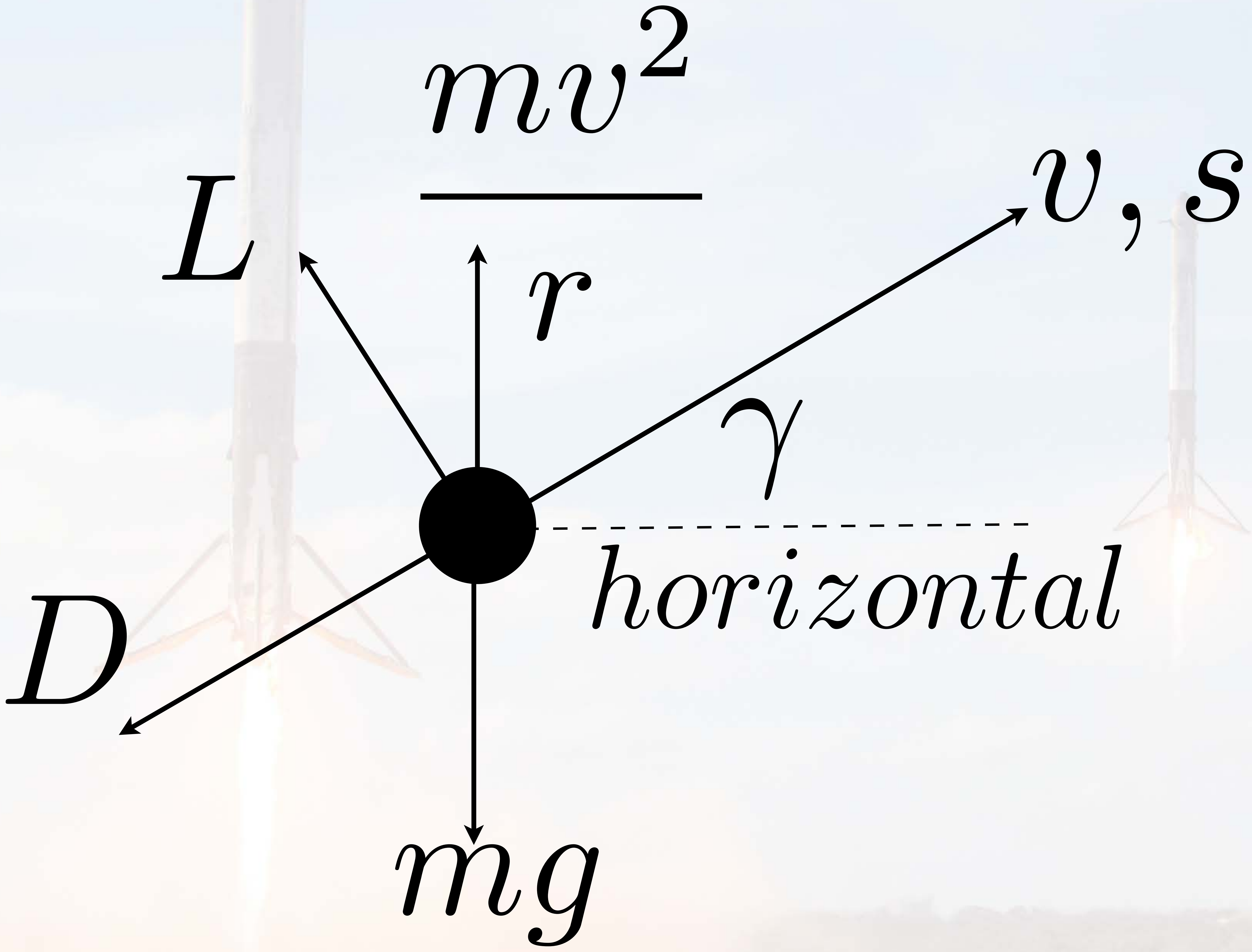
Lifting Entry

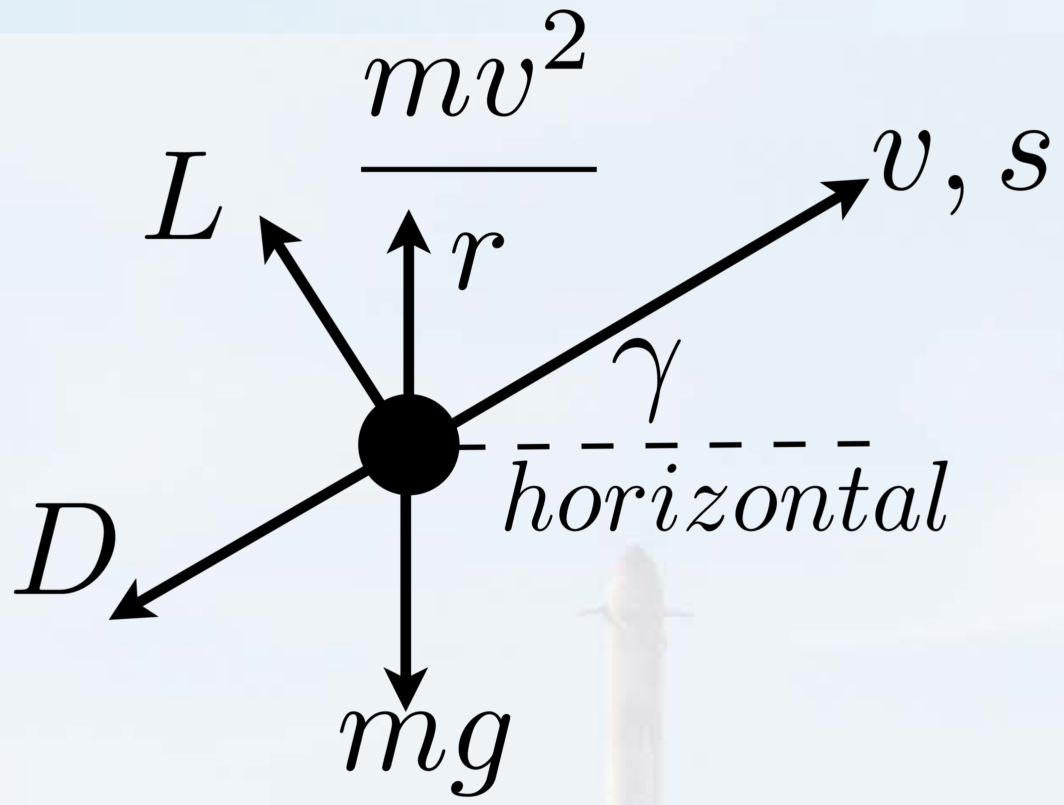
- Basic planar dynamics of motion, again
- Yet another equilibrium glide
- Hypersonic phugoid motion
- Planar state equations
- Top-level discussion of bank angle and cross-range

© 2024 David L. Akin - All rights reserved
http://spacecraft.ssl.umd.edu

Lifting Entry – Free-Body Diagram



Dynamics of Lifting Entry



Along the velocity vector,

$$m\frac{dv}{dt} = m\frac{v^2}{r}\sin\gamma - mg\sin\gamma - D$$

Perpendicular to the velocity vector,

$$mv\frac{d\gamma}{dt} = L + m\frac{v^2}{r}\cos\gamma - mg\cos\gamma$$

(unbalanced lift rotates flight path angle)

Equations of Planar Lifting Entry

$$\frac{dv}{dt} = \left(\frac{v^2}{r} - g\right)\sin\gamma - \frac{D}{m}$$

$$v\frac{d\gamma}{dt} = \frac{L}{m} + \left(\frac{v^2}{r} - g\right)\cos\gamma$$

Equilibrium Glide

- Forces perpendicular to velocity vector are balanced

$$\Longrightarrow \frac{d\gamma}{dt} = 0;\ \ \gamma = \text{constant}$$

- Typically very shallow glide

$$\Longrightarrow \text{assume } \gamma \rightarrow 0;\ \ \sin(\gamma) \rightarrow 0;\ \ \cos(\gamma) \rightarrow 1$$

Equilibrium Glide Equations

$$\frac{dv}{dt} = -\frac{D}{m}$$

$$D = \frac{1}{2}\rho v^2 c_D A$$

$$\rho = \rho_o e^{-\frac{h}{h_s}}$$

$$\frac{dv}{dt} = -\frac{1}{2}\rho_o v^2 \frac{c_D A}{m} e^{-\frac{h}{h_s}}$$

$$0 = \frac{L}{m} + \left(\frac{v^2}{r} - g\right)$$

$$\frac{v^2}{r} - g = -\frac{1}{2}\rho_o v^2 \frac{c_L A}{m} e^{-\frac{h}{h_s}}$$

Dynamics Perpendicular to Velocity

$$\frac{c_L}{c_D} = \frac{L}{D}$$

L/D set by vehicle aerodynamics, flight velocity, and angle of attack (assumed constant)

$$\frac{v^2}{r} - g = -\frac{1}{2}\rho_o v^2 \frac{L}{D}\frac{c_D A}{m} e^{-\frac{h}{h_s}}$$

$$\frac{v^2}{r} = -\frac{1}{2}\rho_o v^2 \frac{L/D}{\beta} e^{-\frac{h}{h_s}} + g$$

$$v^2 = -\frac{1}{2}\rho_o r v^2 \frac{L/D}{\beta} e^{-\frac{h}{h_s}} + gr$$

More Lift-Direction Dynamics

$$\text{Let } e^{-\frac{h}{h_s}} \equiv \sigma \quad \left(= \text{density ratio} \equiv \frac{\rho}{\rho_o} \right)$$

$$v^2 + \frac{1}{2}\rho_o r v^2 \frac{L/D}{\beta}\sigma = gr$$

$$v^2 \left(1 + \frac{1}{2}\rho_o r \frac{L/D}{\beta}\sigma \right) = gr$$

$$v = \sqrt{\frac{gr}{1 + \frac{\rho_o r \sigma (L/D)}{2\beta}}}$$

Velocity during Entry

$$v_{c_o} = \sqrt{\frac{\mu}{r_o}} = \sqrt{g_o r_o} \qquad (\mu = g_o r_o^2)$$

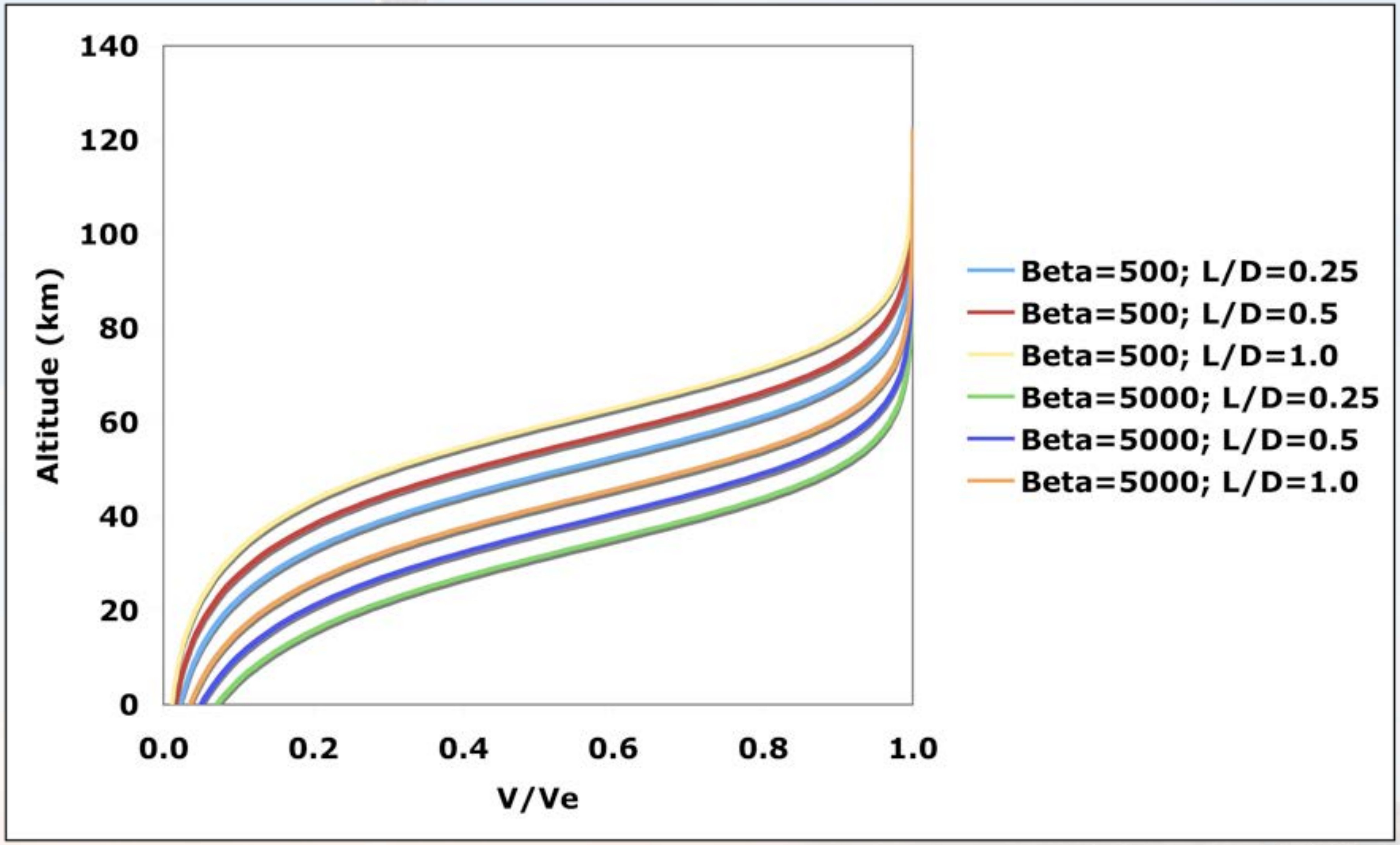
$$\frac{v}{v_{c_o}} = \sqrt{\frac{1}{1 + \frac{\rho_o r_o \sigma (L/D)}{2\beta}}}$$

$$v_{c_o} \cong v_e \quad \text{(within 1-2\% for Earth)}$$

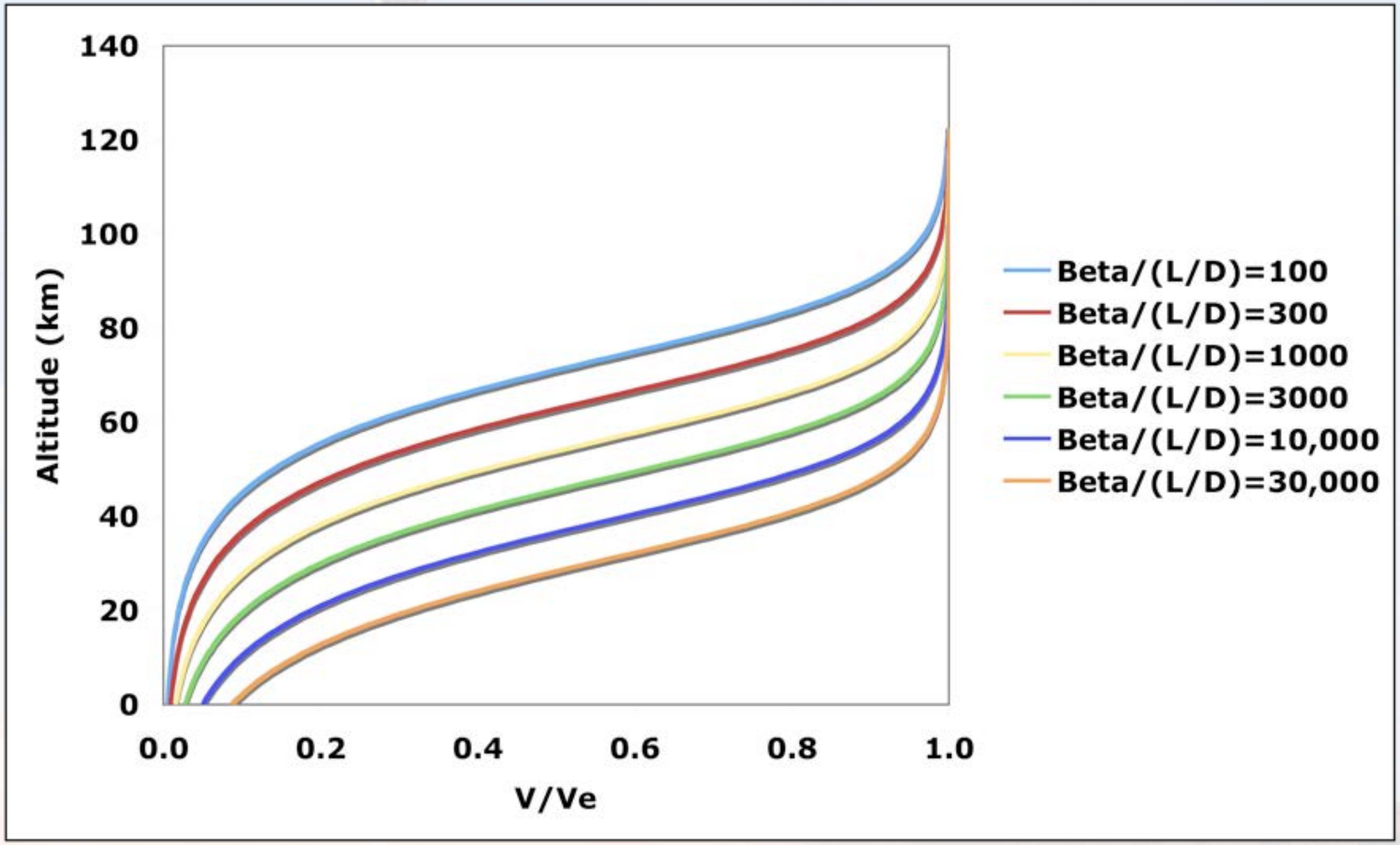
$$\frac{v}{v_e} = \left[1 + \frac{\rho_o r_o}{2\beta}\frac{L}{D} e^{-\frac{h}{h_s}}\right]^{-\frac{1}{2}}$$

Entry trajectory as a function of altitude, ratio $\left(\frac{\beta}{L/D}\right)$

Equilibrium Glide Velocity Trends



Trends with "Lifting Ballistic Coefficient"



Nondimensional Form of Equation

$$\frac{v}{v_e} = \left[1 + \frac{\rho_o r_o}{2\beta}\frac{L}{D}e^{-\frac{h}{h_s}}\right]^{-\frac{1}{2}}$$

Remember $\widehat{\beta} \equiv \dfrac{\beta}{\rho_o h_s}$

$$\frac{v}{v_e} = \left[1 + \frac{\rho_o h_s}{2\beta}\frac{r_o}{h_s}\frac{L}{D}e^{-\frac{h}{h_s}}\right]^{-\frac{1}{2}}$$

$$\frac{v}{v_e} = \left[1 + \frac{1}{2\widehat{\beta}}\frac{r_o}{h_s}\frac{L}{D}e^{-\frac{h}{h_e}\frac{h_e}{h_s}}\right]^{-\frac{1}{2}}$$

Deceleration

$$\frac{L}{m} = g - \frac{v^2}{r} = g - \frac{gv^2}{gr} = g\left[1 - \left(\frac{v}{v_{c_o}}\right)^2\right]$$

$$\frac{dv}{dt} = -\frac{D}{m} = -\frac{L}{L/D}\frac{1}{m} = -\frac{1}{L/D}\frac{L}{m}$$

$$\frac{dv}{dt} = -\frac{g}{L/D}\left[1 - \left(\frac{v}{v_{c_o}}\right)^2\right]$$

Deceleration

$$\text{Let } n \equiv \frac{1}{g}\frac{dv}{dt} \quad \text{(deceleration in g's)}$$

$$n = -\frac{1}{L/D}\left[1 - \left(\frac{v}{v_{c_o}}\right)^2\right]$$

$$n = -\frac{1}{L/D}\left[1 - \frac{1}{1 + \frac{\rho_o r_o}{2\beta}\frac{L}{D}e^{-\frac{h}{h_s}}}\right]$$

$$\left(\text{Note: } 1 - \frac{1}{1+K} = \frac{1+K}{1+K} - \frac{1}{1+K} = \frac{K}{1+K}\right)$$

Deceleration

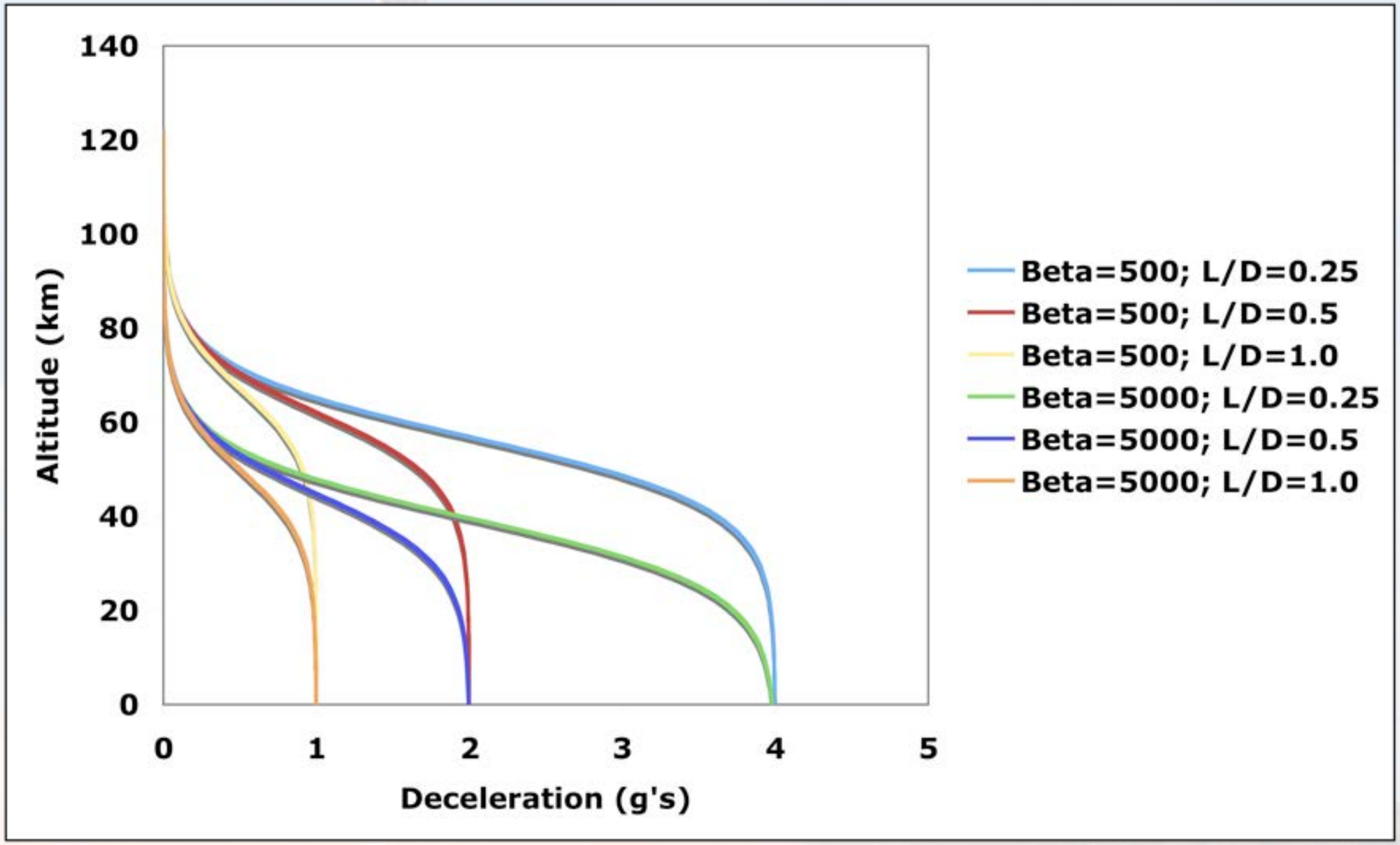
$$n = -\frac{1}{L/D}\left[\frac{\frac{\rho_o r_o}{2\beta}\frac{L}{D}e^{-\frac{h}{h_s}}}{1+\frac{\rho_o r_o}{2\beta}\frac{L}{D}e^{-\frac{h}{h_s}}}\right]$$

$$n = -\frac{\frac{\rho_o r_o}{2\beta}e^{-\frac{h}{h_s}}}{1+\frac{\rho_o r_o}{2\beta}\frac{L}{D}e^{-\frac{h}{h_s}}}$$

$$n = \frac{-1}{\frac{2\beta}{\rho_o r_o}e^{+\frac{h}{h_s}}+\frac{L}{D}}$$

n monotonically increases with decreasing altitude

Deceleration Trends with Altitude



Limiting Deceleration

$$n_{limit} = \frac{-1}{\frac{2\beta}{\rho_o r_o} + \frac{L}{D}}$$

$$\beta \sim \mathrm{O}(10^3);\ \ \rho_o \sim \mathrm{O}(1);\ \ r_o \sim \mathrm{O}(10^6) \Longrightarrow \frac{2\beta}{\rho_o r_o} \sim \mathrm{O}(10^{-3})$$

$$n_{limit} \cong \frac{-1}{L/D}$$

Lift significantly moderates peak g's on entry (L/D=0.25 --> n_{limit}=4 g's)

Time for Entry

$$\frac{dv}{dt} = -\frac{g}{L/D}\left[1 - \left(\frac{v}{v_{c_o}}\right)^2\right]$$

$$dt = \frac{-(L/D)dv}{g\left[1 - \left(\frac{v}{v_{c_o}}\right)^2\right]}$$

$$\int_0^t dt = \int_{v_e}^0 \frac{-(L/D)dv}{g\left[1 - \left(\frac{v}{v_{c_o}}\right)^2\right]}$$

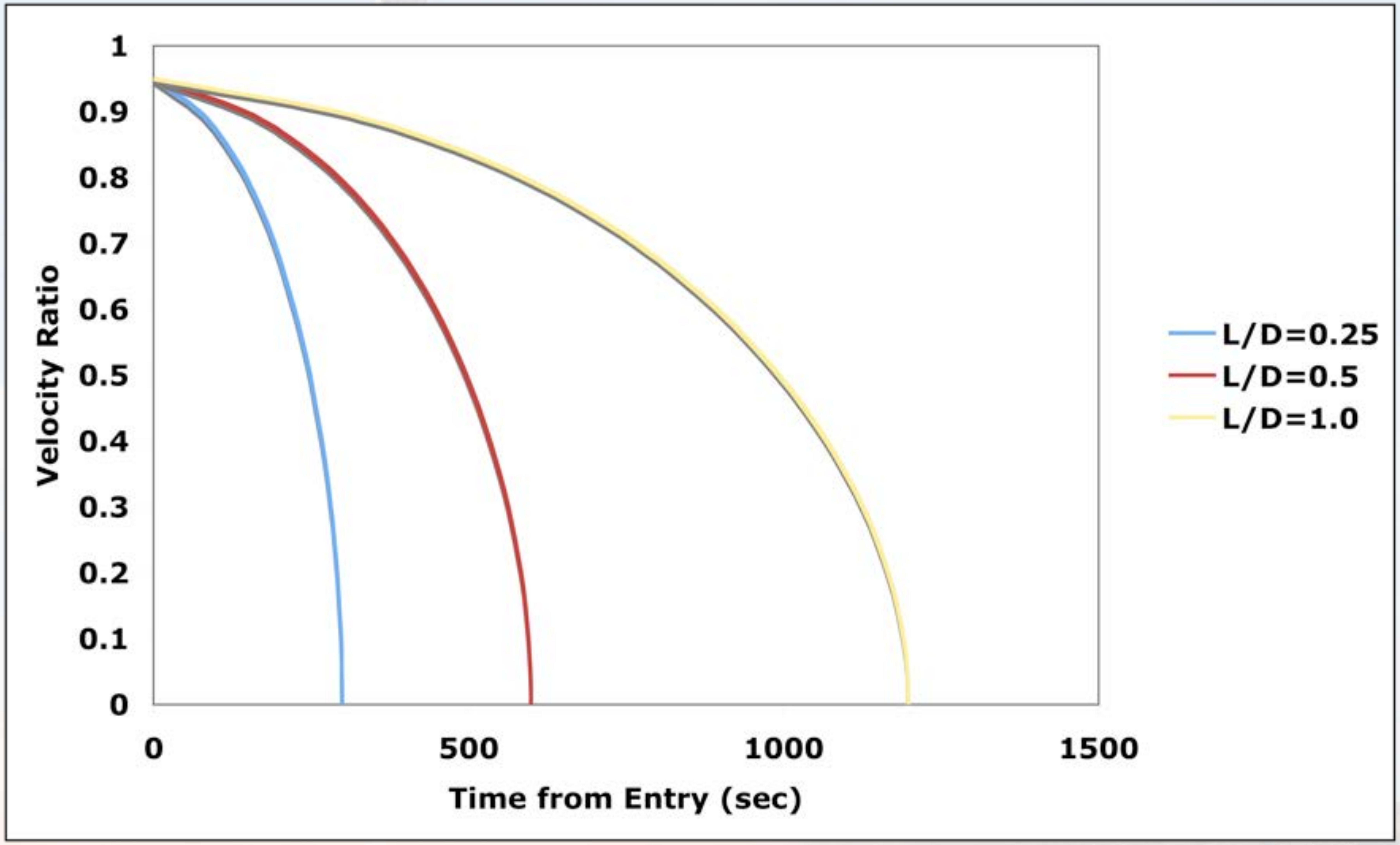
Time for Entry

$$\Delta t = \frac{1}{2}\sqrt{\frac{r_o}{g_o}}\frac{L}{D}\ln\frac{1+\left(\frac{v}{v_{c_o}}\right)^2}{1-\left(\frac{v}{v_{c_o}}\right)^2}$$

$$\text{Time for entry} \propto \frac{L}{D}$$

Not a function of β

Time From Entry Interface



Distance Along Flight Path

For shallow entry, $\dfrac{ds}{dt} \cong v \qquad ds = v\ dt$

$$ds = \frac{-(L/D)v\ dv}{g\left[1 - \left(\frac{v}{v_{c_o}}\right)^2\right]}$$

Let $u \equiv 1 - \left(\dfrac{v}{v_{c_o}}\right)^2 \qquad du = \dfrac{-2v}{v_{c_o}^2}dv \qquad v\ dv = \dfrac{-v_{c_o}^2}{2}du$

$$\int ds = \frac{L/D}{2g}\int \frac{v_{c_o}^2 du}{u}$$

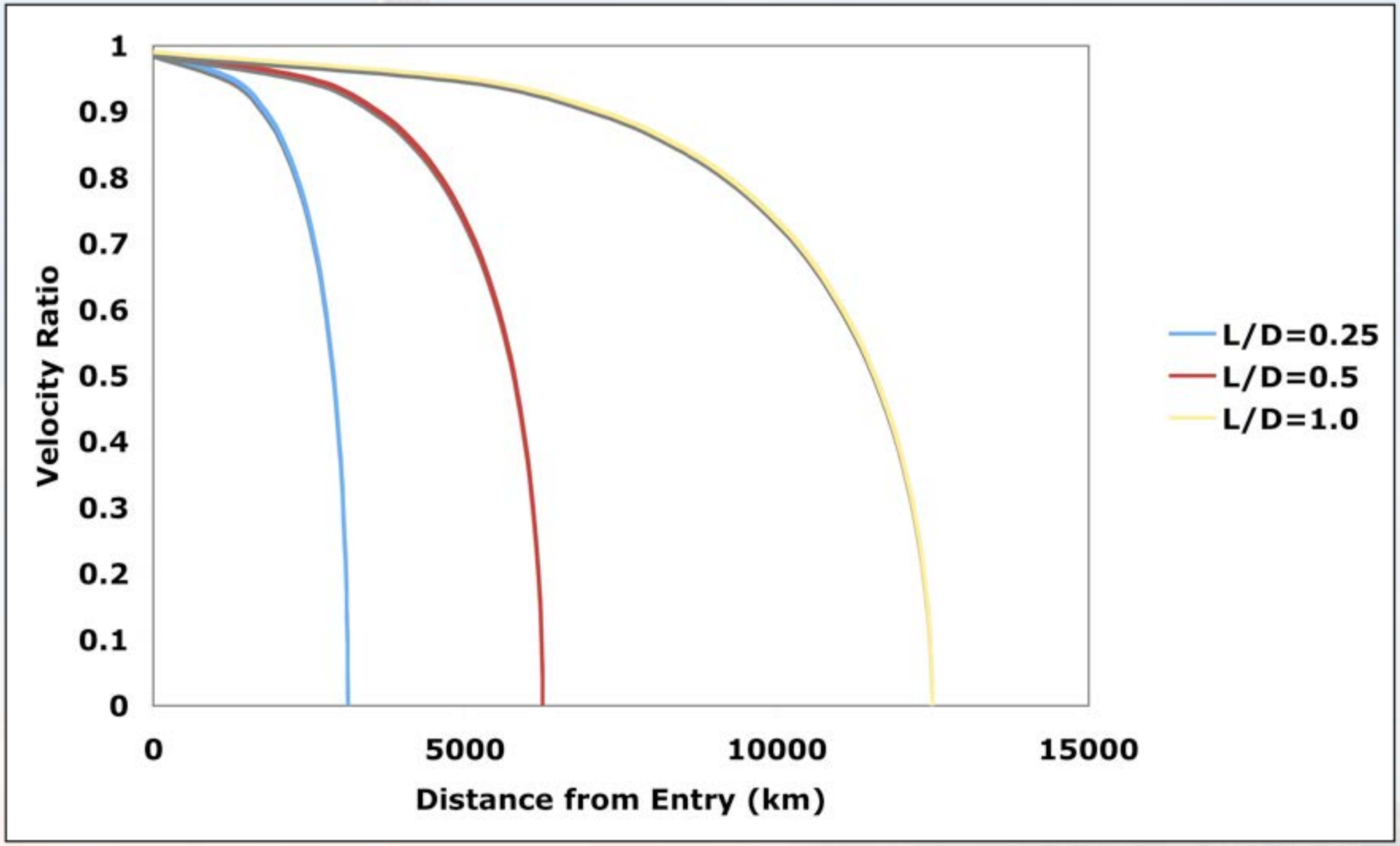
Distance Along Flight Path

$$\Delta s = \frac{(L/D)v_{c_o}^2}{2g}\ln u = \frac{(L/D)g_o r_o}{2g_o}\ln\left[1-\left(\frac{v}{v_{c_o}}\right)^2\right]$$

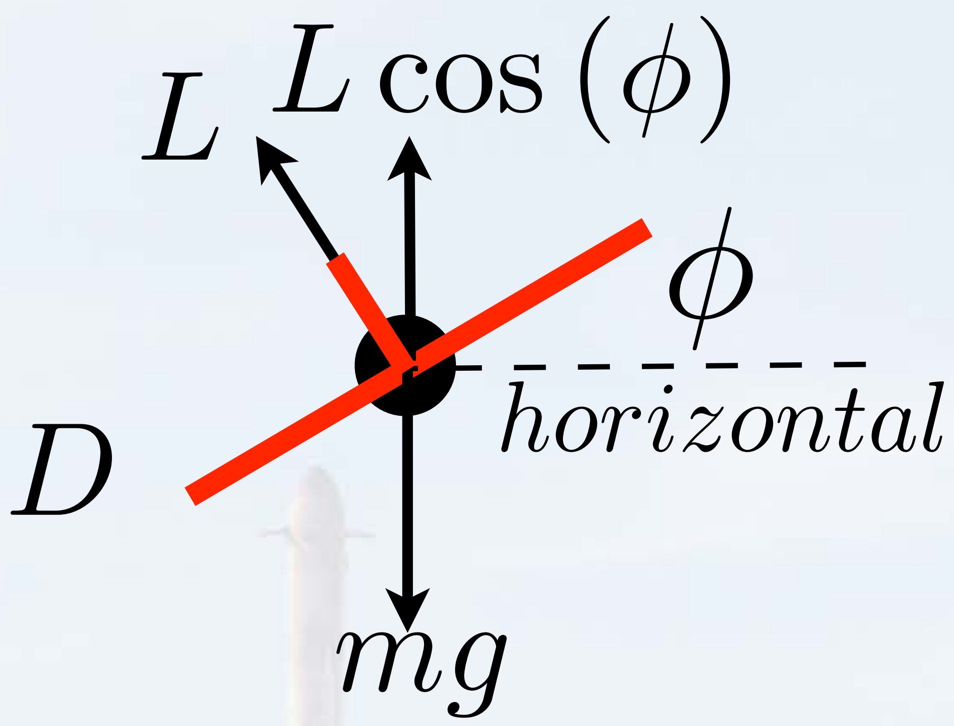
$$\Delta s = \frac{r_o}{2}\frac{L}{D}\ln\left[1-\left(\frac{v}{v_{c_o}}\right)^2\right]$$

Not a function of β or g

Distance from Entry



Bank Angle



$$\phi \equiv \text{Bank Angle}$$

$$\text{Level turn: } L\cos\phi = mg$$

$$\frac{L}{mg} = \frac{1}{\cos\phi} \equiv n_{bank}$$

(g's you pull in the banked turn)

Crossrange (without derivations)

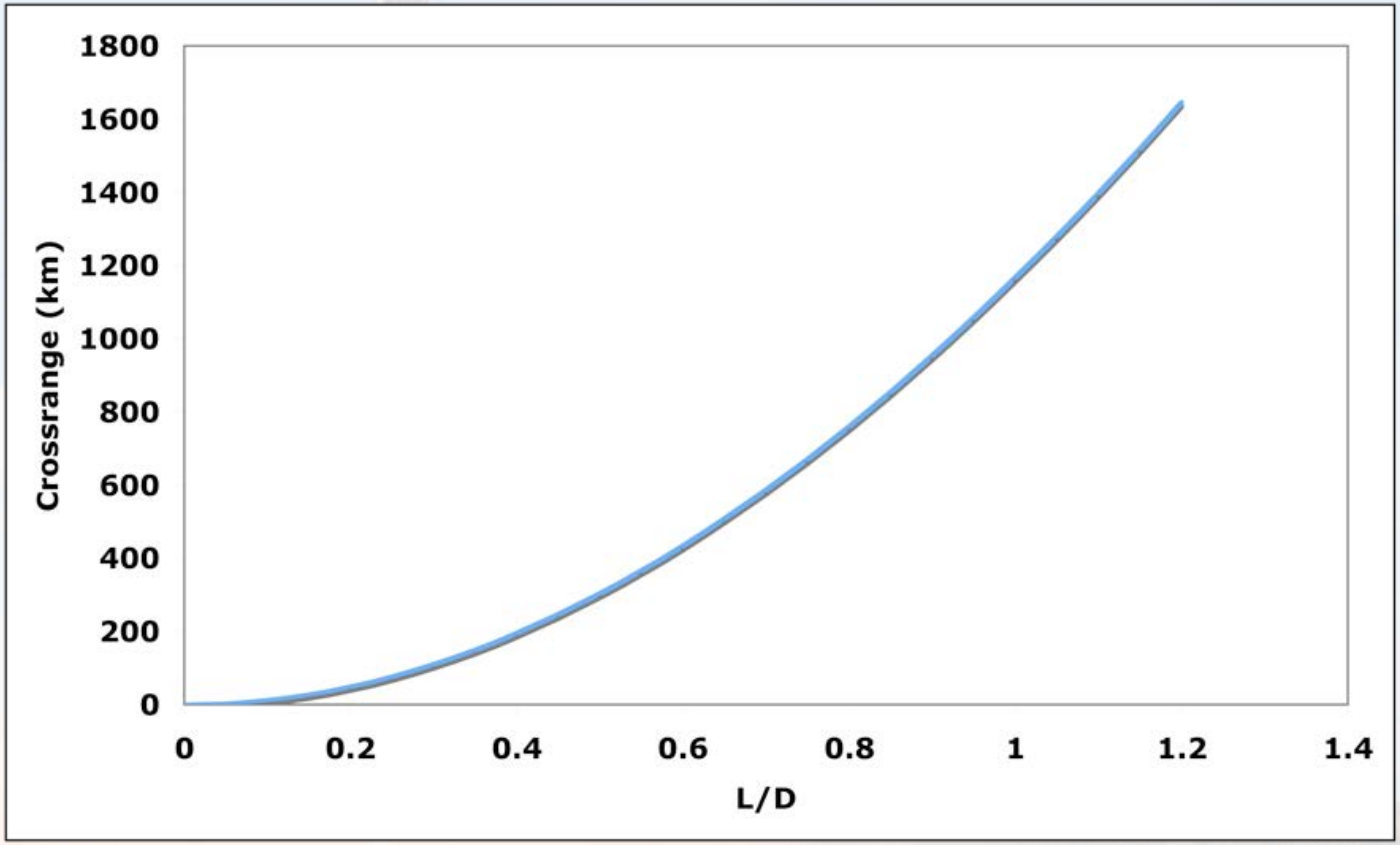
- Use of lift to deviate laterally from planar groundtrack
- Optimum bank angle to maximize crossrange

$$\phi_{opt} \cong \cot^{-1}\sqrt{1+0.106\left(\frac{L}{D}\right)^2}$$

- Maximum achievable crossrange

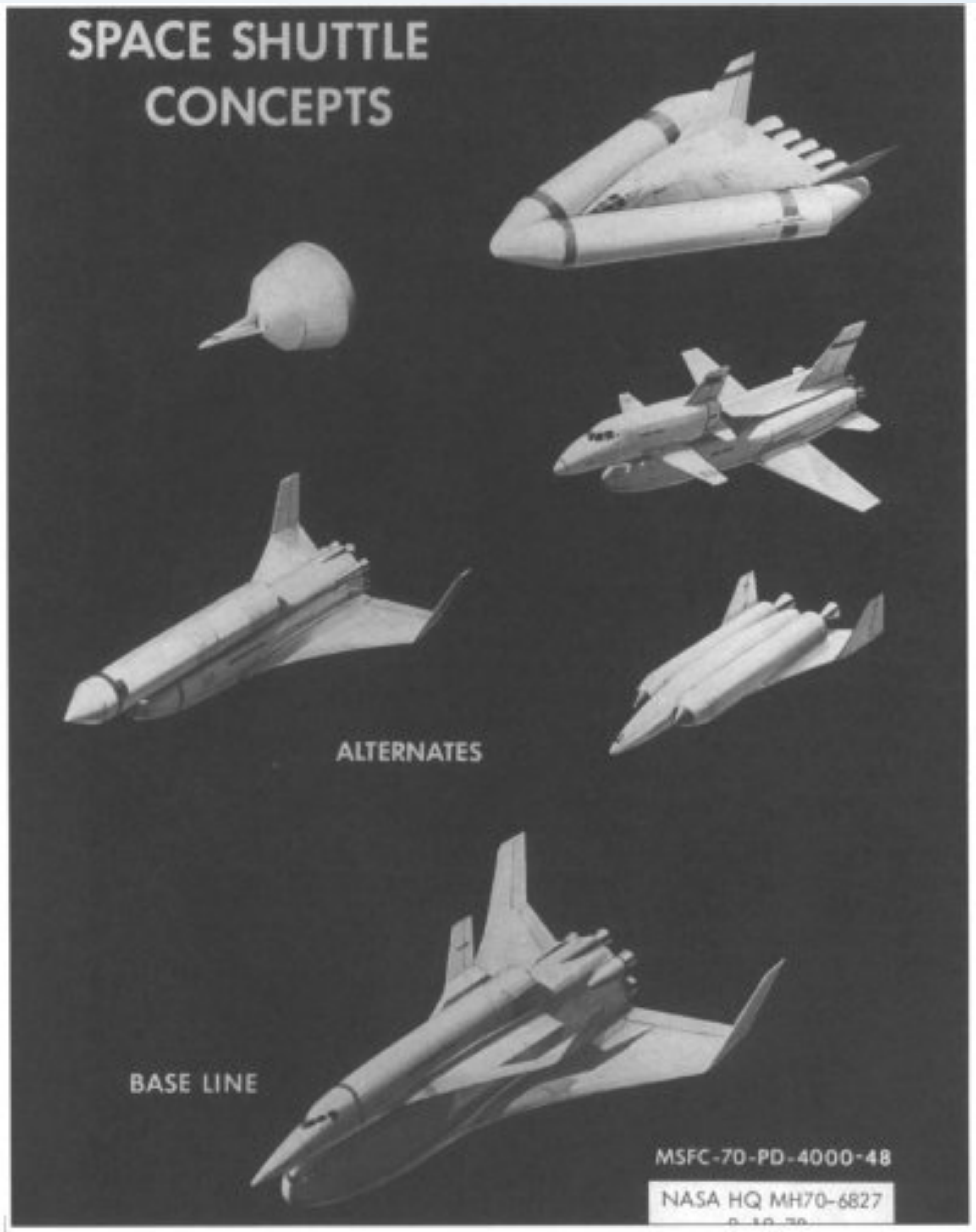
$$y_{max} \cong \frac{r_o}{5.2}\left(\frac{L}{D}\right)^2 \frac{1}{\sqrt{1+0.106\left(\frac{L}{D}\right)^2}}$$

Crossrange Based on L/D



Early Shuttle Design Configurations

- Delta wing configuration
  - High hypersonic lift
  - High landing velocity
  - High crossrange
- Straight-wing configuration
  - High subsonic lift
  - Low(er) landing velocity
  - Low crossrange

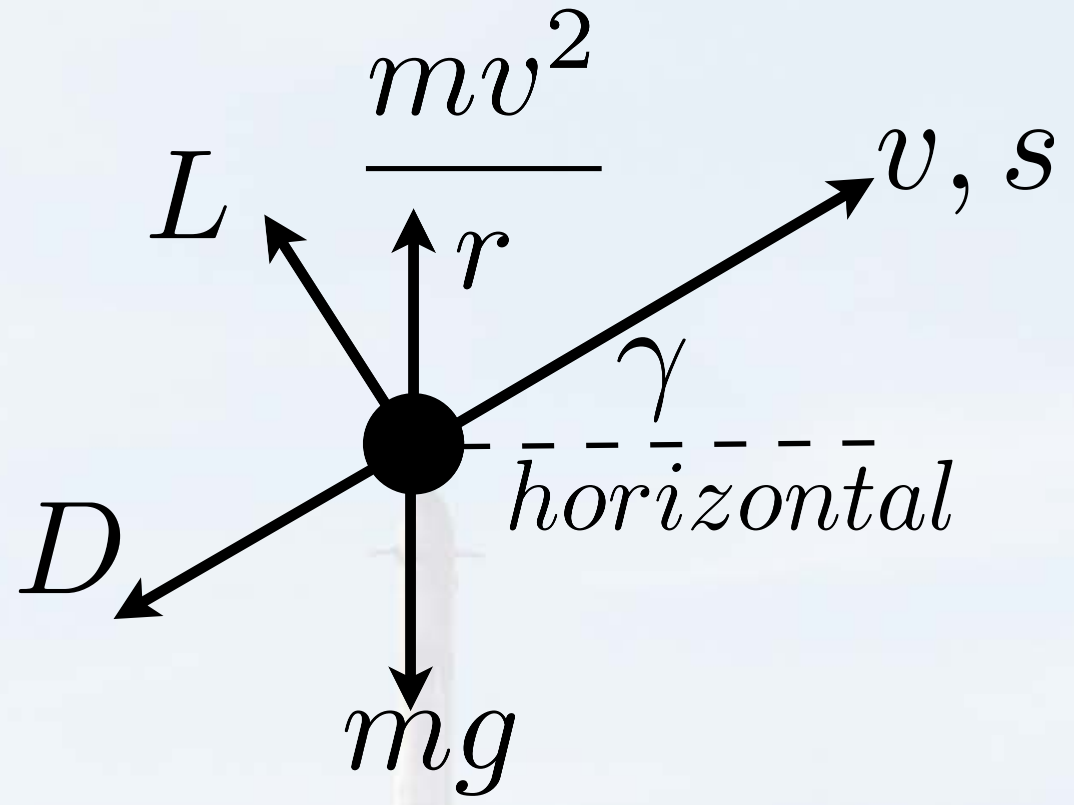


Basic Equations of Motion

From last time,

$$v\frac{d\gamma}{dt} = \frac{L}{m} + \left(\frac{v^2}{r} - g\right)\cos\gamma$$

$$\frac{dv}{dt} = \left(\frac{v^2}{r} - g\right)\sin\gamma - \frac{D}{m}$$



Assume (at entry velocities close to orbital) $g \cong \frac{v^2}{r}$

$$v\frac{d\gamma}{dt} = \frac{L}{m} = \frac{\rho v^2}{2}\frac{c_L A}{m} = \frac{\rho v^2}{2\beta}\frac{L}{D} \qquad (1)$$

$$\frac{dv}{dt} = -\frac{D}{m} = -\frac{\rho v^2}{2\beta} \qquad (2)$$

Solving for Velocity

Divide (2) by (1)

$$\frac{\frac{dv}{dt}}{v\frac{d\gamma}{dt}} = \frac{-\frac{\rho v^2}{2\beta}}{\frac{\rho v^2}{2\beta}\frac{L}{D}} \implies \frac{dv}{v} = -\frac{d\gamma}{L/D} \tag{3}$$

$$\int_{v_e}^{v} \frac{dv}{v} = -\frac{1}{L/D}\int_{\gamma_e}^{\gamma} d\gamma$$

$$\ln\frac{v}{v_e} = \frac{-(\gamma-\gamma_e)}{L/D} \implies \frac{v}{v_e} = e^{-\frac{\gamma-\gamma_e}{L/D}} \tag{4}$$

Differential Elements

As before,

$$\frac{dh}{dt} = v \sin \gamma \tag{5}$$

$$\rho = \rho_o e^{-\frac{h}{h_s}} \tag{6}$$

Differentiating (6),

$$\frac{d\rho}{dt} = -\frac{\rho_o}{h_s} e^{-\frac{h}{h_s}} \frac{dh}{dt} = -\frac{\rho}{h_s} \frac{dh}{dt}$$

$$\frac{d\rho}{dt} = -\frac{\rho}{h_s} v \sin \gamma \tag{7}$$

Differential Elements (2)

Solve (7) for v

$$v = \frac{-h_s}{\sin\gamma}\left(\frac{1}{\rho}\frac{d\rho}{dt}\right) \tag{8}$$

Substitute (8) into (1) and rewrite as

$$\frac{d\gamma}{dt} = \frac{\rho v}{2\beta}\frac{L}{D} = \frac{\rho}{2\beta}\frac{L}{D}\left(\frac{-h_s}{\sin\gamma}\right)\frac{1}{\rho}\frac{d\rho}{dt}$$

$$\frac{d\gamma}{dt} = \frac{-h_s}{2\beta\sin\gamma}\frac{L}{D}d\rho \tag{9}$$

Solving for Flight Path Angle

$$\int_{\gamma_e}^{\gamma} \sin\gamma d\gamma = -\frac{h_s}{2\beta}\frac{L}{D}\int_0^{\rho} d\rho \tag{9}$$

$$\cos\gamma - \cos\gamma_e = \frac{h_s}{2\beta}\frac{L}{D}\rho \tag{10}$$

$$\cos\gamma = \frac{h_s}{2\beta}\frac{L}{D}\rho + \cos\gamma_e \tag{11}$$

$$\gamma = -\cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho + \cos\gamma_e\right) \tag{12}$$

Note that the negative sign was inserted because $\cos^{-1}$ is ambiguous as to direction, and the flight path angle on entry should be <0.

Flight Path Angle and Velocity Equations

$$\gamma = -\cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho_o e^{-\frac{h}{h_s}} + \cos\gamma_e\right) \qquad (13)$$

Rewrite (4) as

$$v = v_e\ exp\left(-\frac{\gamma - \gamma_e}{L/D}\right) = v_e\ exp\left(\frac{\gamma_e - \gamma}{L/D}\right)$$

and substitute into (13)

$$v = v_e\ exp\left\{\frac{1}{L/D}\left[\gamma_e + \cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho_o e^{-\frac{h}{h_s}} + \cos\gamma_e\right)\right]\right\} (14)$$

Deceleration

Along the flight path, $\left|\frac{D}{m}\right| = \frac{\rho v^2}{2\beta}$

Perpendicular to the flight path, $\left|\frac{L}{m}\right| = \frac{\rho v^2}{2\beta}\frac{L}{D}$

Total deceleration (*not* in g's)

$$n = \sqrt{\left(\frac{D}{m}\right)^2 + \left(\frac{L}{m}\right)^2} = \frac{1}{m}\sqrt{D^2 + L^2} = \frac{\rho v^2}{2\beta}\sqrt{1 + \left(\frac{L}{D}\right)^2}$$

$$n = \frac{\rho_o v^2}{2\beta}\sqrt{1 + \left(\frac{L}{D}\right)^2} e^{-\frac{h}{h_s}} \qquad (15)$$

Fiddling with Algebra

$$n = \frac{\rho_o}{2\beta}\sqrt{1+\left(\frac{L}{D}\right)^2} e^{-\frac{h}{h_s}} v^2$$

Substitute in (14)

$$n = \frac{\rho_o v_e^2}{2\beta}\sqrt{1+\left(\frac{L}{D}\right)^2} exp\left\{\frac{2}{L/D}\left[\gamma_e + \cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho_o e^{-\frac{h}{h_s}} + \cos\gamma_e\right)\right]\right\}$$

$$X \equiv \left[\gamma_e + \cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho_o e^{-\frac{h}{h_s}} + \cos\gamma_e\right)\right] \quad (16)$$

$$n = \frac{\rho_o v_e^2}{2\beta}\sqrt{1+\left(\frac{L}{D}\right)^2} e^{\frac{2X}{L/D}} \quad (17)$$

More Algebra

$$\text{Set } \frac{dn}{dh} = 0$$

$$0 = \frac{\rho_o v_e^2}{2\beta}\sqrt{1+\left(\frac{L}{D}\right)^2}\left[e^{-\frac{h}{h_s}} e^{\frac{2X}{L/D}} \frac{2}{L/D}\frac{dX}{dh} + \left(\frac{-1}{h_s}\right) e^{-\frac{h}{h_s}} e^{\frac{2X}{L/D}}\right] \quad (18)$$

Factoring out common terms,

$$\frac{1}{h_s} = \frac{2}{L/D}\frac{dX_m}{dh} \quad (19)$$

Even More Algebra

From (13),

$$\gamma = -\cos^{-1}\left(\frac{h_s}{2\beta}\frac{L}{D}\rho_o e^{-\frac{h}{h_s}} + \cos\gamma_e\right)$$

$$X = \gamma_e - \gamma = -\cos^{-1}(\cos\gamma) + \gamma_e$$

$$Y \equiv \cos\gamma$$

$$X = \gamma_e - \cos^{-1} Y \qquad (20)$$

Trigonometry, for a Change

Trig identity -

$$\frac{d(\cos^{-1} u)}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \tag{21}$$

$$\frac{dX}{dh} = -\frac{d(\cos^{-1} Y)}{dh} = \frac{1}{\sqrt{1-Y^2}}\frac{dY}{dh}$$

$$\frac{dX}{dh} = \frac{1}{\sqrt{1-\cos^2\gamma}}\frac{d(\cos\gamma)}{dh}$$

Back to the Algebra

From (13),

$$\cos\gamma = \frac{h_s}{2\beta}\frac{L}{D}\rho_o e^{-\frac{h}{h_s}} + \cos\gamma_e$$

$$\frac{dX}{dh} = \frac{1}{\sqrt{1-\cos^2\gamma}}\frac{d}{dh}\left(\frac{h_s\rho_o}{2\beta}\frac{L}{D}e^{-\frac{h}{h_s}} + \cos\gamma_e\right) \quad (22)$$

$$\frac{dX}{dh} = \frac{1}{\sin\gamma}\left[\frac{h_s\rho_o}{2\beta}\frac{L}{D}\left(\frac{-1}{h_s}\right)e^{-\frac{h}{h_s}}\right]$$

$$\frac{dX}{dh} = \frac{-\rho_o}{2\beta\sin\gamma}\frac{L}{D}e^{-\frac{h}{h_s}} \quad (23)$$

Maximum Deceleration Case

If we go back to the n_{max} case, (19) gives

$$\frac{1}{h_s} = \frac{2}{L/D}\frac{dX_m}{dh}$$

$$\frac{1}{h_s} = \frac{2}{L/D}\frac{-\rho_o}{2\beta\sin\gamma_m}\frac{L}{D}e^{-\frac{h_m}{h_s}}$$

$h_m,\ \gamma_m$ are values at n_{max}

$$\frac{1}{h_s} = \frac{-\rho_o}{\beta\sin\gamma_m}\frac{L}{D}e^{-\frac{h_m}{h_s}} \tag{24}$$

$$\sin\gamma_m = \frac{-\rho_o h_s}{\beta}\frac{L}{D}e^{-\frac{h_m}{h_s}} \tag{25}$$

Let's Go Back to Algebra

$$\text{Let } \Phi \equiv -\frac{\rho_o h_s}{2\beta}\frac{L}{D}$$

$$\cos\gamma_m = \Phi e^{-\frac{h_m}{h_s}} + \cos\gamma_e$$

$$\sin\gamma_m = \Phi e^{-\frac{h_m}{h_s}}$$

$$\text{Let } H \equiv e^{-\frac{h_m}{h_s}}$$

$$\left(\Phi e^{-\frac{h_m}{h_s}} + \cos\gamma_e\right)^2 + \left(\Phi e^{-\frac{h_m}{h_s}}\right)^2 = 1$$

$$(\Phi H + \cos\gamma_e)^2 + (\Phi H)^2 = 1$$

Algebra is Fun, Don't You Think?

$$2\Phi^2 H^2 + 2\Phi H \cos\gamma_e + \cos^2\gamma_e - 1 = 0$$

$$2\Phi^2 H^2 + 2\Phi H \cos\gamma_e - \sin^2\gamma_e = 0$$

$$H = \frac{-2\Phi\cos\gamma_e \pm \sqrt{4\Phi^2\cos^2\gamma_e + 4(2\Phi^2)\sin^2\gamma_e}}{4\Phi^2}$$

$$H = -\frac{1}{2\Phi}\left(\cos\gamma_e \pm \sqrt{\cos^2\gamma_e + 2\sin^2\gamma_e}\right)$$

$$H = -\frac{1}{2\Phi}\left(\cos\gamma_e \pm \sqrt{1 + \sin^2\gamma_e}\right)$$

Maximum Deceleration Equations

Skipping some painful algebra and trig,

$$h_m = h_s \ln\left\{\frac{-\rho_o h_s}{2\beta \sin\gamma_e}\left[\sqrt{4+\left(\frac{L}{D}\right)^2 \csc^2\gamma_e} - \frac{L}{D}\cot\gamma_e\right]\right\}$$

$$\cos\gamma_m = \cos\gamma_e - \frac{(L/D)\sin\gamma_e}{\sqrt{4+(L/D)^2\csc^2\gamma_e} - (L/D)\cot\gamma_e}$$

$$v_m = v_e e^{-\frac{\gamma_m - \gamma_e}{L/D}}$$

$$n_{max} = \frac{\rho_o v_e^2}{2\beta}\sqrt{1+\left(\frac{L}{D}\right)^2} v_m^2$$

Phugoid Oscillations

Assume a shallow, linear entry trajectory:

$$\gamma \ll 1 \qquad D \approx 0 \qquad \dot{\gamma}_o \approx 0$$

$$\dot{h} = v \sin\gamma \cong v\gamma$$

$$\frac{v\dot{\gamma}}{g} = \frac{L}{mg} - \left(1 - \frac{v^2}{v_c^2}\right)$$

Small perturbations $\Longrightarrow \quad h = h_1 + \Delta h$

$$\gamma = \gamma_1 + \Delta\gamma \approx \Delta\gamma$$

Perturbation Analysis

$$\dot{h} = v\gamma$$

$$\dot{h} = \dot{h}_1 + \Delta\dot{h} = \gamma_1 v_1 + \Delta\dot{h} = \Delta\dot{h}$$

$$\Delta\dot{h} = (v_1 + \Delta v_1)(\gamma_1 + \Delta\gamma) = v_1\Delta\gamma + \Delta v_1\Delta\gamma$$

$$\boxed{\Delta\dot{h} = v_1\Delta\gamma}$$

$$\text{Neglecting drag} \implies v \cong \text{constant}$$

$$\frac{L}{m} = \frac{L_1 + \Delta L}{m} = \frac{v^2 A c_L}{2m}\rho_o e^{-\frac{h_1+\Delta h}{h_s}}$$

$$= \frac{v^2 A c_L}{2m}\rho_o e^{-\frac{h_1}{h_s}} e^{-\frac{\Delta h}{h_s}} = \frac{L_1}{m} e^{-\frac{\Delta h}{h_s}}$$

Perturbation Analysis

Using Taylor's series expansion,

$$\frac{L}{m} \cong \frac{L_1}{m}\left(1 - \frac{\Delta h}{h_s}\right)$$

$$\frac{\Delta L}{m} = -\frac{L_1}{m}\frac{\Delta h}{h_s}$$

$$\frac{v_1}{g}\Delta\dot{\gamma} = \frac{\Delta L}{mg} + \left[\frac{L_1}{mg} - \left(1 - \frac{v_1^2}{v_c^2}\right)\right]$$

$$v_1\Delta\dot{\gamma} = \frac{\Delta L}{m} = \Delta\ddot{h}$$

Perturbed Lift

On an equilibrium glide,

$$\dot{\gamma} = 0 \implies \frac{L}{m} = g - \frac{v^2}{r}$$

$$\frac{L_1}{mg} = 1 - \frac{v_1^2}{gr} = 1 - \frac{v_1^2}{v_c^2}$$

$$\Delta\ddot{h} = \frac{\Delta L}{m} = -\frac{L_1}{m}\frac{\Delta h}{h_s} = -\left(1 - \frac{v_1^2}{v_c^2}\right)\frac{g\Delta h}{h_s}$$

$$\boxed{\Delta\ddot{h} + \left(1 - \frac{v_1^2}{v_c^2}\right)\frac{g}{h_s}\Delta h = 0}$$

$\Longleftarrow$ Simple harmonic motion (undamped)

Phugoid Parameters

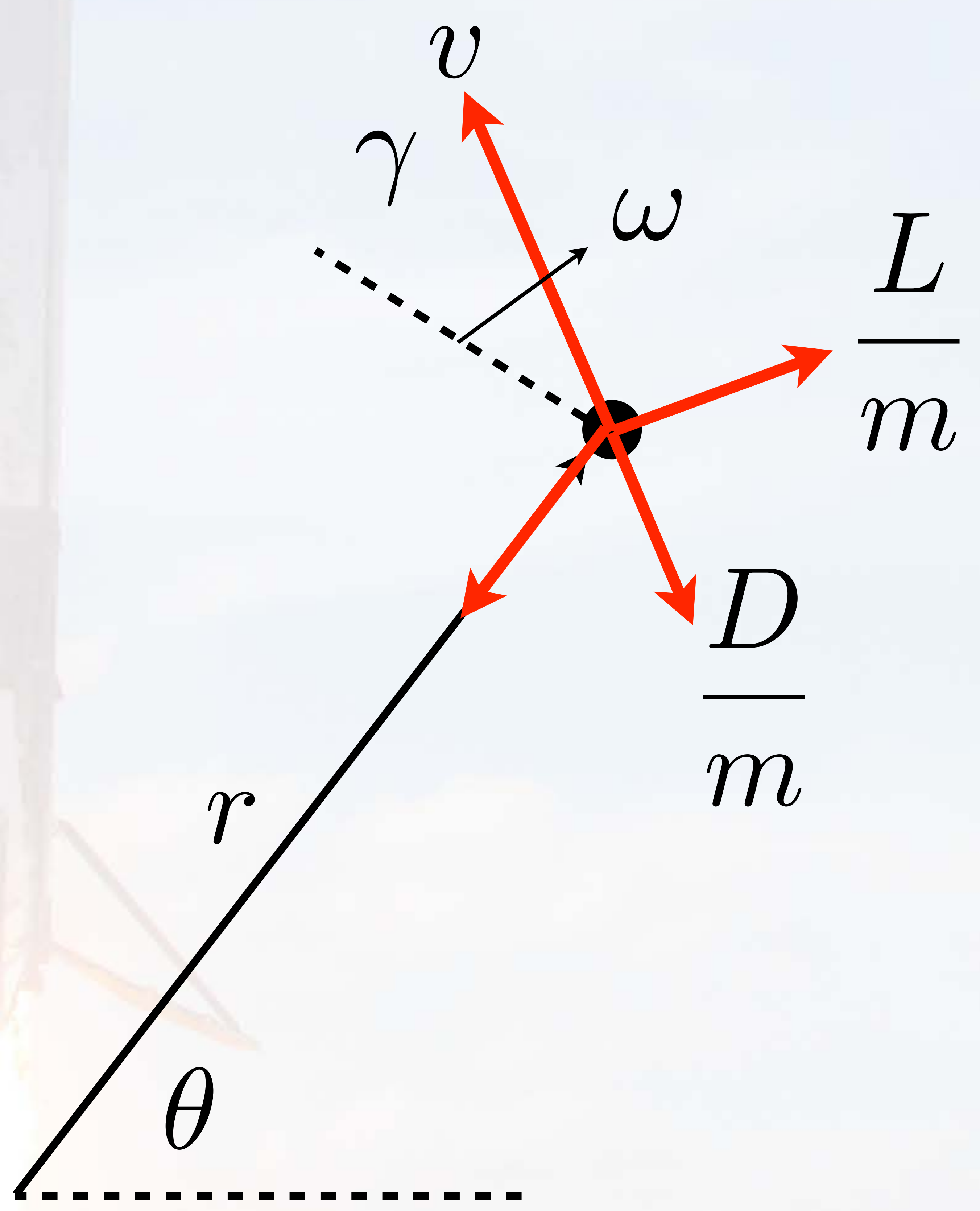
$$\text{Frequency: } \omega^2 = \left(1 - \frac{v_1^2}{v_c^2}\right)\frac{g}{h_s}$$

$$\text{Period: } P = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\left(1 - \frac{v_1^2}{v_c^2}\right)\frac{g}{h_s}}} \cong \frac{169\ sec}{\sqrt{1 - \frac{v_1^2}{v_c^2}}}$$

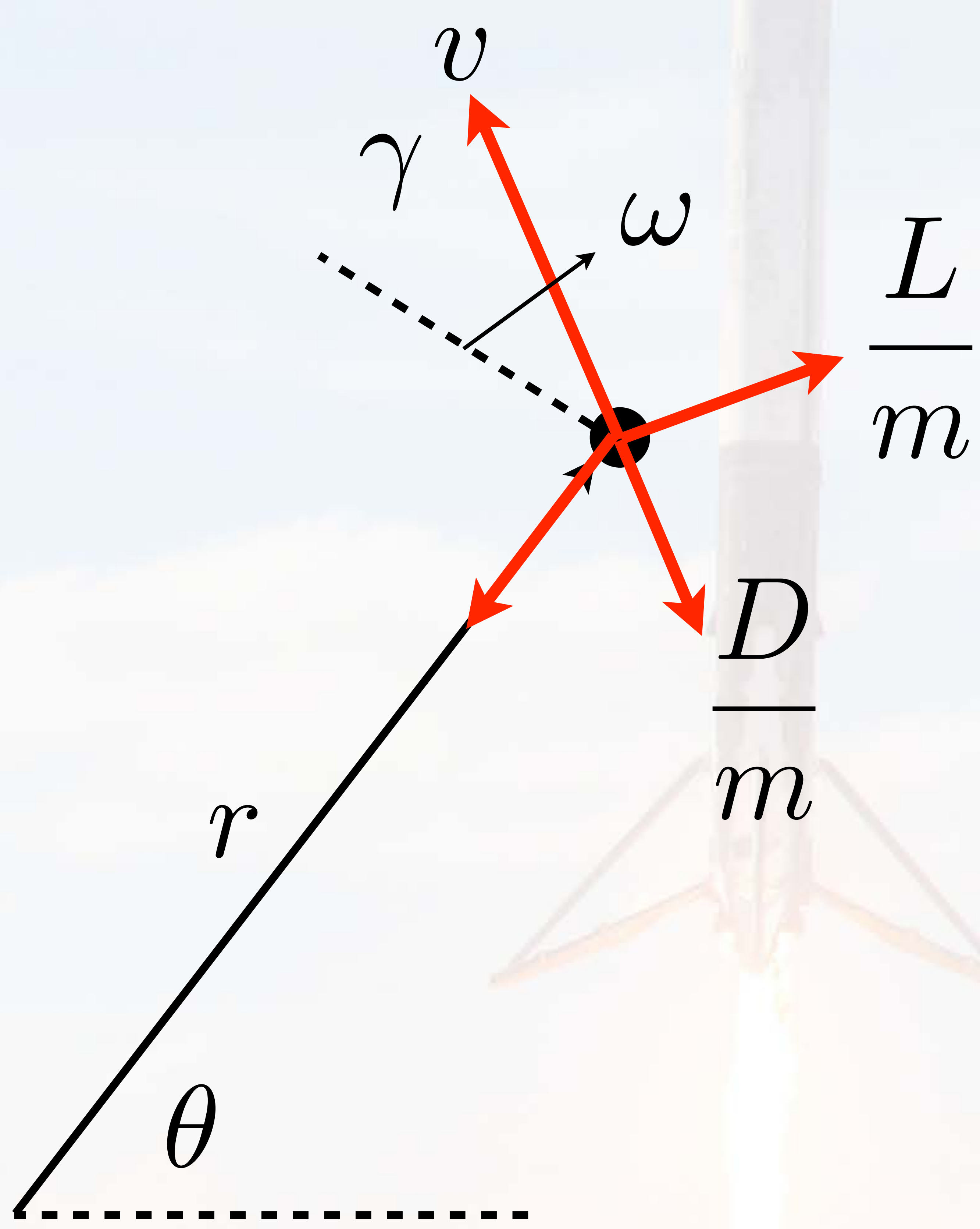
$$v_c \sim 8000\ \frac{m}{sec} \implies$$

$v_1 \left(\frac{m}{sec}\right)$	P
7750	11m21s
6000	4m15s
4000	3m15s
2000	2m55s

Free-Body Diagram with Spherical Planet



Planar State Equations



$$\omega = \dot{\gamma} - \dot{\theta}$$

Sum of transverse accelerations

$$\frac{L}{m} - g\cos\gamma = \omega v$$

Sum of parallel accelerations

$$-\frac{D}{m} - g\sin\gamma = \dot{v}$$

Planar State Equations (2)

$$\dot{r} = \dot{h} = v \sin\gamma$$

$$r\dot{\theta} = v\cos\gamma$$

$$\omega = \dot{\gamma} - \dot{\theta} = \dot{\gamma} - \frac{v}{r}\cos\gamma$$

$$\frac{L}{m} - g\cos\gamma = \left(\dot{\gamma} - \frac{v}{r}\cos\gamma\right)v$$

$$\frac{L}{m} - \left(g - \frac{v^2}{r}\right)\cos\gamma = \dot{\gamma}v$$

$$\frac{L}{m} - \left(1 - \frac{v^2}{rg}\right)g\cos\gamma = \dot{\gamma}v$$

The Canonical Planar State Equations

$$v\dot{\gamma} = \frac{L}{m} - \left(1 - \frac{v^2}{v_c^2}\right) g\cos\gamma$$

$$\dot{v} = -\frac{D}{m} - g\sin\gamma$$

$$\dot{r} = \dot{h} = v\sin\gamma$$

$$r\dot{\theta} = v\cos\gamma$$

Coupled first-order ODEs

Associated Parameters to State Eqns

$$\frac{L}{m} = \frac{1}{2}\frac{\rho v^2 A c_L}{m} = \frac{\rho v^2}{2}\frac{A c_D}{m}\frac{c_L}{c_D} = \frac{\rho v^2}{2\beta}\frac{L}{D}$$

$$\frac{D}{m} = \frac{1}{2}\frac{\rho v^2 A c_D}{m} = \frac{\rho v^2}{2\beta}$$

$$\rho = \rho_o e^{-\frac{h}{h_s}}$$

$$h = r - r_o$$

$$g = g_o\left(\frac{r_o}{r}\right)^2$$