Orbital Mechanics

- Energy and velocity in orbit
- Elliptical orbit parameters
- Orbital elements
- Coplanar orbital transfers
- Noncoplanar transfers
- Time and flight path angle as a function of orbital position
- Relative orbital motion ("proximity operations")
Energy in Orbit

- Kinetic Energy
  \[ K.E. = \frac{1}{2}mv^2 \Rightarrow \frac{K.E.}{m} = \frac{v^2}{2} \]

- Potential Energy
  \[ P.E. = -\frac{mu}{r} \Rightarrow \frac{P.E.}{m} = -\frac{\mu}{r} \]

- Total Energy
  \[ \text{Const.} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \]

\[<--\text{Vis-Viva Equation}\]
Implications of Vis-Viva

- **Circular orbit (r=a)**
  \[
  v_{circular} = \sqrt{\frac{\mu}{r}}
  \]

- **Parabolic escape orbit (a tends to infinity)**
  \[
  v_{escape} = \sqrt{\frac{2\mu}{r}}
  \]

- **Relationship between circular and parabolic orbits**
  \[
  v_{escape} = \sqrt{2}v_{circular}
  \]
Some Useful Constants

- **Gravitation constant** $\mu = GM$
  - Earth: $398,604 \text{ km}^3/\text{sec}^2$
  - Moon: $4667.9 \text{ km}^3/\text{sec}^2$
  - Mars: $42,970 \text{ km}^3/\text{sec}^2$
  - Sun: $1.327 \times 10^{11} \text{ km}^3/\text{sec}^2$

- **Planetary radii**
  - $r_{\text{Earth}} = 6378 \text{ km}$
  - $r_{\text{Moon}} = 1738 \text{ km}$
  - $r_{\text{Mars}} = 3393 \text{ km}$
Classical Parameters of Elliptical Orbits
Basic Orbital Parameters

- Semi-latus rectum (or parameter)
  \[ p = a(1 - e^2) \]

- Radial distance as function of orbital position
  \[ r = \frac{p}{1 + e \cos \theta} \]

- Periapse and apoapse distances
  \[ r_p = a(1 - e) \quad r_a = a(1 + e) \]

- Angular momentum
  \[ \vec{h} = \vec{r} \times \vec{v} \quad h = \sqrt{\mu p} \]
The Classical Orbital Elements

\[ \begin{align*}
\Omega &: \text{longitude of the ascending node} \\
\omega &: \text{argument of periapsis} \\
\bar{\omega} &= \Omega + \omega : \text{longitude of periapsis} \\
f &: \text{true anomaly} \\
L &= \bar{\omega} + f : \text{true longitude}
\end{align*} \]

The Hohmann Transfer

$v_{perigee}$

$r_1$

$r_2$

$v_{1}$

$v_{2}$

$v_{apogee}$
First Maneuver Velocities

- Initial vehicle velocity
  \[ v_1 = \sqrt{\frac{\mu}{r_1}} \]

- Needed final velocity
  \[ v_{\text{perigee}} = \sqrt{\frac{\mu}{r_1}} \frac{2r_2}{r_1 + r_2} \]

- Delta-V
  \[ \Delta v_1 = \sqrt{\frac{\mu}{r_1}} \left( \sqrt{\frac{2r_2}{r_1 + r_2}} - 1 \right) \]
Second Maneuver Velocities

- Initial vehicle velocity
  \[ v_{\text{apogee}} = \sqrt{\frac{\mu}{r_2}} \sqrt{\frac{2r_1}{r_1 + r_2}} \]

- Needed final velocity
  \[ v_2 = \sqrt{\frac{\mu}{r_2}} \]

- Delta-V
  \[ \Delta v_2 = \sqrt{\frac{\mu}{r_2}} \left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right) \]
Limitations on Launch Inclinations
Differences in Inclination
Choosing the Wrong Line of Apsides
Simple Plane Change

\[ v_{\text{apogee}} - v_1 = \Delta v_2 \]

\[ v_{\text{perigee}} \]
Optimal Plane Change

\[ v_{\text{perigee}} \quad \Delta v_1 \quad v_1 \quad \Delta v_2 \quad v_{\text{apogee}} \quad v_2 \]
First Maneuver with Plane Change $\Delta i_1$

- Initial vehicle velocity
  \[ v_1 = \sqrt{\frac{\mu}{r_1}} \]

- Needed final velocity
  \[ v_p = \sqrt{\frac{\mu}{r_1}} \left( \frac{2r_2}{r_1 + r_2} \right) \]

- Delta-V
  \[ \Delta v_1 = \sqrt{v_1^2 + v_p^2 - 2v_1 v_p \cos(\Delta i_1)} \]
Second Maneuver with Plane Change $\Delta i_2$

- Initial vehicle velocity
  \[ v_a = \sqrt[\frac{\mu}{r_2}} \left( \frac{2r_1}{r_1 + r_2} \right) \]

- Needed final velocity
  \[ v_2 = \sqrt{\frac{\mu}{r_2}} \]

- Delta-V
  \[ \Delta v_2 = \sqrt{v_2^2 + v_a^2 - 2v_2 v_a \cos(\Delta i_2)} \]
Sample Plane Change Maneuver

Optimum initial plane change = 2.20°
Bielliptic Transfer
Coplanar Transfer Velocity Requirements

Noncoplanar Bielliptic Transfers
Calculating Time in Orbit
Time in Orbit

- **Period of an orbit**
  
  $$P = 2\pi \sqrt{\frac{a^3}{\mu}}$$

- **Mean motion (average angular velocity)**
  
  $$n = \sqrt{\frac{\mu}{a^3}}$$

- **Time since pericenter passage**
  
  $$M = nt = E - e \sin E$$

\(M\)=mean anomaly
Dealing with the Eccentric Anomaly

• Relationship to orbit

\[ r = a(1 - e \cos E) \]

• Relationship to true anomaly

\[ \tan \frac{\theta}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2} \]

• Calculating \( M \) from time interval: iterate

\[ E_{i+1} = nt + e \sin E_i \]

until it converges
Patched Conics

- Simple approximation to multi-body motion (e.g., traveling from Earth orbit through solar orbit into Martian orbit)
- Treats multibody problem as “hand-offs” between gravitating bodies --> reduces analysis to sequential two-body problems
- Caveat Emptor: There are a number of formal methods to perform patched conic analysis. The approach presented here is a very simple, convenient, and not altogether accurate method for performing this calculation. Results will be accurate to a few percent, which is adequate at this level of design analysis.
Example: Lunar Orbit Insertion

- $v_2$ is velocity of moon around Earth
- Moon overtakes spacecraft with velocity of $(v_2 - v_{\text{apogee}})$
- This is the velocity of the spacecraft relative to the moon while it is effectively "infinitely" far away (before lunar gravity accelerates it) = "hyperbolic excess velocity"
Planetary Approach Analysis

- Spacecraft has $v_h$ hyperbolic excess velocity, which fixes total energy of approach orbit.

- Vis-viva provides velocity of approach

\[ v = 2 \sqrt{\frac{v_h^2}{2} + \frac{\mu}{r}} \]

- Choose transfer orbit such that approach is tangent to desired final orbit at periapse

\[ \Delta v = 2 \sqrt{\frac{v_h^2}{2} + \frac{\mu}{r_{\text{orbit}}}} - \sqrt{\frac{\mu}{r_{\text{orbit}}}} \]
### ΔV Requirements for Lunar Missions

<table>
<thead>
<tr>
<th>From:</th>
<th>To:</th>
<th>Low Earth Orbit</th>
<th>Lunar Transfer Orbit</th>
<th>Low Lunar Orbit</th>
<th>Lunar Descent Orbit</th>
<th>Lunar Landing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low Earth Orbit</td>
<td></td>
<td></td>
<td>3.107 km/sec</td>
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<tr>
<td>Lunar Transfer Orbit</td>
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<td>3.107 km/sec</td>
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<td>0.837 km/sec</td>
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<td>3.140 km/sec</td>
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<tr>
<td>Low Lunar Orbit</td>
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<td>0.837 km/sec</td>
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<td>0.022 km/sec</td>
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<tr>
<td>Lunar Descent Orbit</td>
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<td></td>
<td>0.022 km/sec</td>
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<td>2.684 km/sec</td>
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<tr>
<td>Lunar Landing</td>
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<td>2.890 km/sec</td>
<td></td>
<td>2.312 km/sec</td>
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</tr>
</tbody>
</table>
Hill's Equations (Proximity Operations)

\[
\begin{align*}
\ddot{x} &= 3n^2 x + 2n\dot{y} + a_{dx} \\
\ddot{y} &= -2nx + a_{dy} \\
\ddot{z} &= -n^2 z + a_{dz}
\end{align*}
\]

Ref: J. E. Prussing and B. A. Conway, *Orbital Mechanics*  
Oxford University Press, 1993
**Clohessy-Wiltshire ("CW") Equations**

\[
x(t) = [4 - 3\cos(nt)]x_o + \frac{\sin(nt)}{n} \dot{x}_o + \frac{2}{n} [1 - \cos(nt)] \dot{y}_o
\]

\[
y(t) = 6[\sin(nt) - nt]x_o + y_o - \frac{2}{n} [1 - \cos(nt)] \dot{x}_o + \frac{4 \sin(nt) - 3nt}{n} \dot{y}_o
\]

\[
z(t) = z_o \cos(nt) + \frac{\dot{z}_o}{n} \sin(nt)
\]

\[
\dot{z}(t) = -z_o n \sin(nt) + \dot{z}_o \sin(nt)
\]
“V-Bar” Approach

Ref: Collins, Meissinger, and Bell, Small Orbit Transfer Vehicle (OTV) for On-Orbit Satellite Servicing and Resupply, 15th USU Small Satellite Conference, 2001
“R-Bar” Approach

- Approach from along the radius vector (“R-bar”)
- Gravity gradients decelerate spacecraft approach velocity - low contamination approach
- Used for Mir, ISS docking approaches

Ref: Collins, Meissinger, and Bell, Small Orbit Transfer Vehicle (OTV) for On-Orbit Satellite Servicing and Resupply, 15th USU Small Satellite Conference, 2001
References for Lecture 3

• Francis J. Hale, *Introduction to Space Flight* Prentice-Hall, 1994