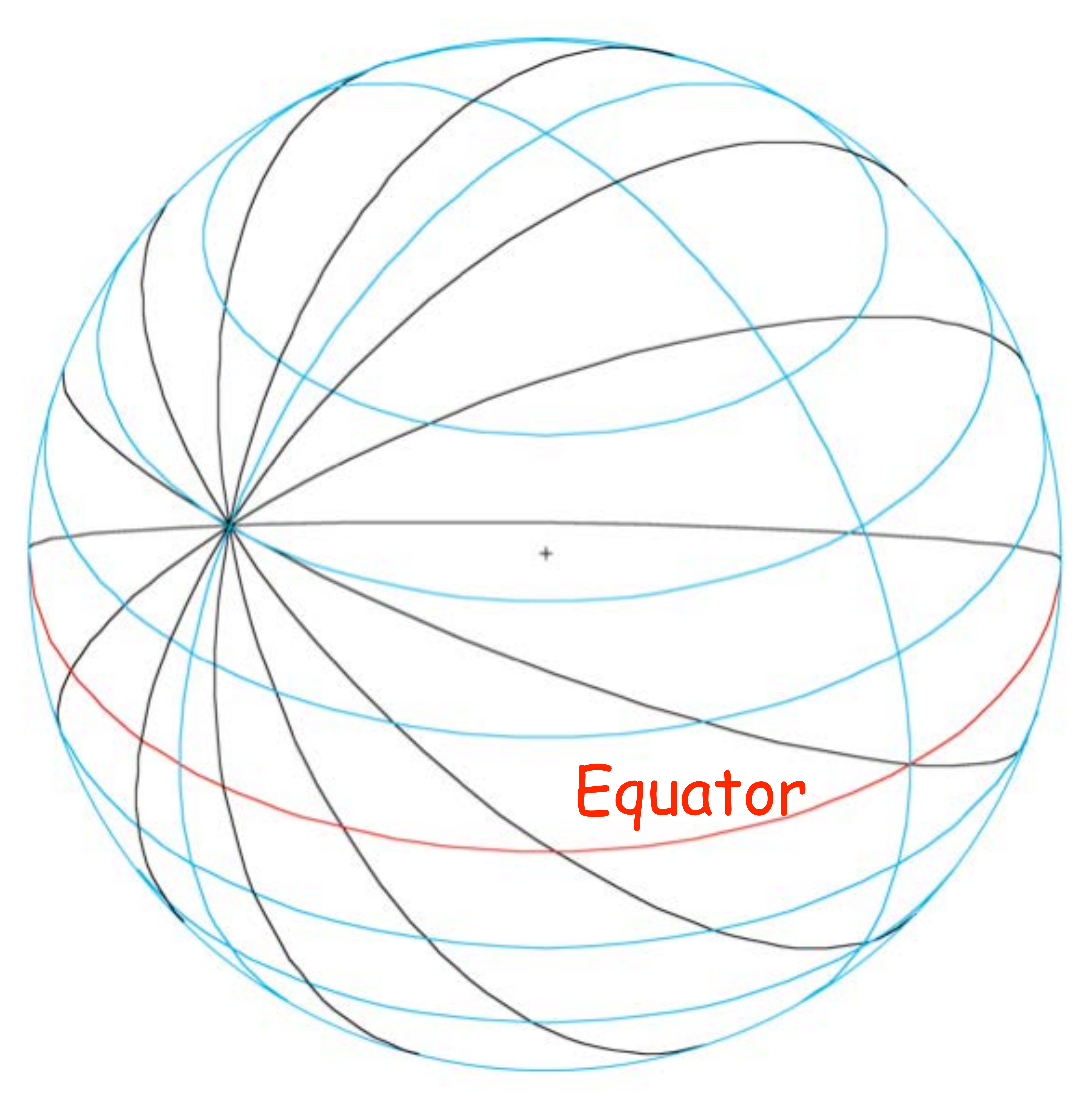
- Needed final velocity

$$v_2 = \sqrt{\frac{\mu}{r_2}}$$

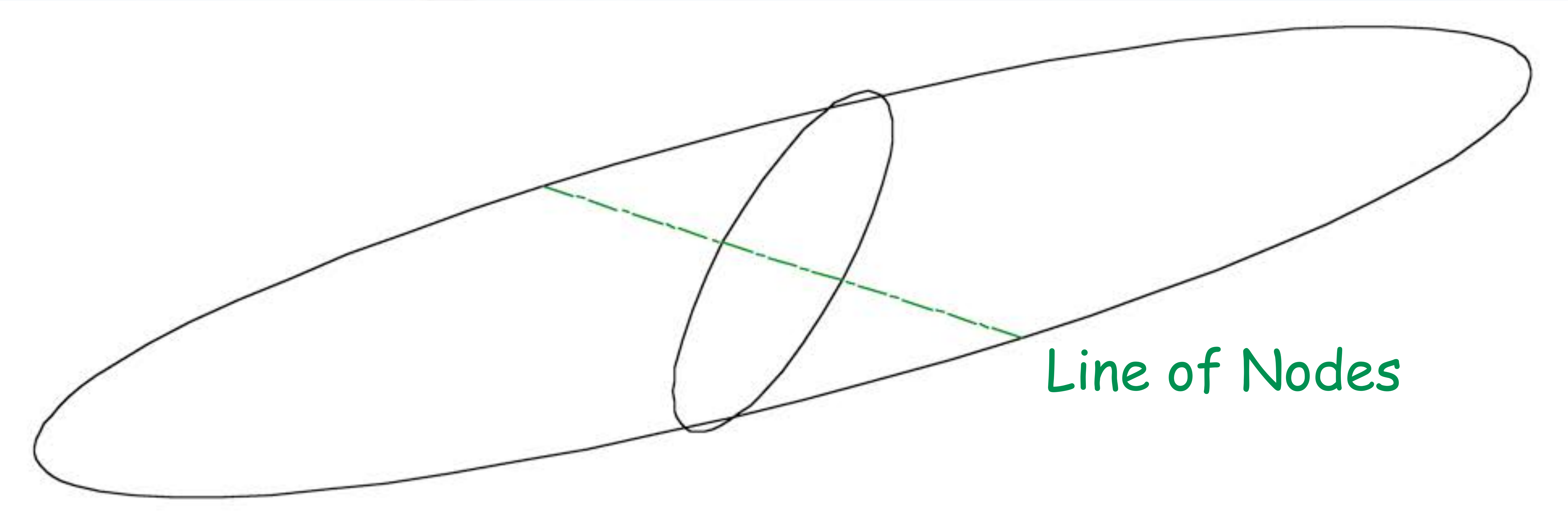
- Required ΔV

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1+r_2}}\right)$$

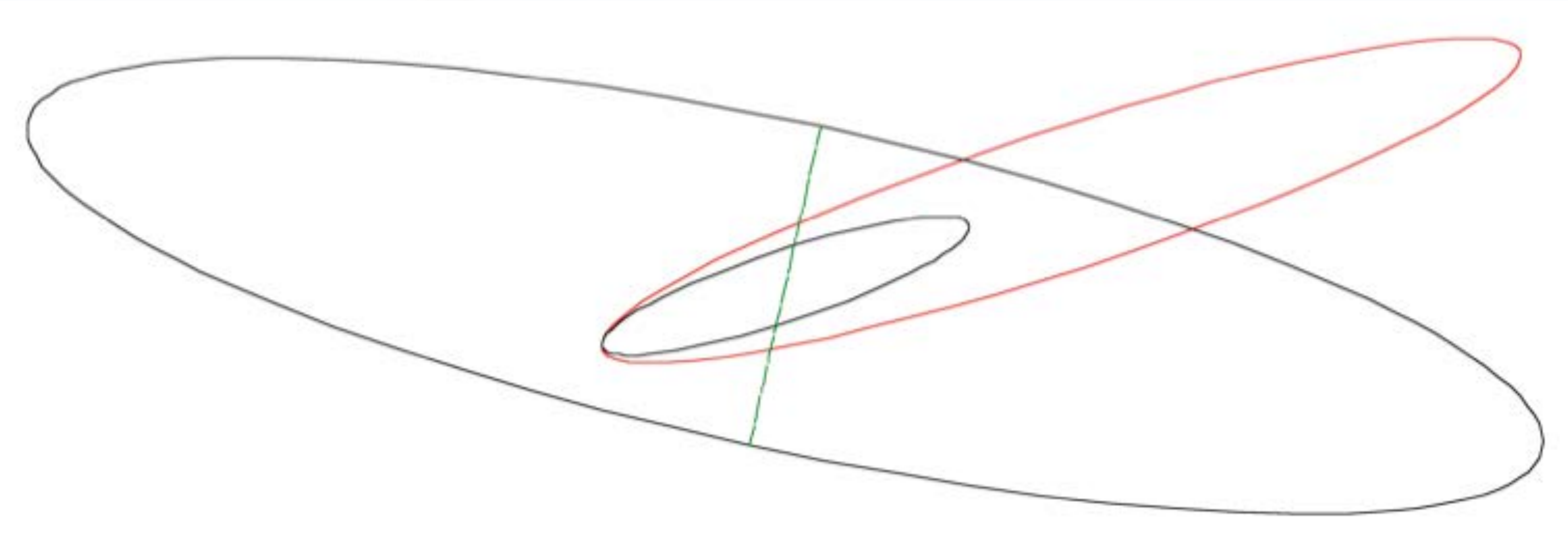
Limitations on Launch Inclinations



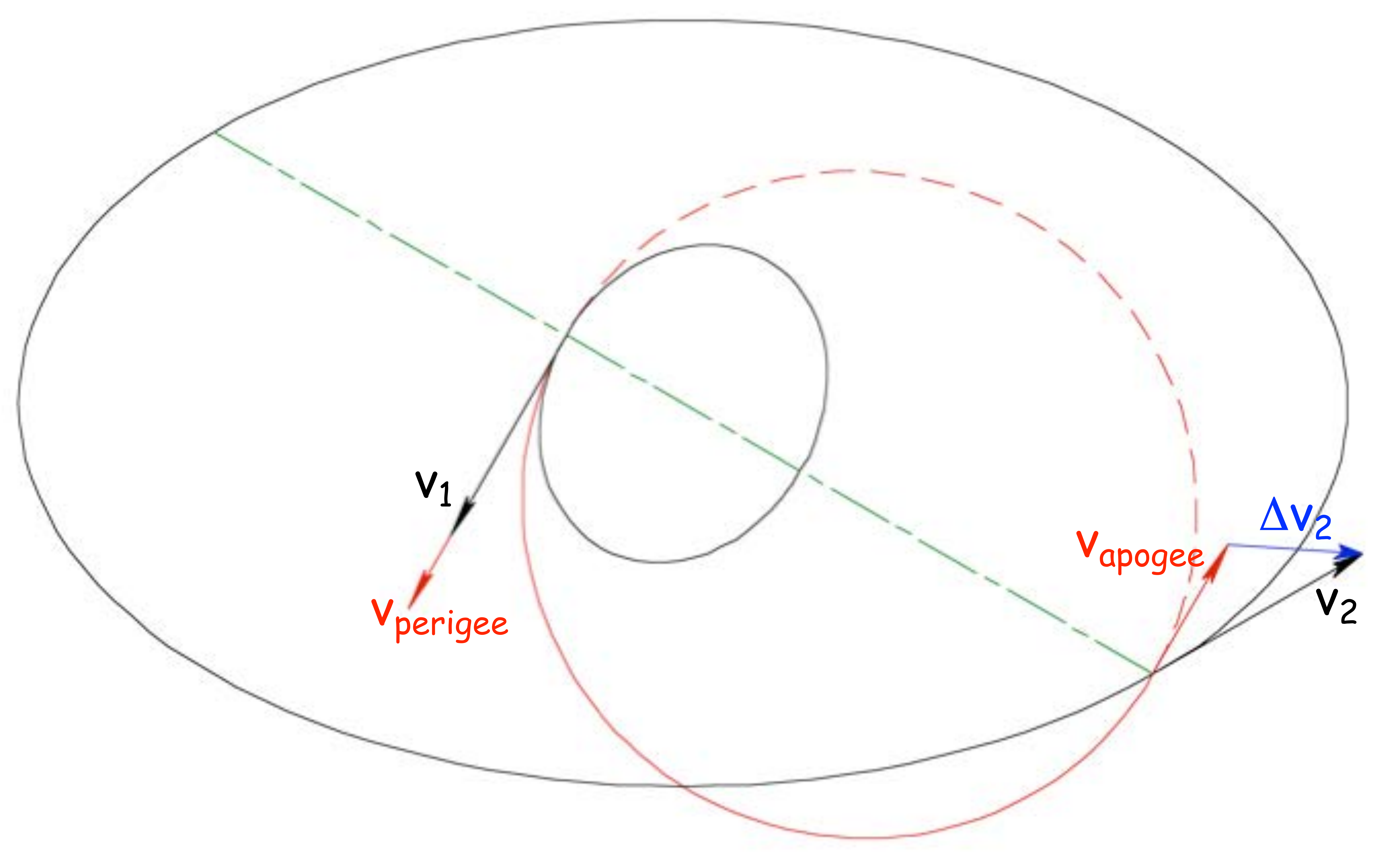
Differences in Inclination



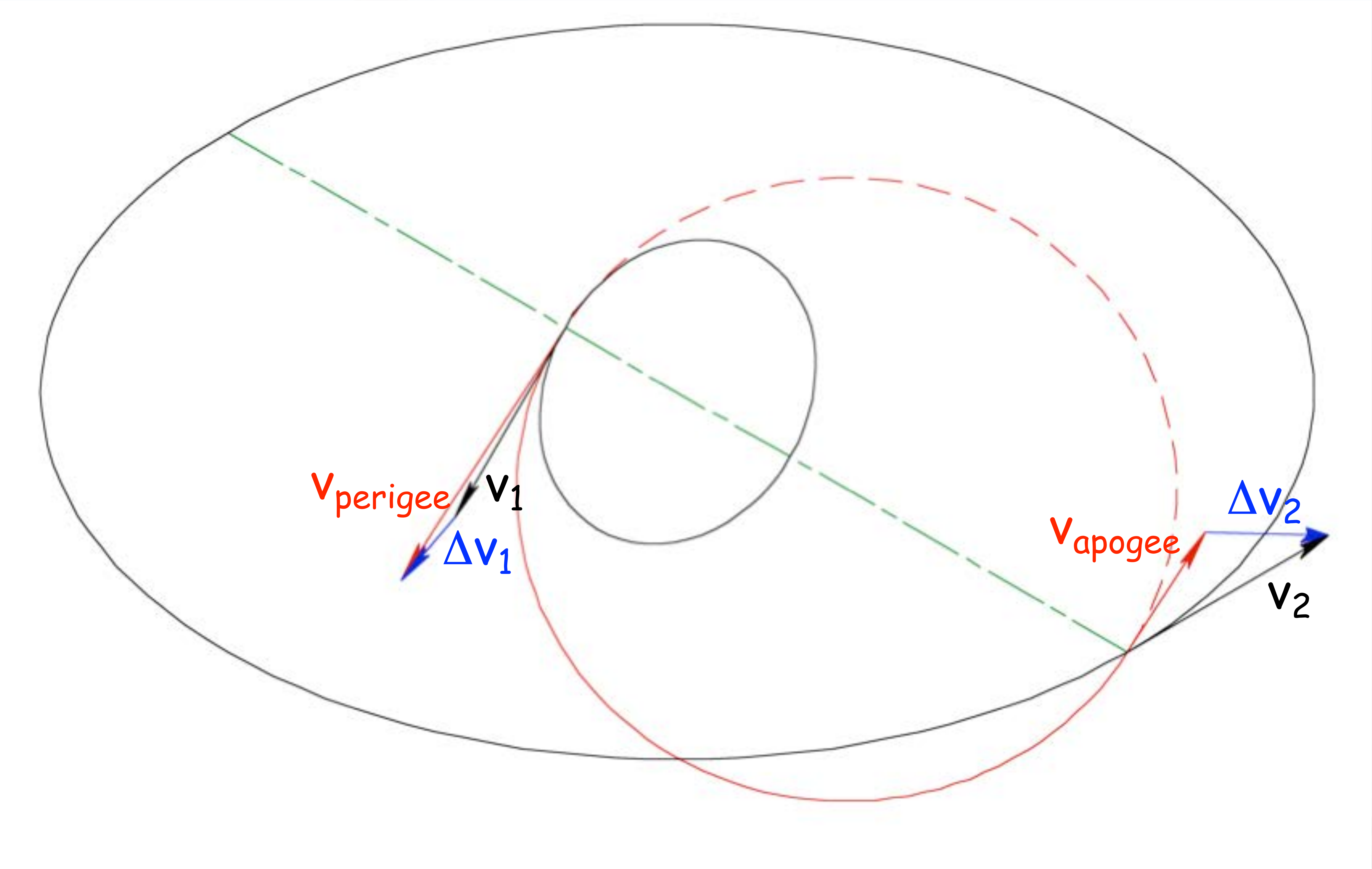
Choosing the Wrong Line of Apsides



Simple Plane Change



Optimal Plane Change



First Maneuver with Plane Change Δi_1

- Initial vehicle velocity

$$v_1 = \sqrt{\frac{\mu}{r_1}}$$

- Needed final velocity

$$v_p = \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{2r_2}{r_1 + r_2}}$$

- Required ΔV

$$\Delta v_1 = \sqrt{v_1^2 + v_p^2 - 2v_1 v_p \cos \Delta i_1}$$

Second Maneuver with Plane Change Δi_2

- Initial vehicle velocity

$$v_a = \sqrt{\frac{\mu}{r_2}}\sqrt{\frac{2r_1}{r_1 + r_2}}$$

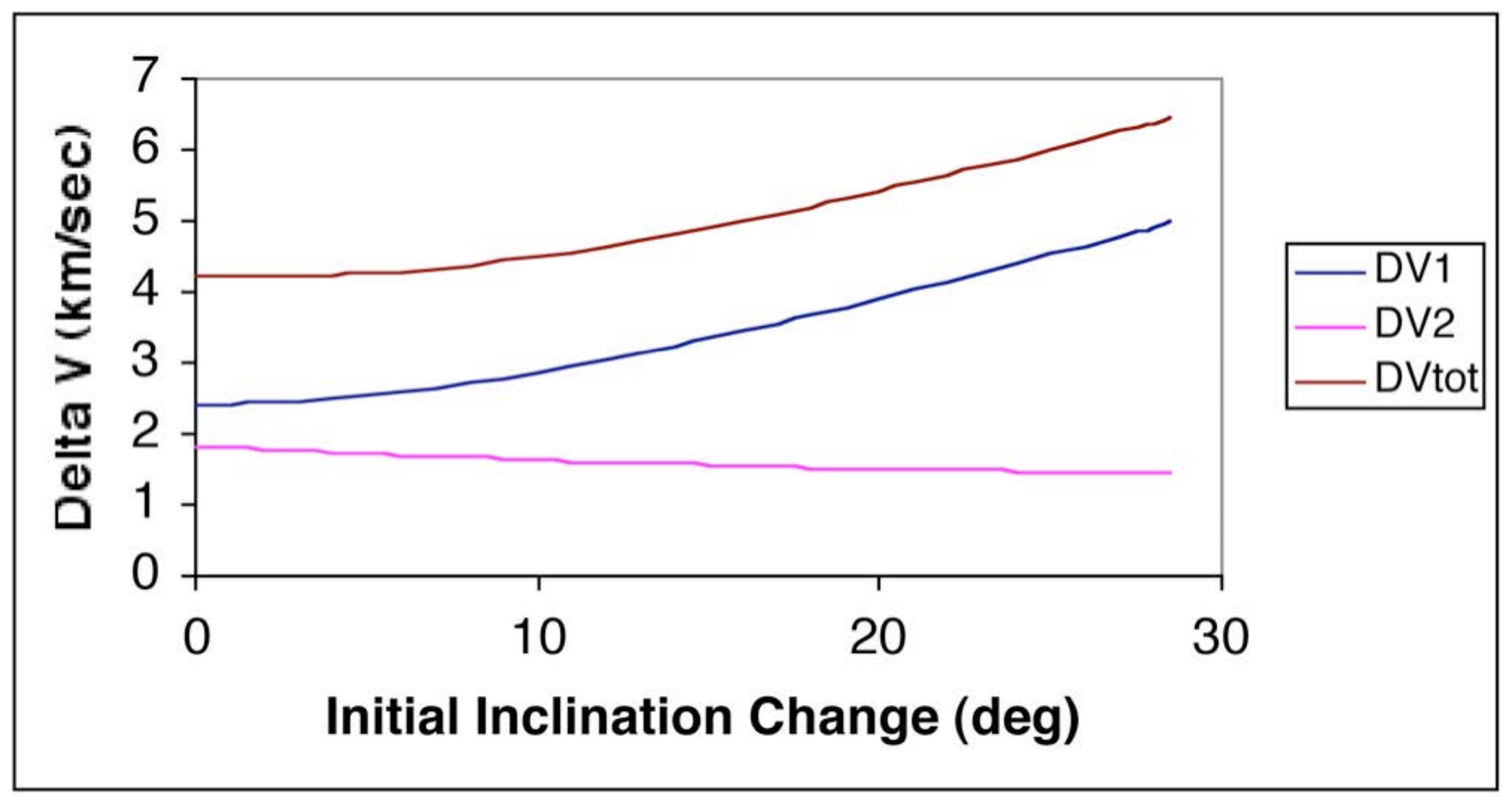
- Needed final velocity

$$v_2 = \sqrt{\frac{\mu}{r_2}}$$

- Required ΔV

$$\Delta v_2 = \sqrt{v_2^2 + v_a^2 - 2v_2 v_a \cos \Delta i_2}$$

Sample Plane Change Maneuver



Optimum initial plane change = 2.20°

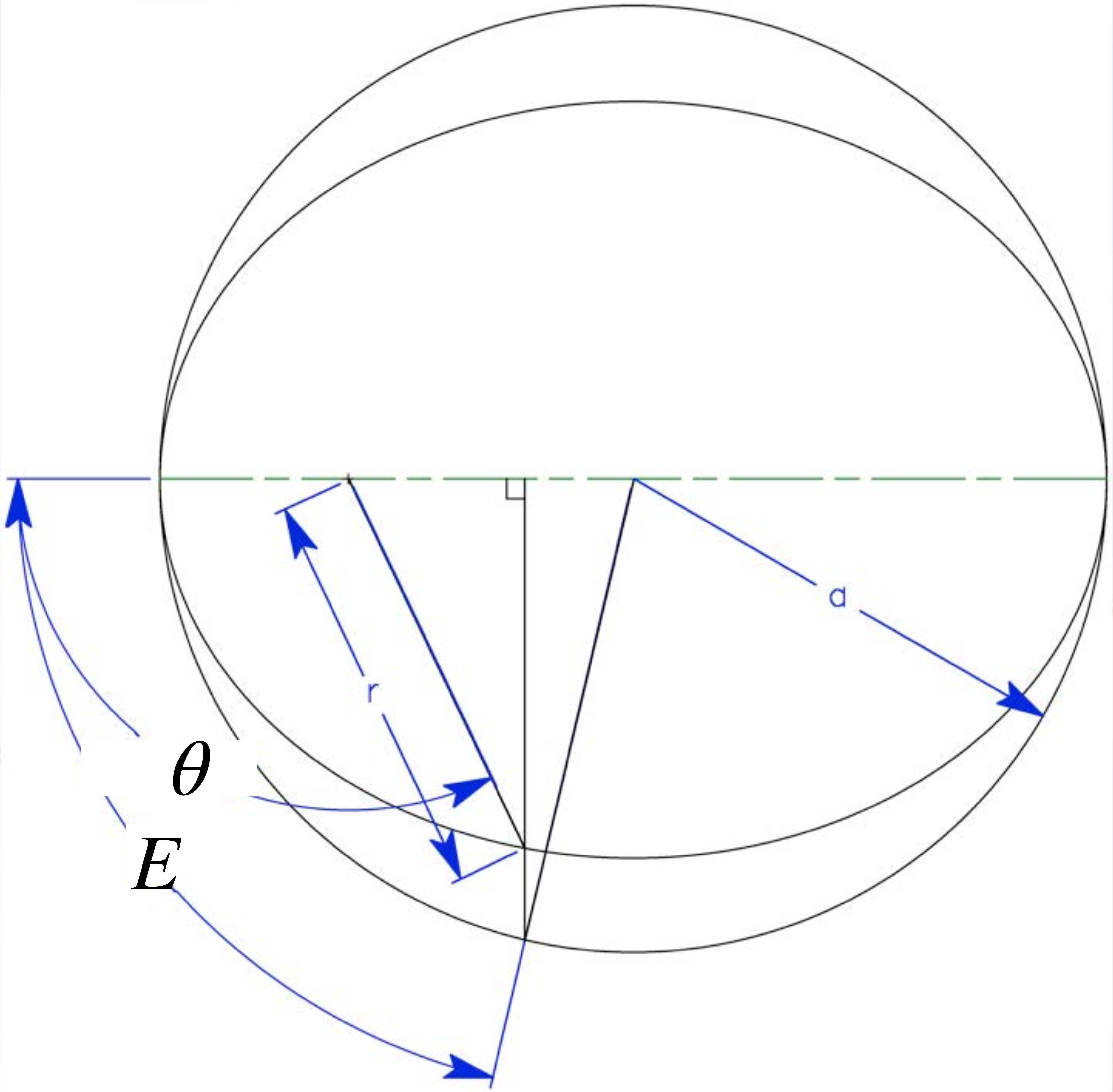
Geo Transfer Orbit – Practical Considerations

- Most launches of geosynchronous communications satellites are to geo transfer orbit (GTO) – ideally elliptical trajectory to GEO altitude at apogee
- Launch vehicle performs the perigee burn; satellite performs apogee circularization with apogee kick motor (AKM) or (more common lately) electric propulsion
- Optimization must take into account different performance and mission implications of LV vs. payload maneuvers

Geo Transfer Orbit – Accommodations

- Typical maneuver: inject into LEO parking orbit and perform GTO injection when passing equator
- If the payload is slightly larger than the launch vehicle capability, can inject into a lower apogee and make up difference with satellite propulsion
- If the launch vehicle has extra margin, can inject into a super synchronous orbit to reduce satellite Δv requirements
- Some LVs can offer "GEO direct" – upper stage stays active with propellant to perform circularization

Calculating Time in Orbit



Time in Orbit

- Period of an orbit

$$P = 2\pi\sqrt{\frac{a^3}{\mu}}$$

- Mean motion (average angular velocity)

$$n = \sqrt{\frac{\mu}{a^3}}$$

- Time since pericenter passage

$$M = nt = E - e\sin E$$

➥M=mean anomaly

Dealing with the Eccentric Anomaly

- Relationship to orbit

$$r = a\left(1 - e\cos E\right)$$

- Relationship to true anomaly

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

- Calculating M from time interval: iterate until it converges

$$E_{i+1} = nt + e\sin E_i$$

Example: Time in Orbit

- Hohmann transfer from LEO to GEO
  - h_1=300 km --> r_1=6378+300=6678 km
  - r_2=42240 km
- Time of transit (1/2 orbital period)

$$a = \frac{1}{2}(r_1 + r_2) = 24,459\ km$$

$$t_{transit} = \frac{P}{2} = \pi\sqrt{\frac{a^3}{\mu}} = 19,034\ sec = 5h17m14s$$

Example: Time-based Position

Find the spacecraft position 3 hours after perigee

$$n = \sqrt{\frac{\mu}{a^3}} = 1.650x10^{-4}\ \frac{rad}{sec}$$

$$e = 1 - \frac{r_p}{a} = 0.7270$$

$$E_{j+1} = nt + e\sin E_j = 1.783 + 0.7270\sin E_j$$

E=0; 1.783; 2.494; 2.222; 2.361; 2.294; 2.328; 2.311; 2.320; 2.316; 2.318; 2.317; 2.317; 2.317

Example: Time-based Position (cont.)

$$E = 2.317$$

$$r = a(1 - e\cos E) = 12,387 \text{ km}$$

Have to be sure to get the position in the proper quadrant - since the time is less than 1/2 the period, the spacecraft has yet to reach apogee --> $0^\circ<\theta<180^\circ$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \Longrightarrow \theta = 160 \text{ deg}$$

Velocity Components in Orbit

$$r = \frac{p}{1 + e\cos\theta}$$

$$v_r = \frac{dr}{dt} = \frac{d}{dt}\left(\frac{p}{1 + e\cos\theta}\right) = \frac{-p(-e\sin\theta\frac{d\theta}{dt})}{(1 + e\cos\theta)^2}$$

$$v_r = \frac{pe\sin\theta}{(1 + e\cos\theta)^2}\frac{d\theta}{dt}$$

$$1 + e\cos\theta = \frac{p}{r} \Rightarrow v_r = \frac{r^2\frac{d\theta}{dt}e\sin\theta}{p}$$

$$\overrightarrow{h} = \overrightarrow{r} \times \overrightarrow{v}$$

Velocity Components in Orbit (cont.)

$$\vec{h} = \vec{r} \times \vec{v}$$

$$h = rv\cos\gamma = r\left(r\frac{d\theta}{dt}\right) = r^2\frac{d\theta}{dt}$$

$$v_r = \frac{r^2\frac{d\theta}{dt}e\sin\theta}{p} = \frac{he\sin\theta}{p} = \frac{\sqrt{p\mu}}{p}e\sin\theta$$

$$v_r = \sqrt{\frac{\mu}{p}}e\sin\theta$$

$$v_\theta = r\frac{d\theta}{dt} = r\frac{h}{r^2} = \frac{h}{r} = \frac{\sqrt{p\mu}}{r}$$

$$v_\theta = \sqrt{\frac{\mu}{p}}(1 + e\cos\theta)$$

$$\tan\gamma = \frac{v_r}{v_\theta} = \frac{e\sin\theta}{1 + e\cos\theta}$$

Patched Conics

- Simple approximation to multi-body motion (e.g., traveling from Earth orbit through solar orbit into Martian orbit)
- Treats multibody problem as “hand-offs” between gravitating bodies --> reduces analysis to sequential two-body problems
- Caveat Emptor: There are a number of formal methods to perform patched conic analysis. The approach presented here is a very simple, convenient, and not altogether accurate method for performing this calculation. Results will be accurate to a few percent, which is adequate at this level of design analysis.

Planetary Approach Analysis

- Spacecraft has v_h hyperbolic excess velocity, which fixes total energy of approach orbit

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} = \frac{v_h^2}{2}$$

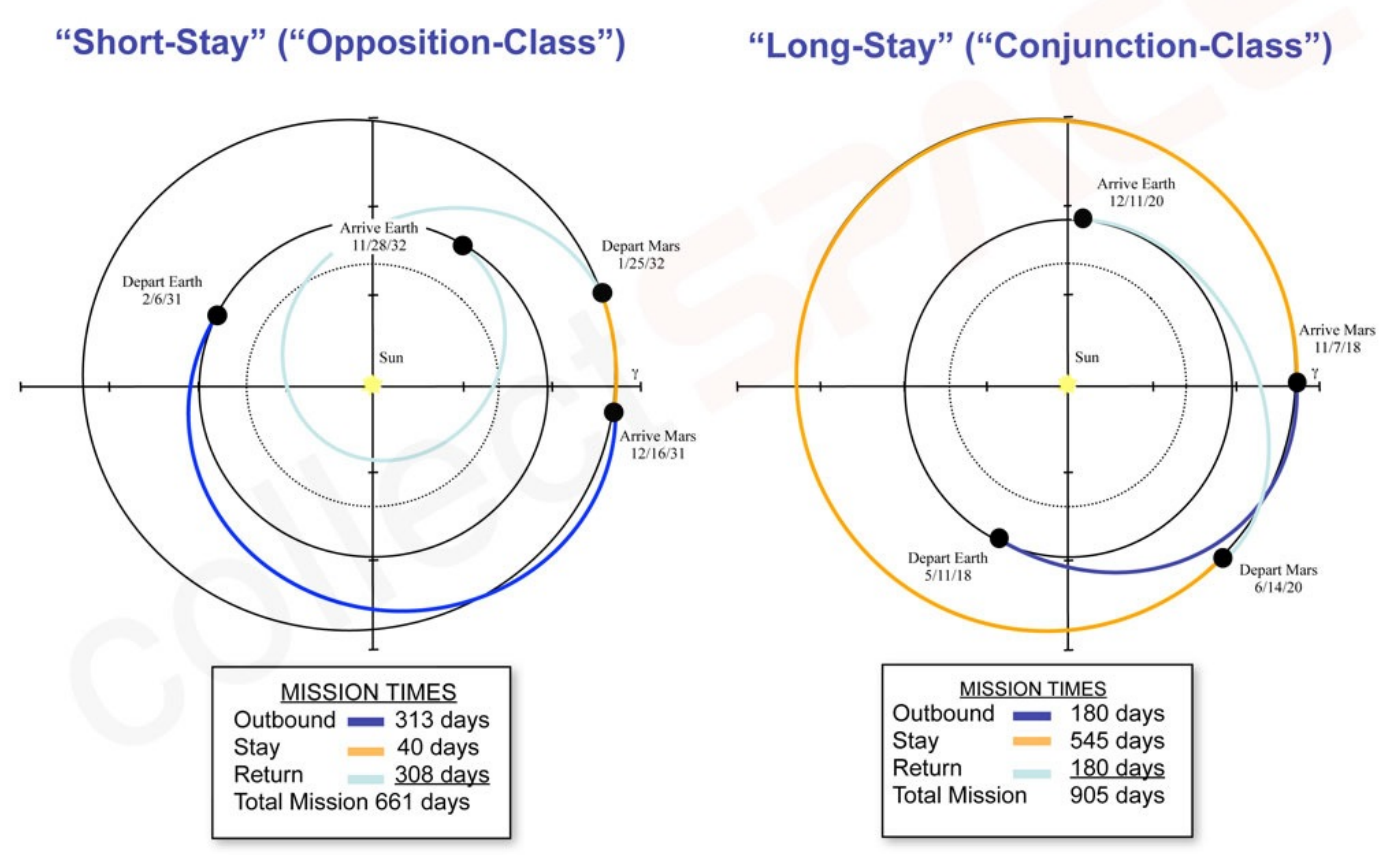
- Vis-viva provides velocity of approach

$$v = \sqrt{v_h^2 + \frac{2\mu}{r}}$$

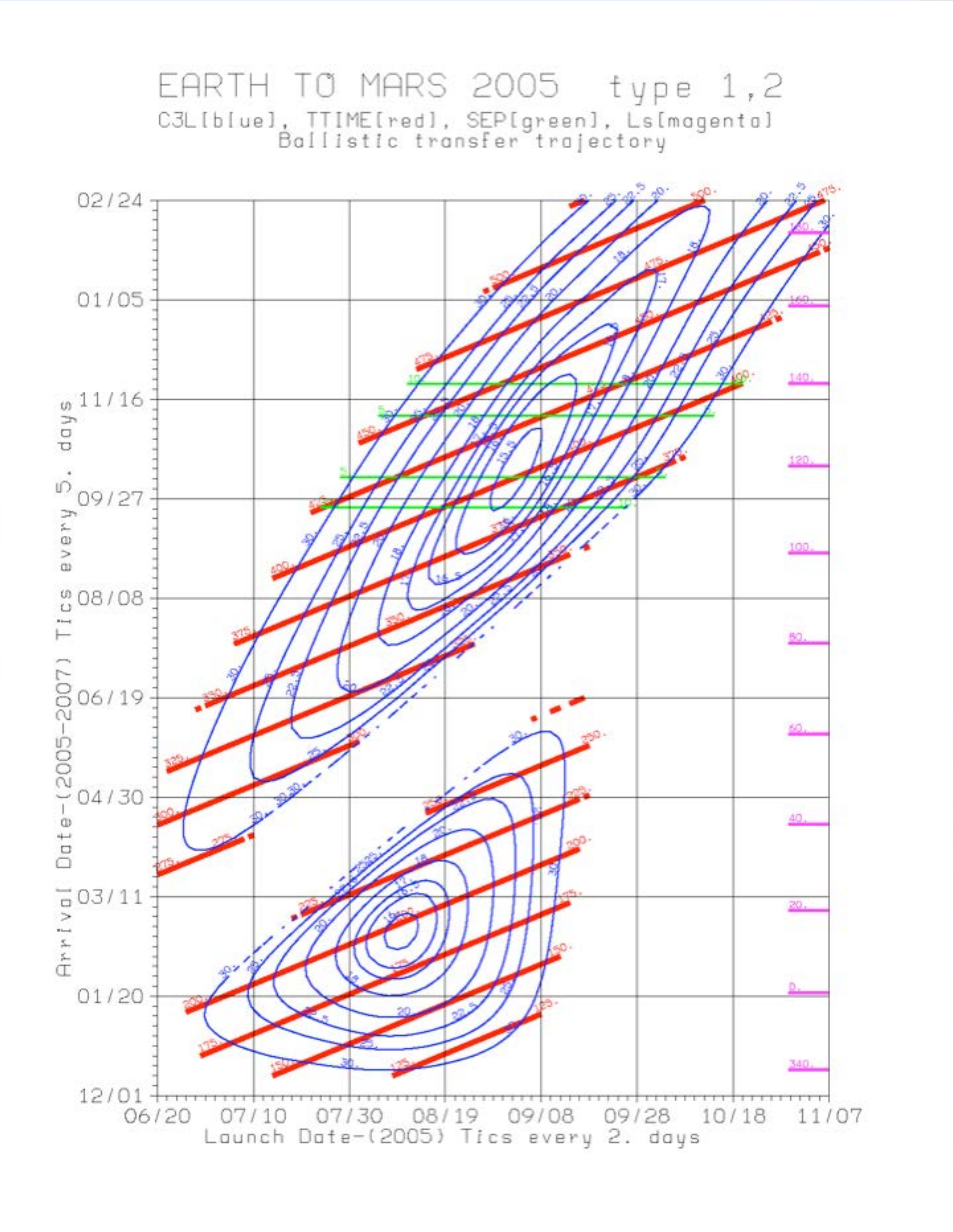
- Choose transfer orbit such that approach is tangent to desired final orbit at periapse

$$\Delta v = \sqrt{v_h^2 + \frac{2\mu}{r}} - \sqrt{\frac{\mu}{r}}$$

Interplanetary Trajectory Types

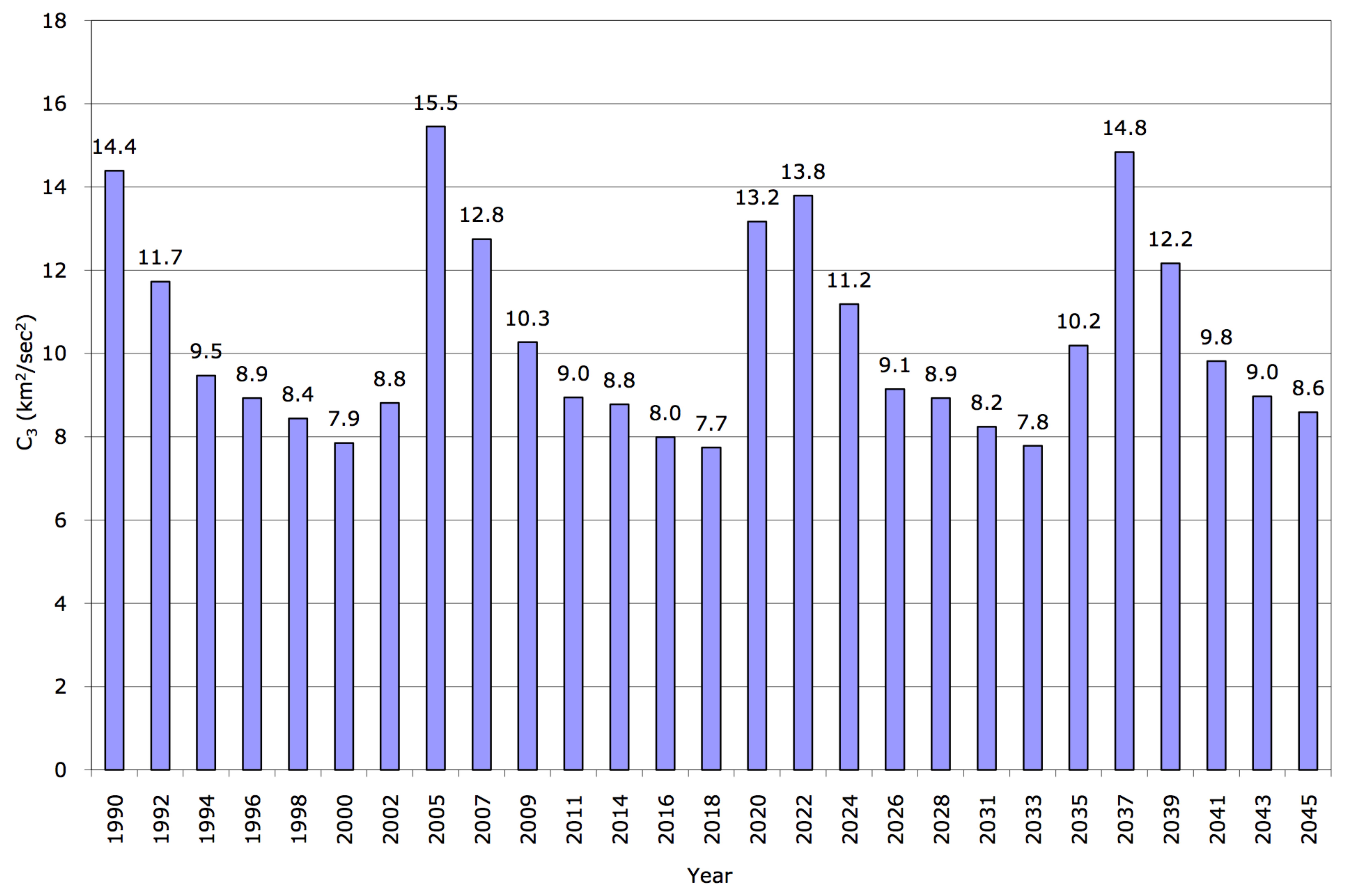


Interplanetary "Pork Chop" Plots



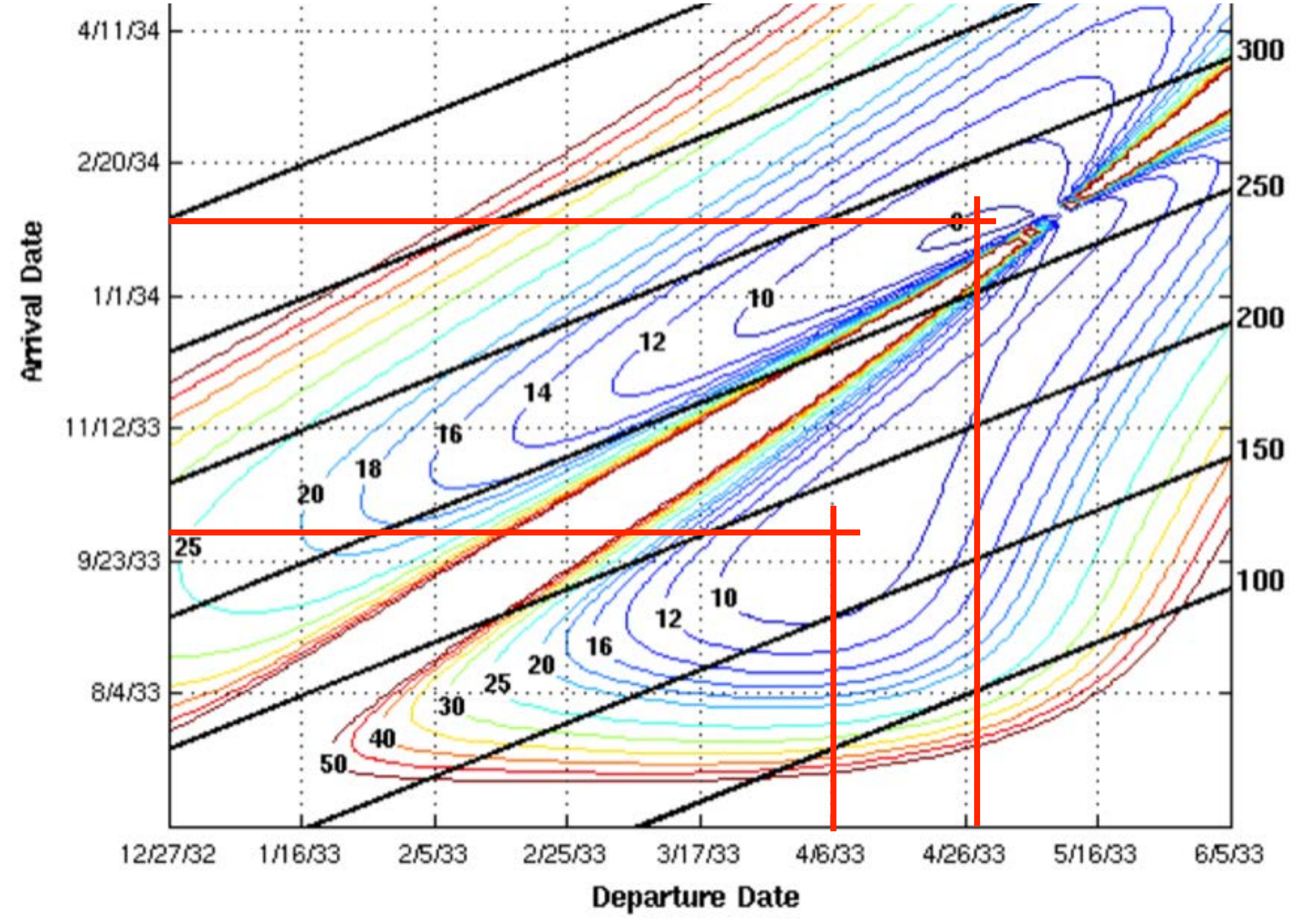
- Summarize a number of critical parameters
  - Date of departure
  - Date of arrival
  - Hyperbolic energy ("C3"= v_h^2)
  - Transfer geometry
- Launch vehicle determines available C3 based on window, payload mass
- Calculated using Lambert's Theorem

C3 for Earth-Mars Transfer 1990-2045



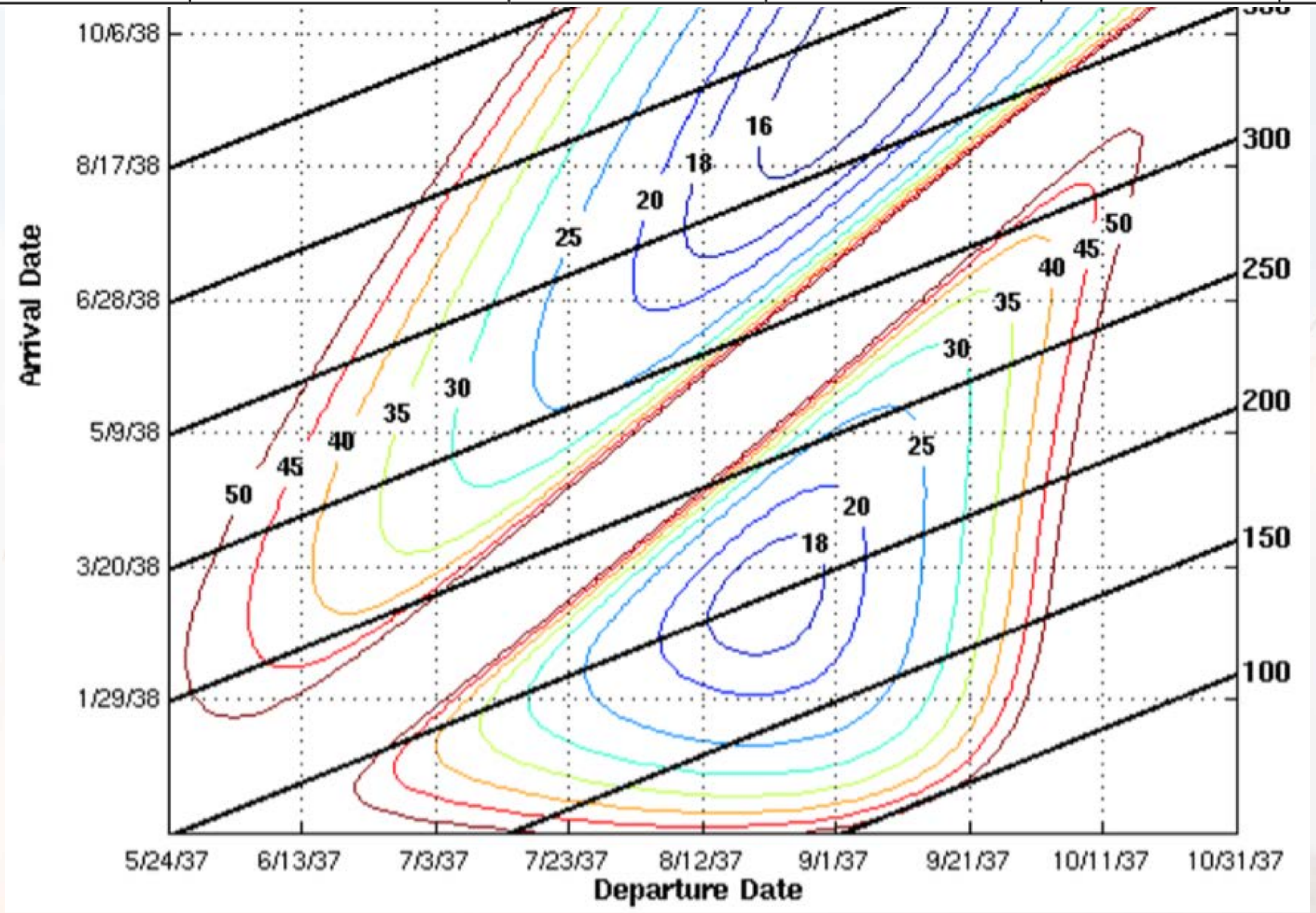
Earth-Mars Transfer 2033

Mission type	Earth departure date (m/d/yr)	Mars arrival date (m/d/yr)	C_3 (km^2/sec^2)	Right ascension (deg)	Declination (deg)	Mars arrival excess speed (km/s)
Type 1	4/6/33	10/1/33	8.412	271	–54.9	3.956
Type 2	4/28/33	1/27/34	7.781	311.4	–11.2	4.377
Type 1	4/20/33	11/6/33	9.266	267.1	–53.2	3.311
Type 2	1/26/33	10/17/33	17.78	278.3	–2.53	3.831



Earth-Mars Transfer 2037

Mission type	Earth departure date (m/d/yr)	Mars arrival date (m/d/yr)	C_3 (km^2/sec^2)	Right ascension (deg)	Declination (deg)	Mars arrival excess speed (km/s)
Type 1	6/2/37	12/17/37	**17.07**	43.45	39.79	3.344
Type 2	6/18/37	7/19/38	**14.84**	74.97	13.59	3.356
Type 1	6/30/37	2/19/38	28.33	26.54	32.34	**2.334**
Type 2	4/13/37	2/7/38	31.13	66.88	1.891	**2.422**



Interplanetary Delta-V

$$\text{Hyperbolic excess velocity} \equiv V_h$$

$$C_3 = V_h^2$$

$$V_{req} = \sqrt{V_{esc}^2 + C_3}$$

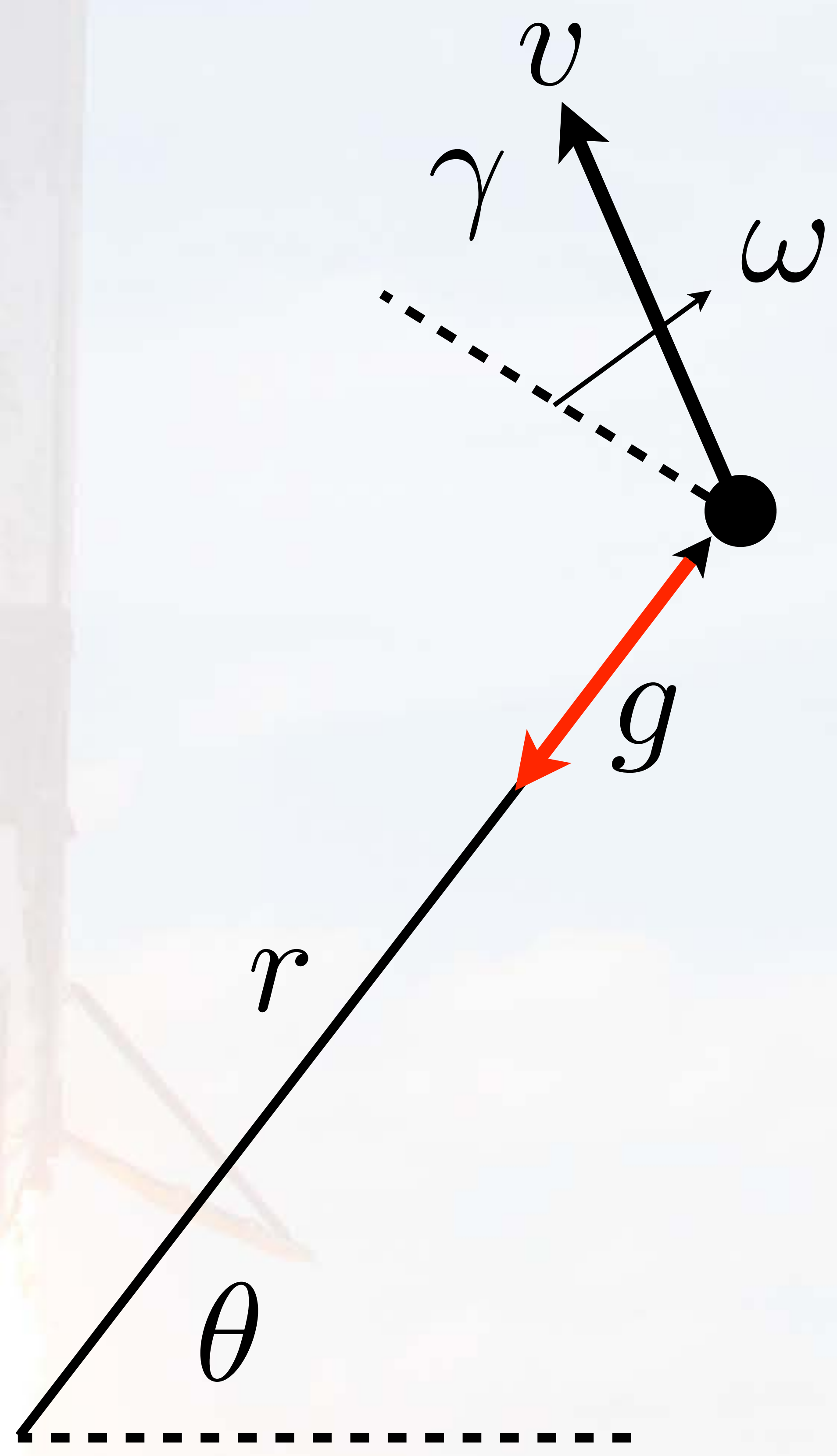
$$\Delta V = \sqrt{V_{esc}^2 + C_3} - V_c$$

$$\text{2033 Window: } \Delta V = 3.55\ km/sec$$

$$\text{2037 Window: } \Delta V = 3.859\ km/sec$$

$$\Delta V \text{ in departure from 300 km LEO}$$

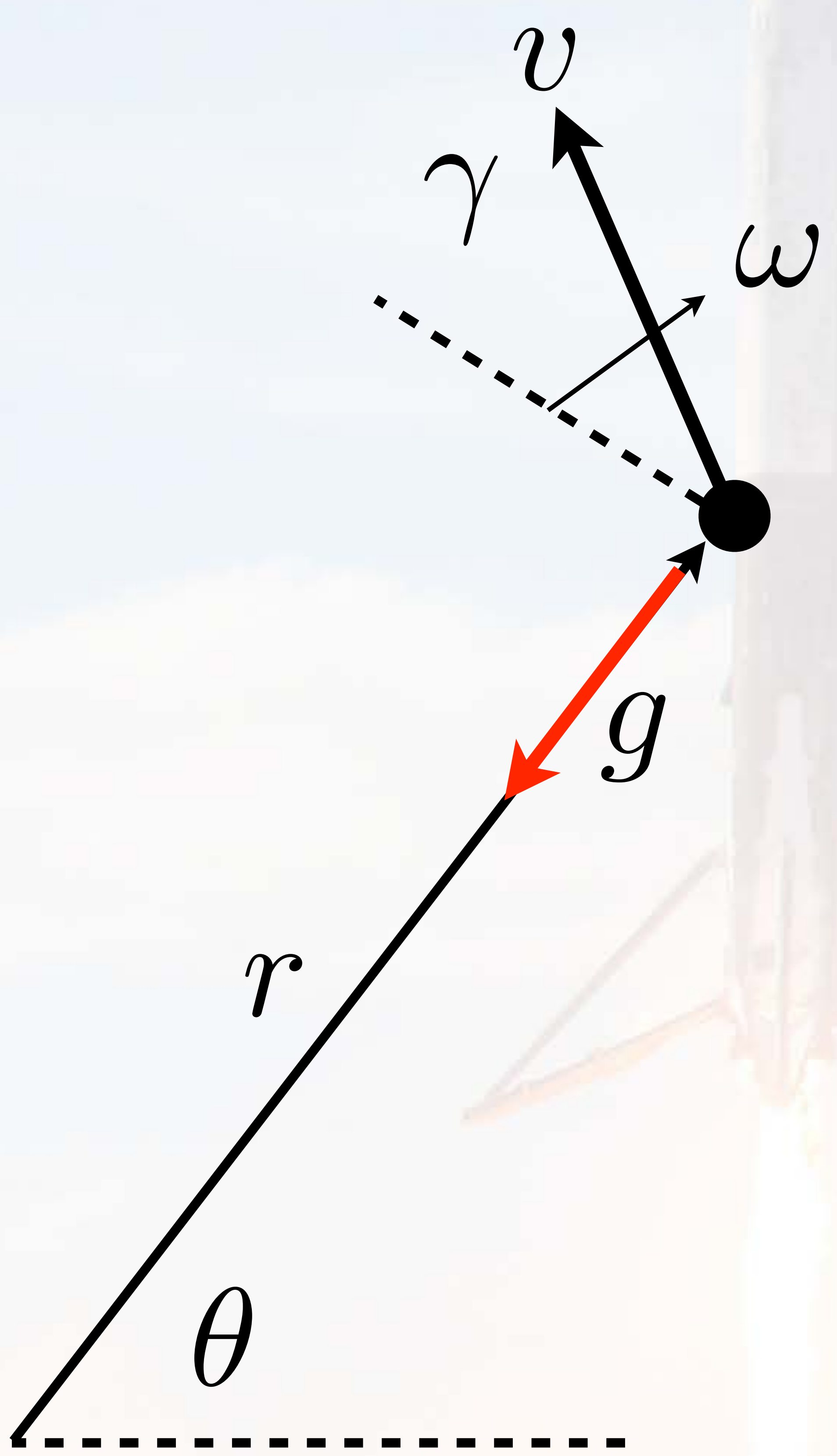
Free-Body Diagram with Spherical Planet



γ = flight path angle

ω = rotational velocity of $\bar{v}$

Orbital Planar State Equations



Inertial angular velocity

$$\omega = \dot{\gamma} - \dot{\theta}$$

Sum of accelerations normal to velocity vector

$$-g\cos\gamma = \omega v$$

Sum of accelerations perpendicular to velocity vector

$$-g\sin\gamma = \dot{v}$$

Orbital Planar State Equations (2)

$$\dot{r} = v \sin\gamma$$

$$r\dot{\theta} = v \cos\gamma$$

$$\omega = \dot{\gamma} - \dot{\theta} = \dot{\gamma} - \frac{v}{r}\cos\gamma$$

$$-g\cos\gamma = \left(\dot{\gamma} - \frac{v}{r}\cos\gamma\right) v$$

$$-\left(g - \frac{v^2}{r}\right)\cos\gamma = \dot{\gamma} v$$

$$-\left(1 - \frac{v^2}{rg}\right) g\cos\gamma = \dot{\gamma} v$$

Canonical Orbital Planar State Equations

$$\dot{\gamma} = -\frac{1}{v}\left(1 - \frac{v^2}{v_c^2}\right) g \cos\gamma$$

$$\dot{v} = -g \sin\gamma$$

$$\dot{r} = v \sin\gamma$$

$$\dot{\theta} = \frac{v}{r} \cos\gamma$$

Coupled first-order ODEs

$$g = g_o \left(\frac{r_o}{r}\right)^2$$

Numerical Integration - 4th Order R-K

Given a series of equations $\dot{\bar{y}} = \bar{f}(t, \bar{x})$

$$\bar{k_1} = \Delta t \ \bar{f}\left(t_n, \bar{y_n}\right)$$

$$\bar{k_2} = \Delta t \ \bar{f}\left(t_n + \frac{\Delta t}{2}, \bar{y_n} + \frac{\bar{k_1}}{2}\right)$$

$$\bar{k_3} = \Delta t \ \bar{f}\left(t_n + \frac{\Delta t}{2}, \bar{y_n} + \frac{\bar{k_2}}{2}\right)$$

$$\bar{k_4} = \Delta t \ \bar{f}\left(t_n + \Delta t, \bar{y_n} + \bar{k_3}\right)$$

$$\bar{y_{n+1}} = \bar{y_n} + \frac{\bar{k_1}}{6} + \frac{\bar{k_2}}{3} + \frac{\bar{k_3}}{3} + \frac{\bar{k_4}}{6} + O(\Delta t^5)$$

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