# Orbital Mechanics

- Energy and velocity in orbit
- Elliptical orbit parameters
- Orbital elements
- Coplanar orbital transfers
- Noncoplanar transfers
- Time and flight path angle as a function of orbital position
- Relative orbital motion ("proximity operations")

# Energy in Orbit

- Kinetic Energy

$$K.E. = \frac{1}{2}mv^2 \Rightarrow \frac{K.E.}{m} = \frac{v^2}{2}$$

- Potential Energy

$$P.E. = -\frac{m\mu}{r} \Rightarrow \frac{P.E.}{m} = -\frac{\mu}{r}$$

- Total Energy

$$Const. = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

<--Vis-Viva Equation

# Implications of Vis-Viva

- Circular orbit (r=a)

$$v_{circular} = \sqrt{\frac{\mu}{r}}$$

- Parabolic escape orbit (a tends to infinity)

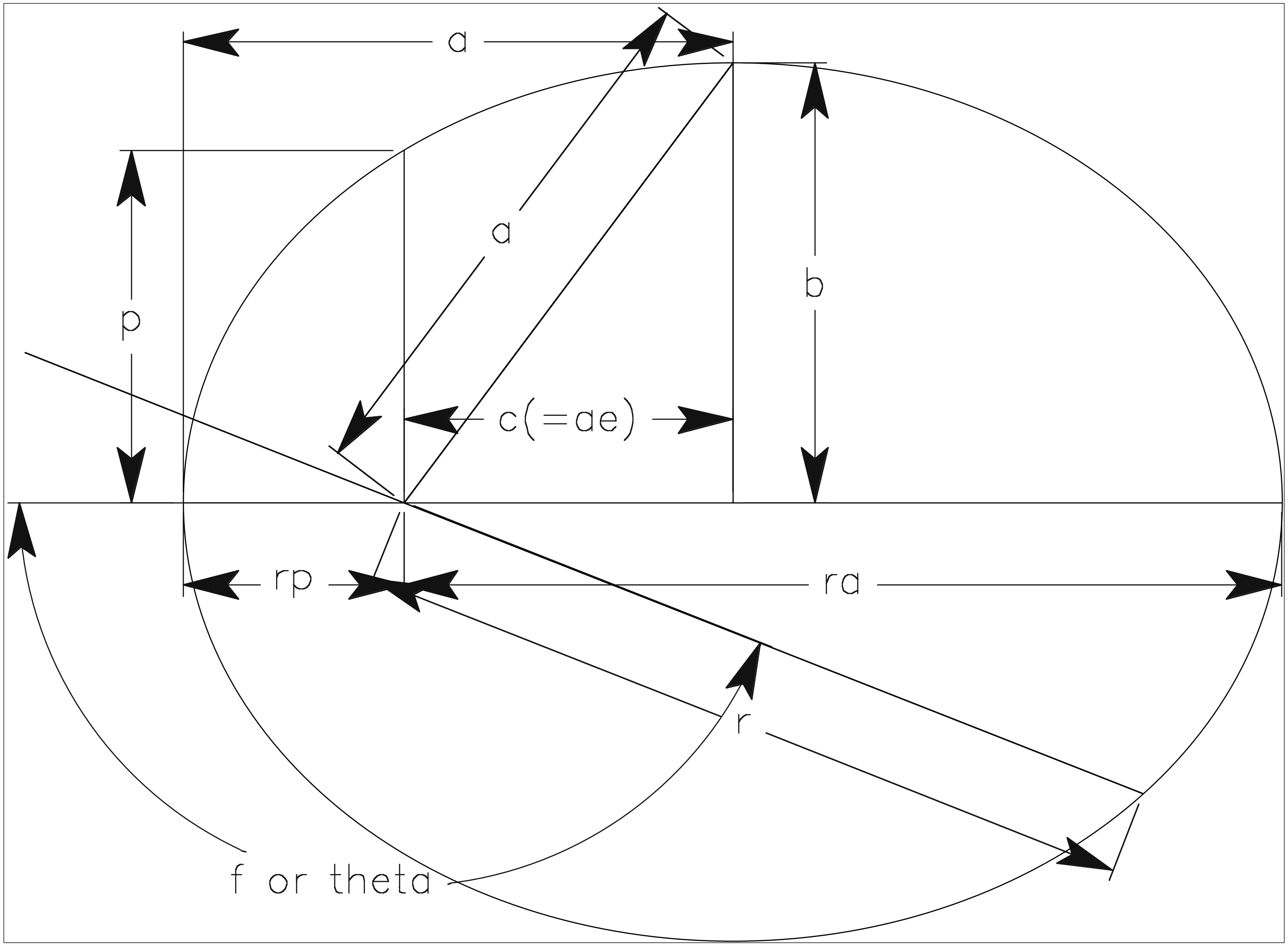
$$v_{escape} = \sqrt{\frac{2\mu}{r}}$$

- Relationship between circular and parabolic orbits $v_{escape} = \sqrt{2} v_{circular}$

# Some Useful Constants

- Gravitation constant $\mu$ = GM
  - Earth: 398,604 $km^3/sec^2$
  - Moon: 4667.9 $km^3/sec^2$
  - Mars: 42,970 $km^3/sec^2$
  - Sun: $1.327 \times 10^{11}$ $km^3/sec^2$
- Planetary radii
  - $r_{Earth}$ = 6378 km
  - $r_{Moon}$ = 1738 km
  - $r_{Mars}$ = 3393 km

# Classical Parameters of Elliptical Orbits



# Basic Orbital Parameters

- Semi-latus rectum (or parameter)

$$p = a(1 - e^2)$$

- Radial distance as function of orbital position

$$r = \frac{p}{1 + e\cos\theta}$$

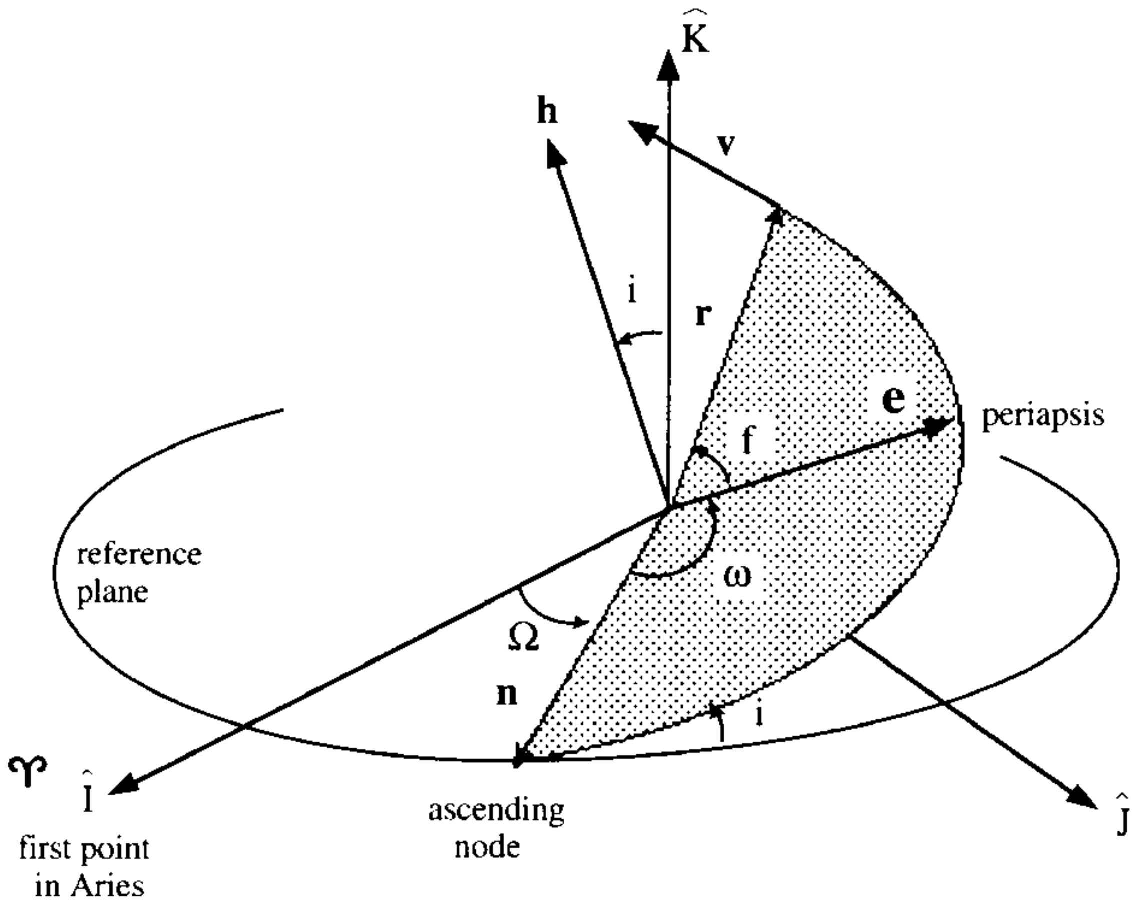
- Periapse and apoapse distances

$$r_p = a(1 - e) \qquad r_a = a(1 + e)$$

- Angular momentum

$$\vec{h} = \vec{r} \times \vec{v} \qquad h = \sqrt{\mu p}$$

# The Classical Orbital Elements




Ω : longitude of the ascending node

ω : argument of periapsis

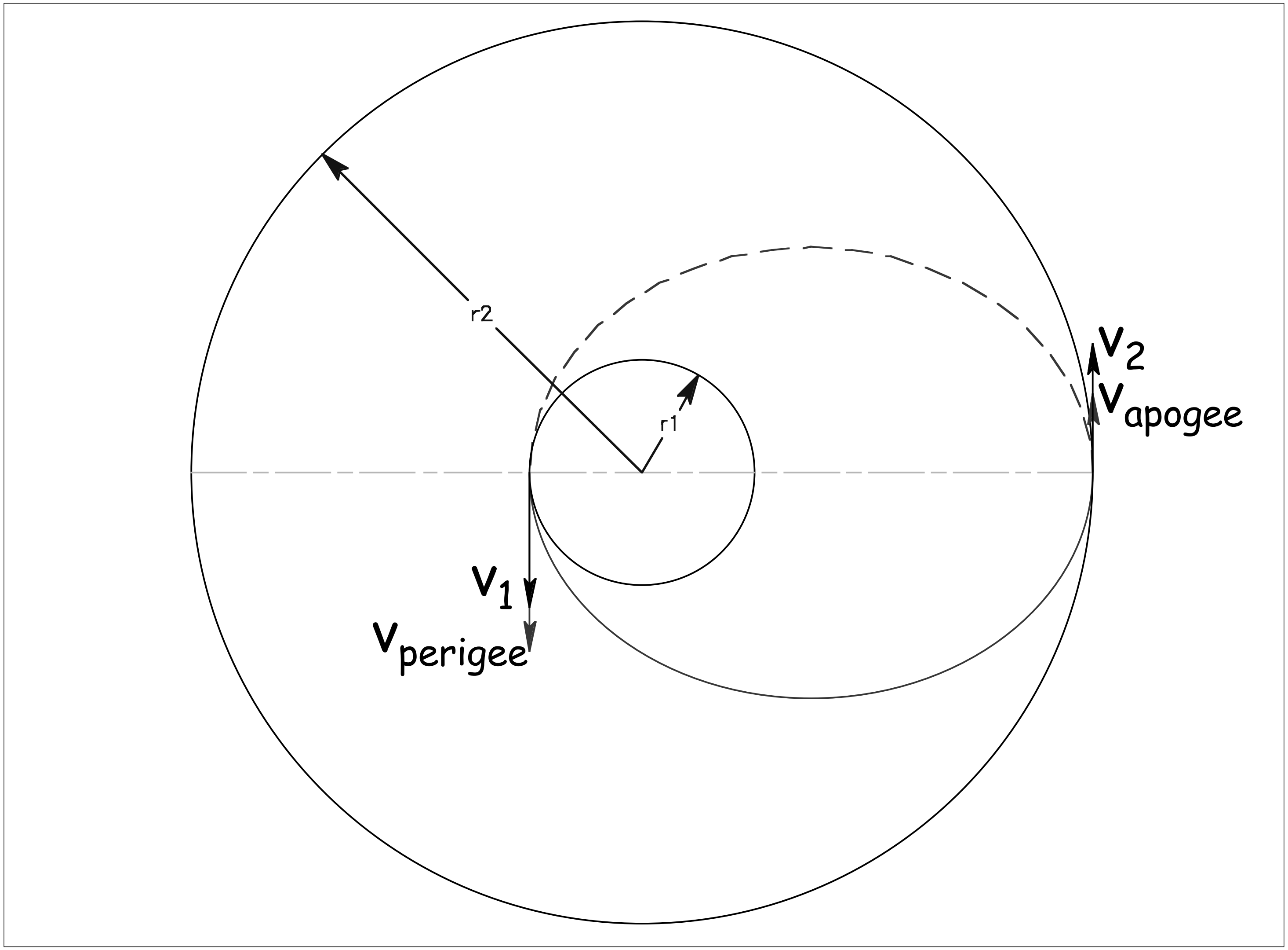
$\tilde{\omega} = \Omega + \omega$ : longitude of periapsis

f : true anomaly

$L = \tilde{\omega} + f$ : true longitude

Ref: J. E. Prussing and B. A. Conway, *Orbital Mechanics* Oxford University Press, 1993

# The Hohmann Transfer



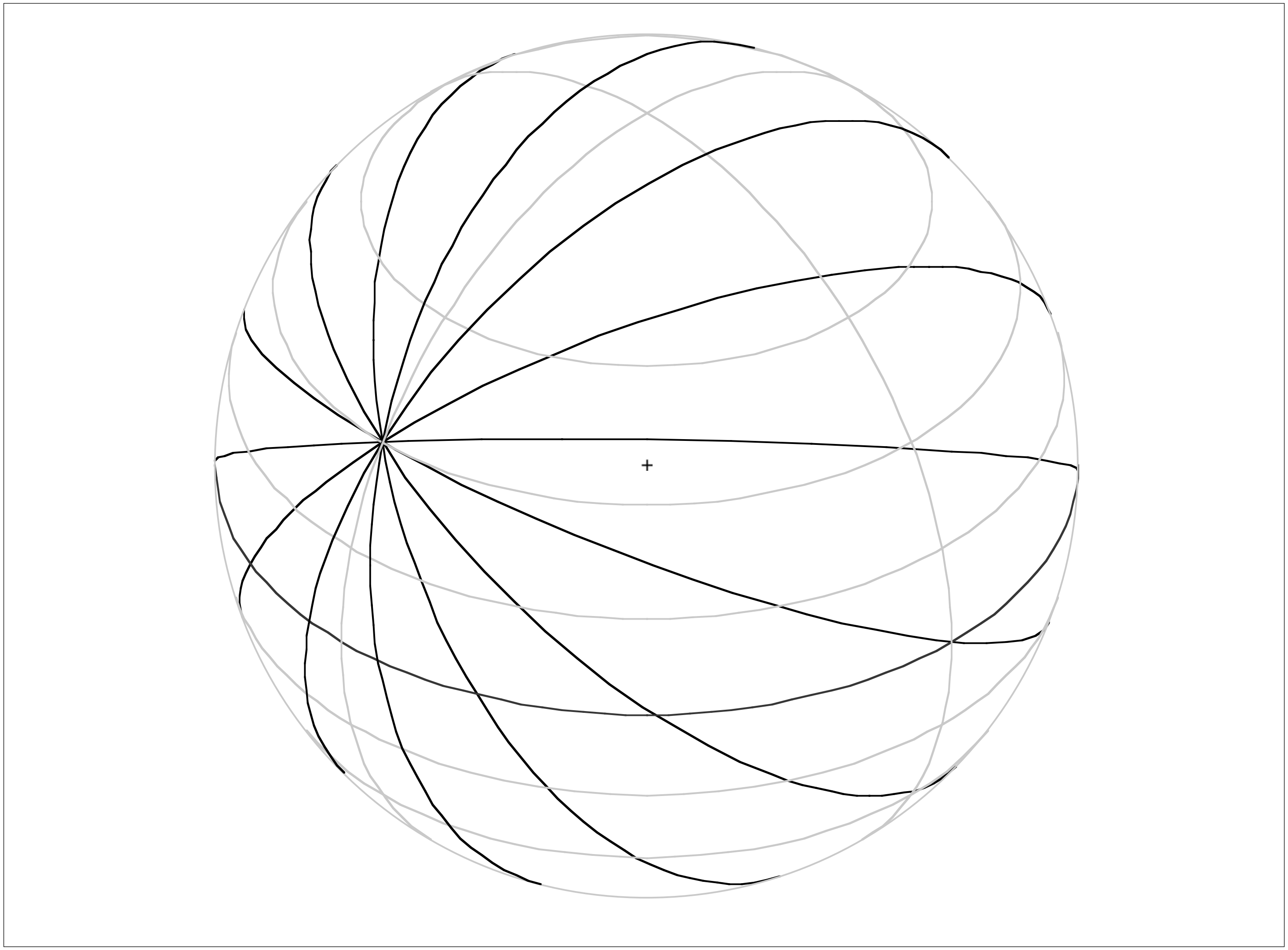
# First Maneuver Velocities

- Initial vehicle velocity $$v_1 = \sqrt{\frac{\mu}{r_1}}$$
- Needed final velocity $$v_{perigee} = \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{2r_2}{r_1 + r_2}}$$
- Delta-V $$\Delta v_1 = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right)$$

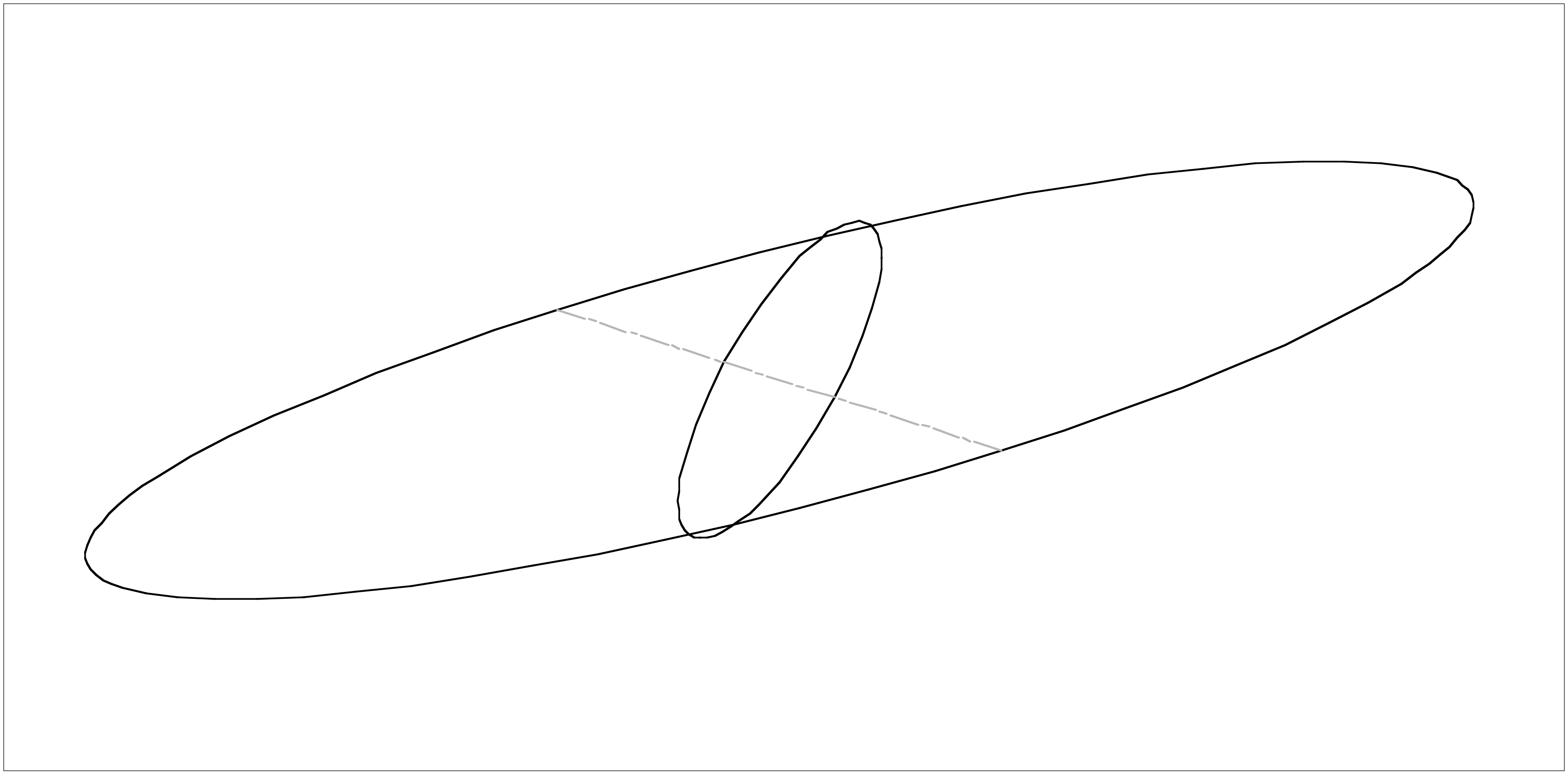
# Second Maneuver Velocities

- Initial vehicle velocity $$v_{apogee} = \sqrt{\frac{\mu}{r_2}}\sqrt{\frac{2r_1}{r_1 + r_2}}$$
- Needed final velocity $$v_2 = \sqrt{\frac{\mu}{r_2}}$$
- Delta-V $$\Delta v_2 = \sqrt{\frac{\mu}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right)$$

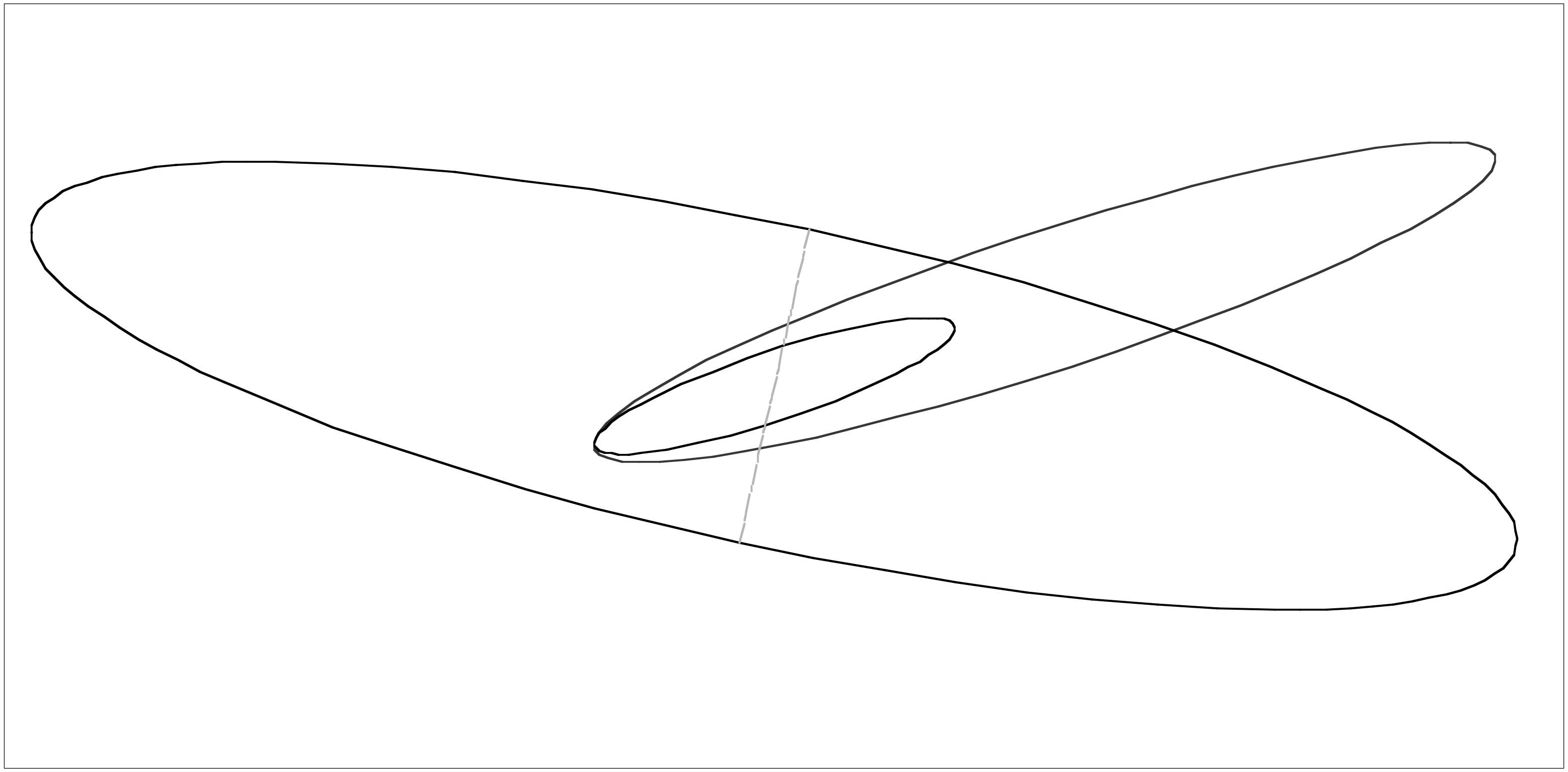
# Limitations on Launch Inclinations



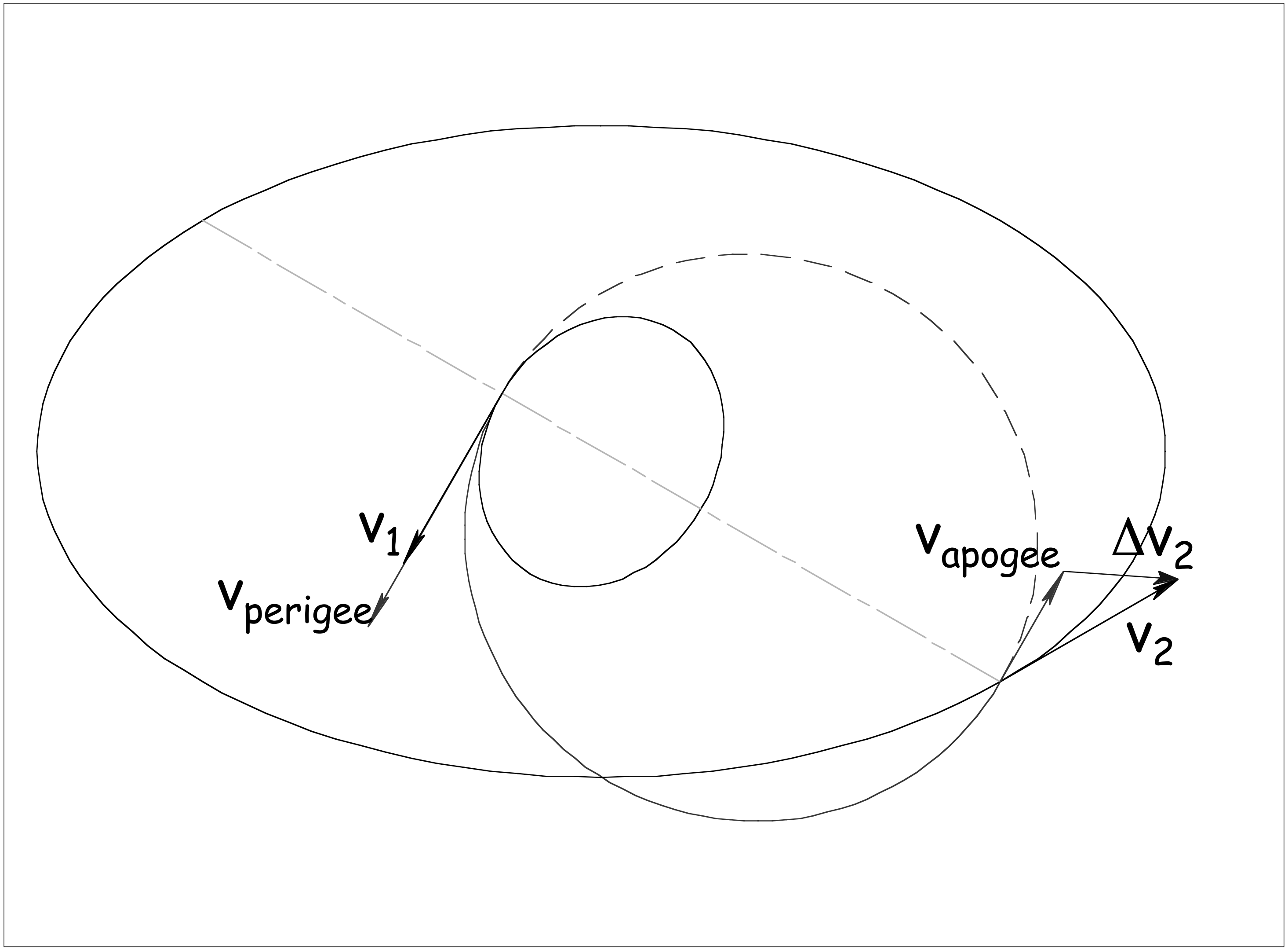
# Differences in Inclination



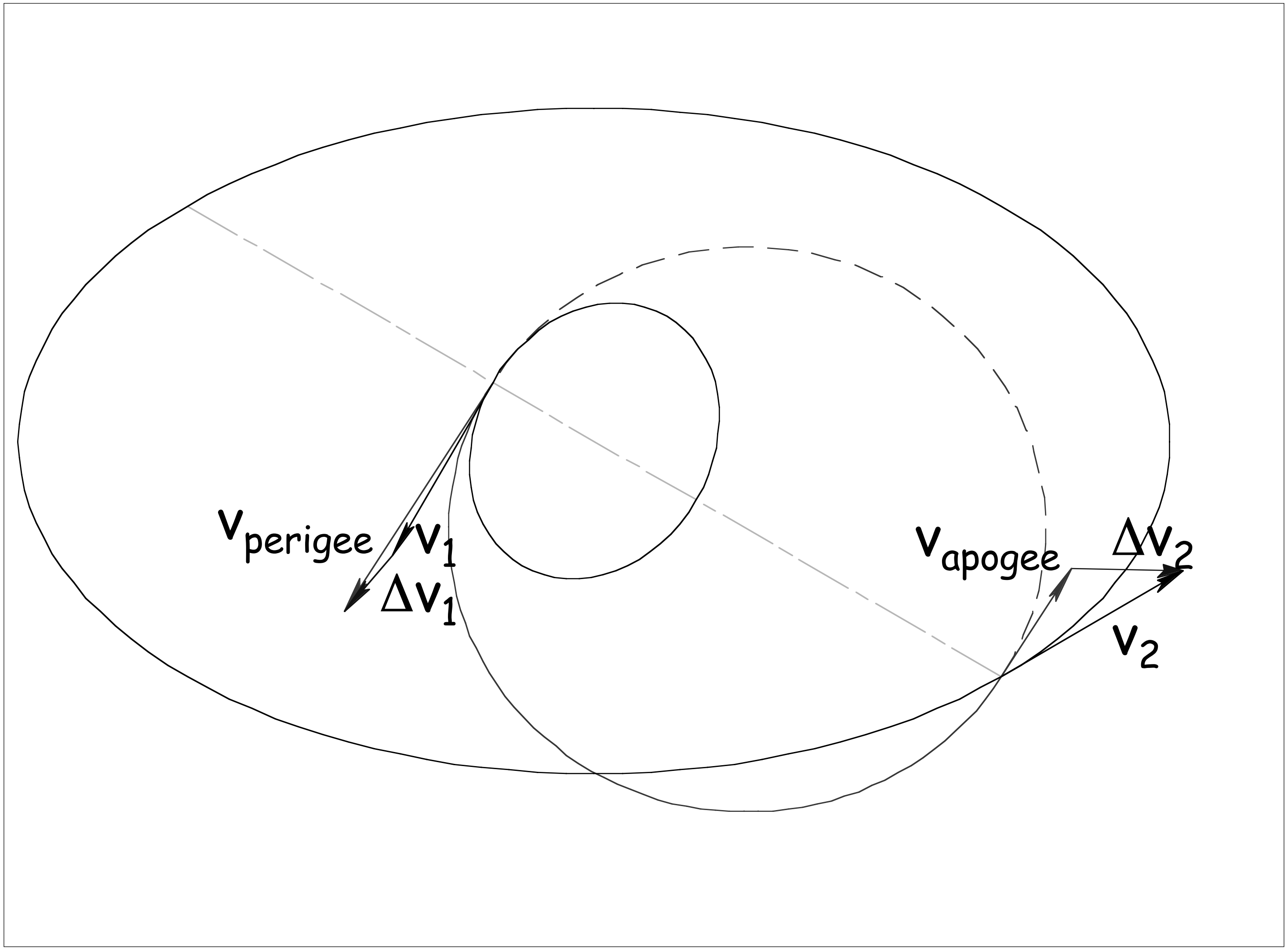
# Choosing the Wrong Line of Apsides



# Simple Plane Change



# Optimal Plane Change



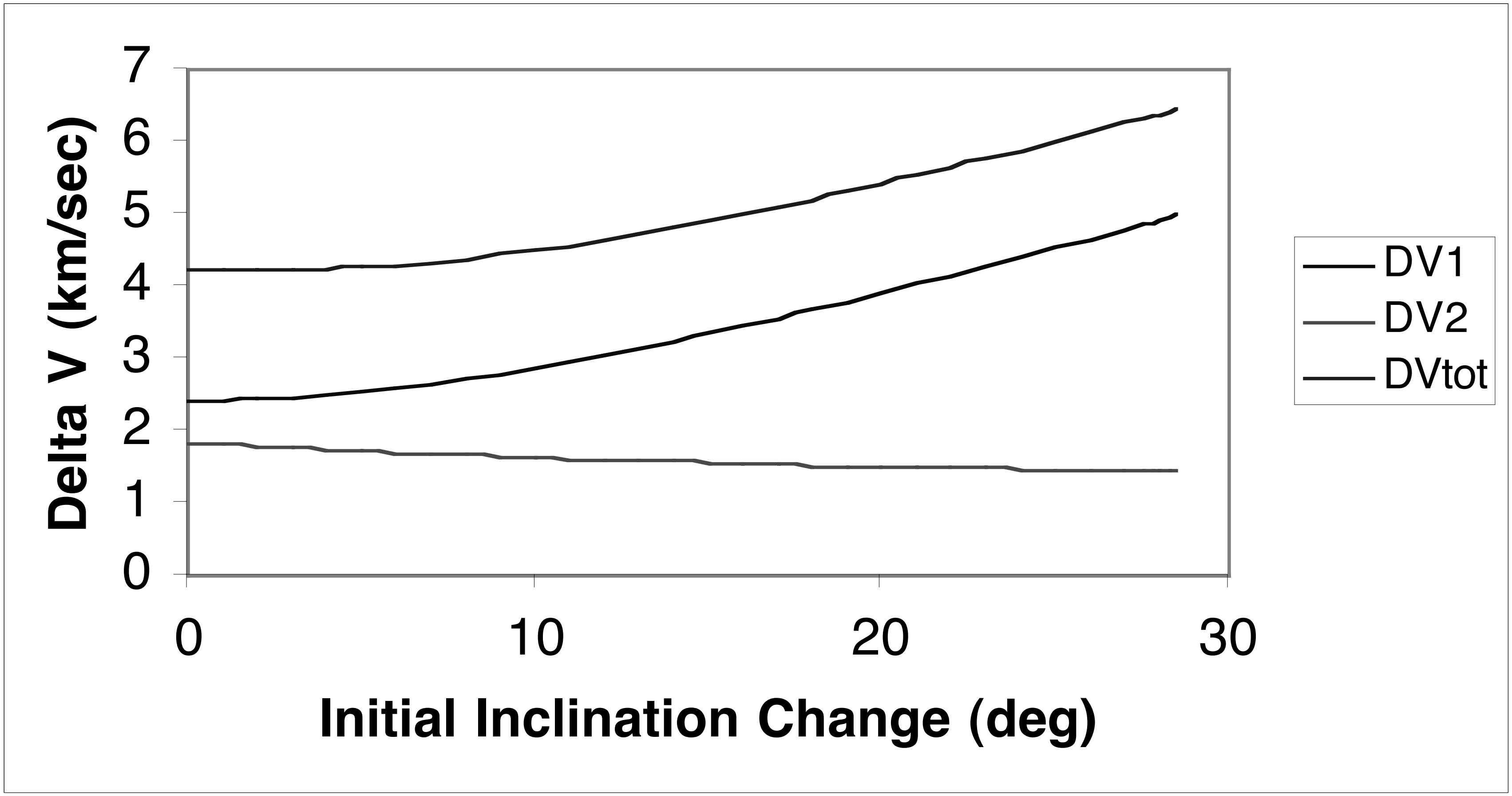
# First Maneuver with Plane Change $\Delta i_1$

- Initial vehicle velocity $$v_1 = \sqrt{\frac{\mu}{r_1}}$$
- Needed final velocity $$v_p = \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{2r_2}{r_1 + r_2}}$$
- Delta-V $$\Delta v_1 = \sqrt{v_1^2 + v_p^2 - 2v_1 v_p \cos(\Delta i_1)}$$

# Second Maneuver with Plane Change $\Delta i_2$

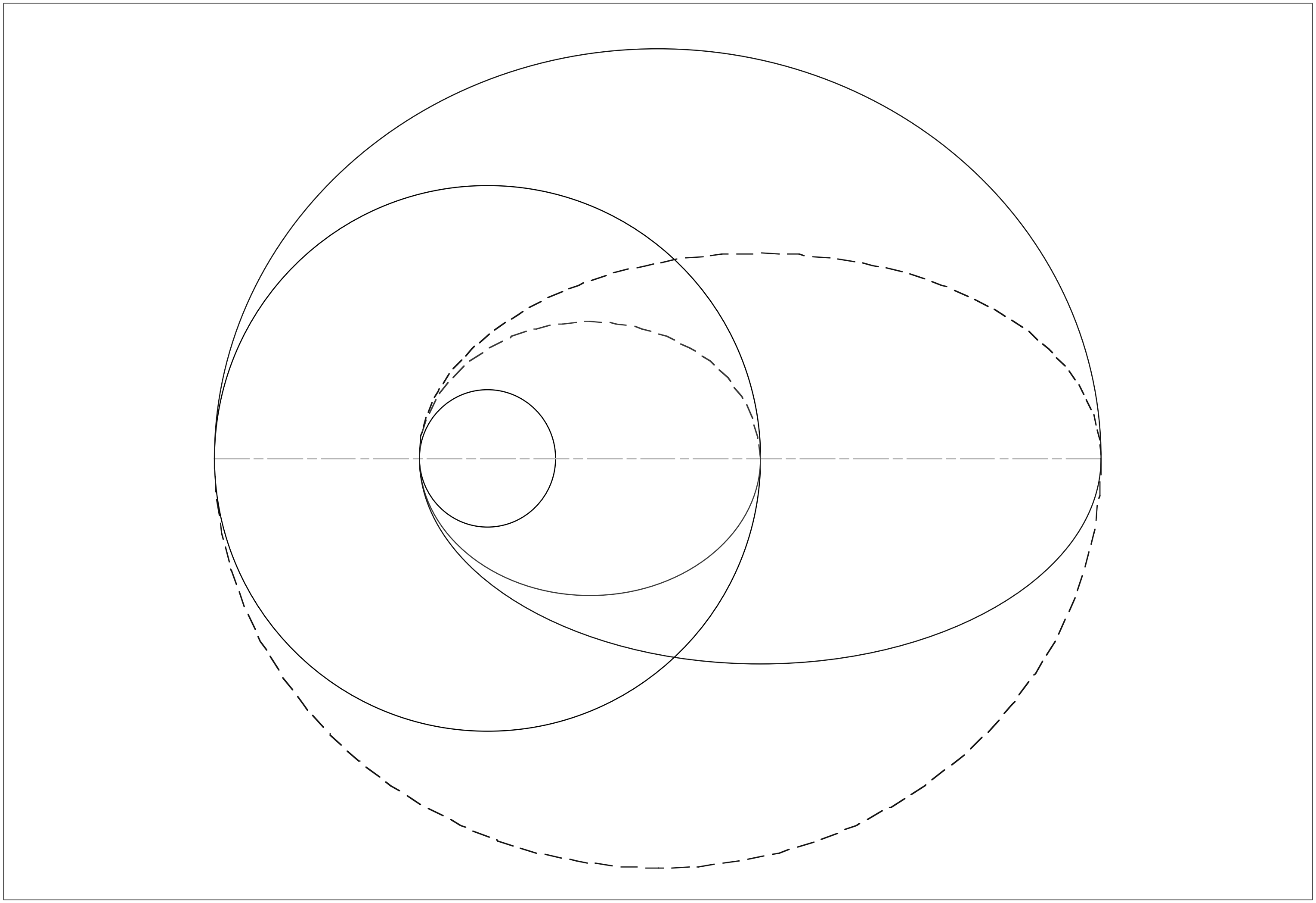
- Initial vehicle velocity $$v_a = \sqrt{\frac{\mu}{r_2}}\sqrt{\frac{2r_1}{r_1 + r_2}}$$
- Needed final velocity $$v_2 = \sqrt{\frac{\mu}{r_2}}$$
- Delta-V $$\Delta v_2 = \sqrt{v_2^2 + v_a^2 - 2v_2 v_a \cos(\Delta i_2)}$$

# Sample Plane Change Maneuver

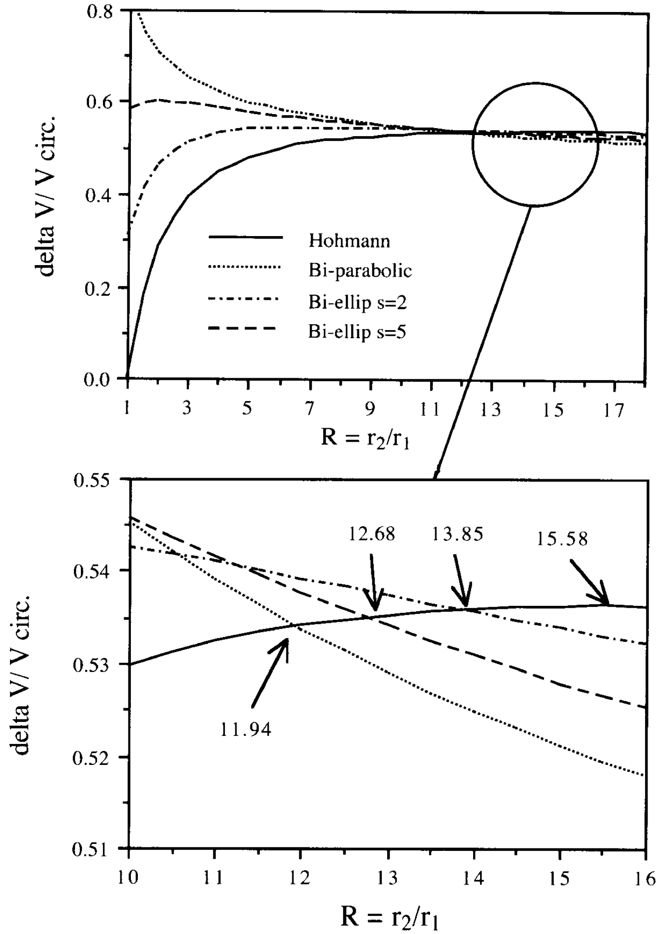



Optimum initial plane change = 2.20°

# Bielliptic Transfer

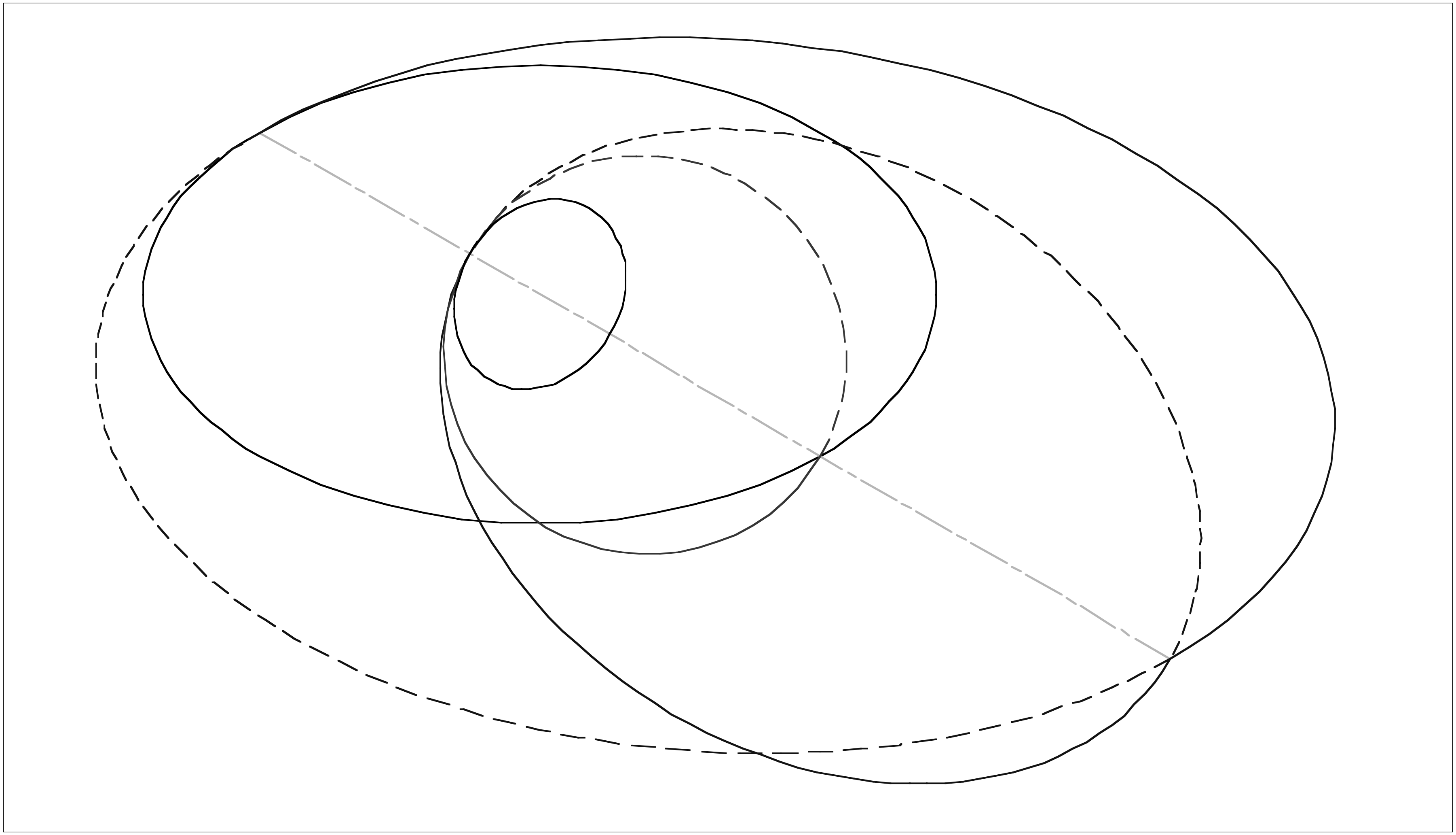


# Coplanar Transfer Velocity Requirements

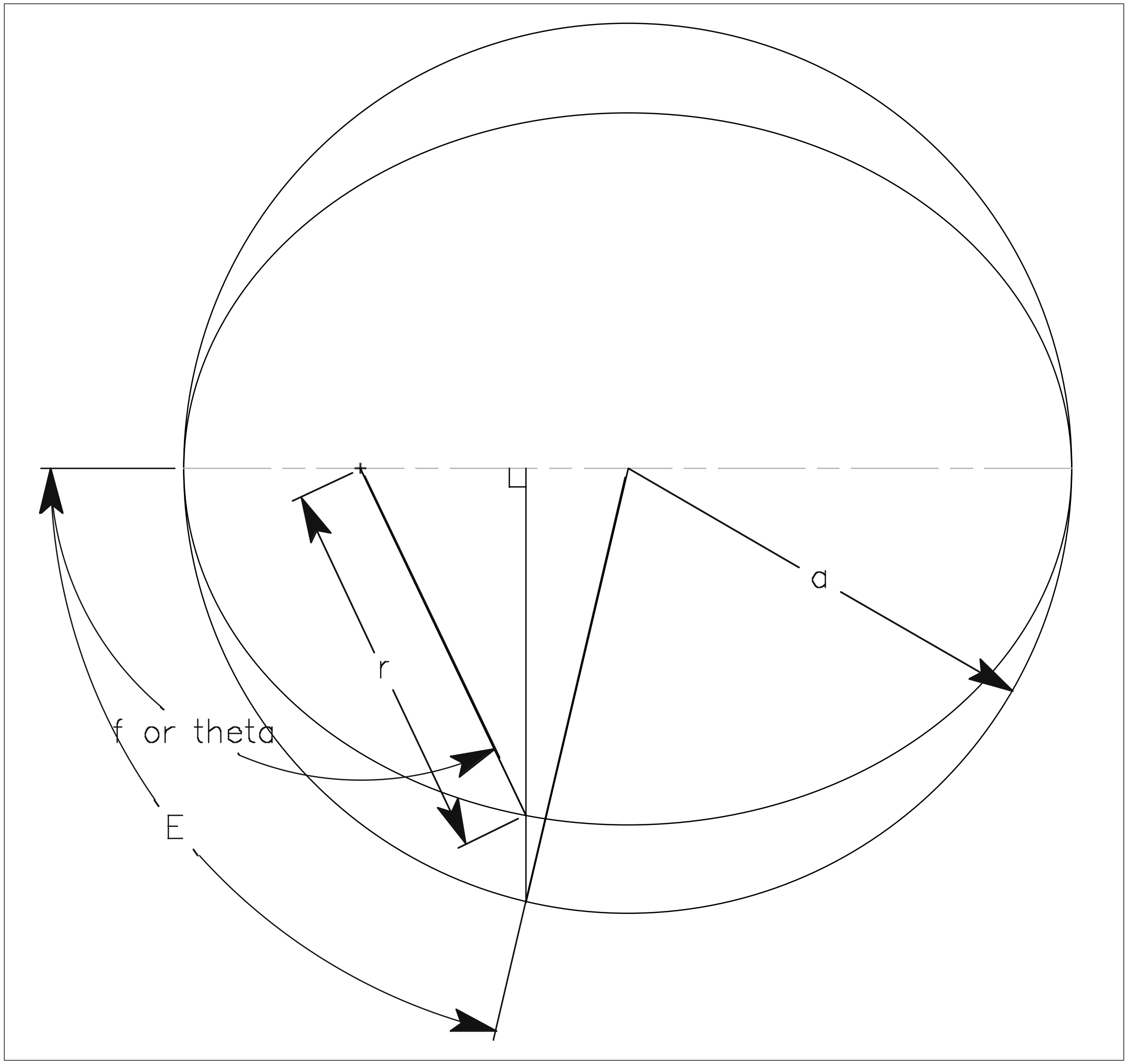



Ref: J. E. Prussing and B. A. Conway, *Orbital Mechanics* Oxford University Press, 1993

# Noncoplanar Bielliptic Transfers



# Calculating Time in Orbit



# Time in Orbit

- Period of an orbit

$$P = 2\pi\sqrt{\frac{a^3}{\mu}}$$

- Mean motion (average angular velocity)

$$n = \sqrt{\frac{\mu}{a^3}}$$

- Time since pericenter passage

$$M = nt = E - e\sin E$$

➥M=mean anomaly

# Dealing with the Eccentric Anomaly

- Relationship to orbit

$$r = a(1 - e\cos E)$$

- Relationship to true anomaly

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

- Calculating M from time interval: iterate

$$E_{i+1} = nt + e\sin E_i$$

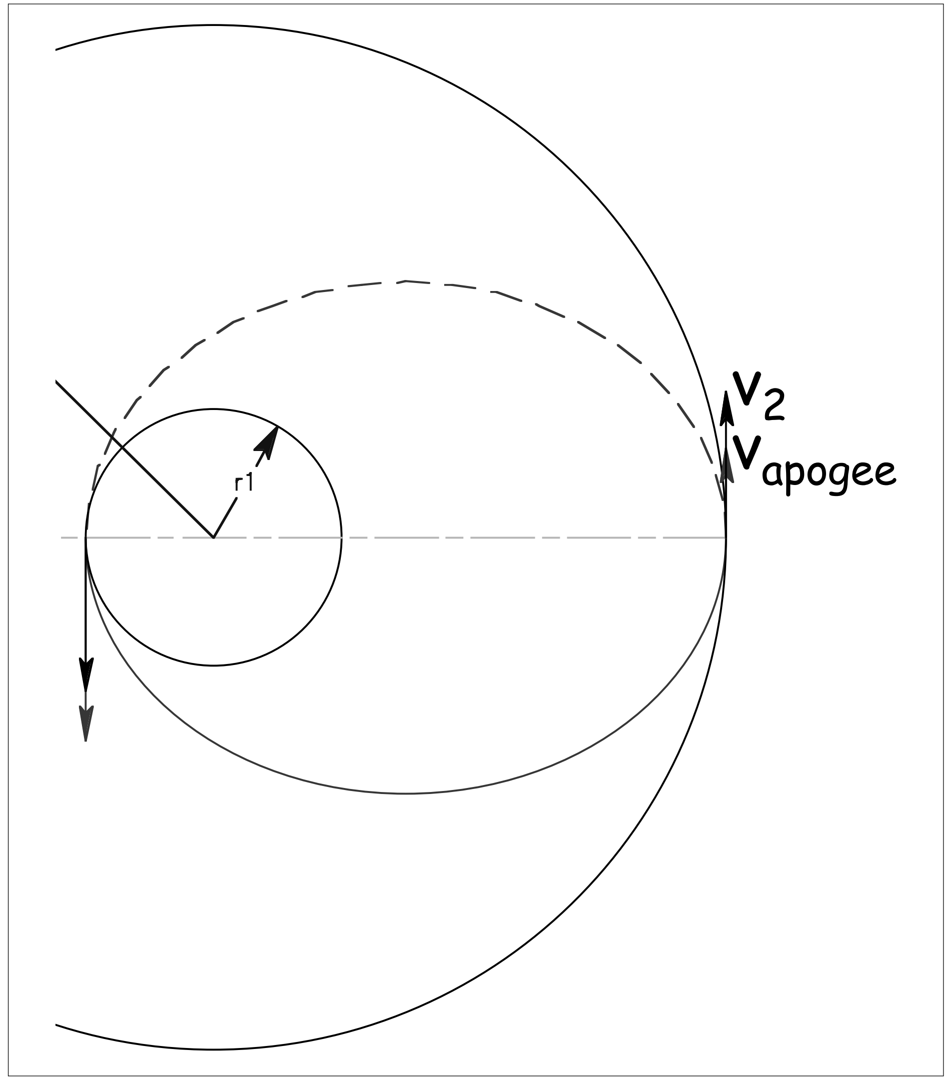
until it converges

# Patched Conics

- Simple approximation to multi-body motion (e.g., traveling from Earth orbit through solar orbit into Martian orbit)
- Treats multibody problem as “hand-offs” between gravitating bodies --> reduces analysis to sequential two-body problems
- Caveat Emptor: There are a number of formal methods to perform patched conic analysis. The approach presented here is a very simple, convenient, and not altogether accurate method for performing this calculation. Results will be accurate to a few percent, which is adequate at this level of design analysis.

# Example: Lunar Orbit Insertion

- $v_2$ is velocity of moon around Earth
- Moon overtakes spacecraft with velocity of ($v_2$-$v_{apogee}$)
- This is the velocity of the spacecraft relative to the moon while it is effectively "infinitely" far away (before lunar gravity accelerates it) = "hyperbolic excess velocity"



# Planetary Approach Analysis

- Spacecraft has $v_h$ hyperbolic excess velocity, which fixes total energy of approach orbit

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} = \frac{v_h^2}{2}$$

- Vis-viva provides velocity of approach

$$v = 2\sqrt{\frac{v_h^2}{2} + \frac{\mu}{r}}$$

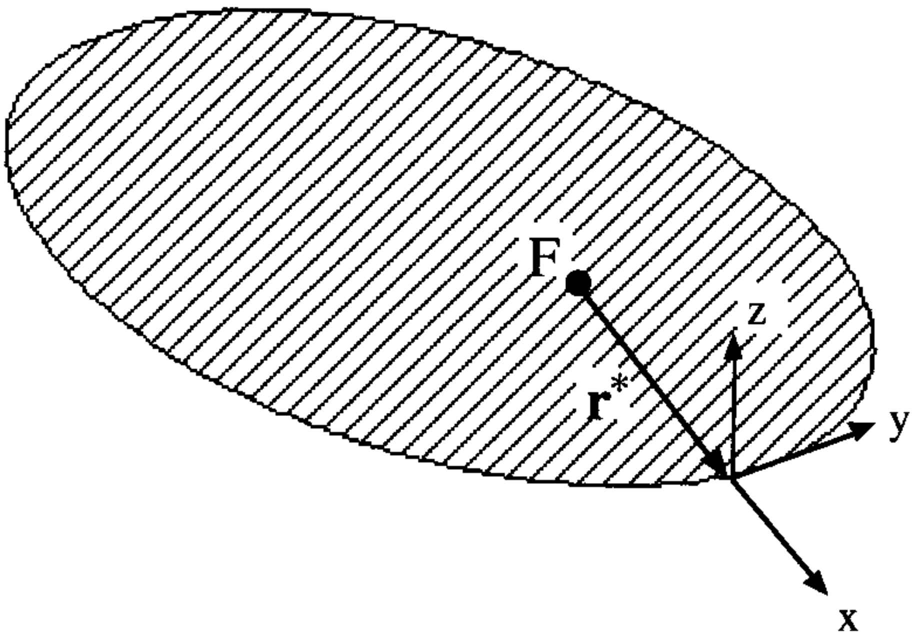
- Choose transfer orbit such that approach is tangent to desired final orbit at periapse

$$\Delta v = 2\sqrt{\frac{v_h^2}{2} + \frac{\mu}{r_{orbit}}} - \sqrt{\frac{\mu}{r_{orbit}}}$$

# ΔV Requirements for Lunar Missions

| To: / From: | Low Earth Orbit | Lunar Transfer Orbit | Low Lunar Orbit | Lunar Descent Orbit | Lunar Landing |
|---|---|---|---|---|---|
| Low Earth Orbit | | 3.107 km/sec | | | |
| Lunar Transfer Orbit | 3.107 km/sec | | 0.837 km/sec | | 3.140 km/sec |
| Low Lunar Orbit | | 0.837 km/sec | | 0.022 km/sec | |
| Lunar Descent Orbit | | | 0.022 km/sec | | 2.684 km/sec |
| Lunar Landing | | 2.890 km/sec | | 2.312 km/sec | |

# Hill's Equations (Proximity Operations)




Ref: J. E. Prussing and B. A. Conway, *Orbital Mechanics*
Oxford University Press, 1993

$$\ddot{x} = 3n^2 x + 2n\dot{y} + a_{dx}$$

$$\ddot{y} = -2n\dot{x} + a_{dy}$$

$$\ddot{z} = -n^2 z + a_{dz}$$

# Clohessy-Wiltshire ("CW") Equations

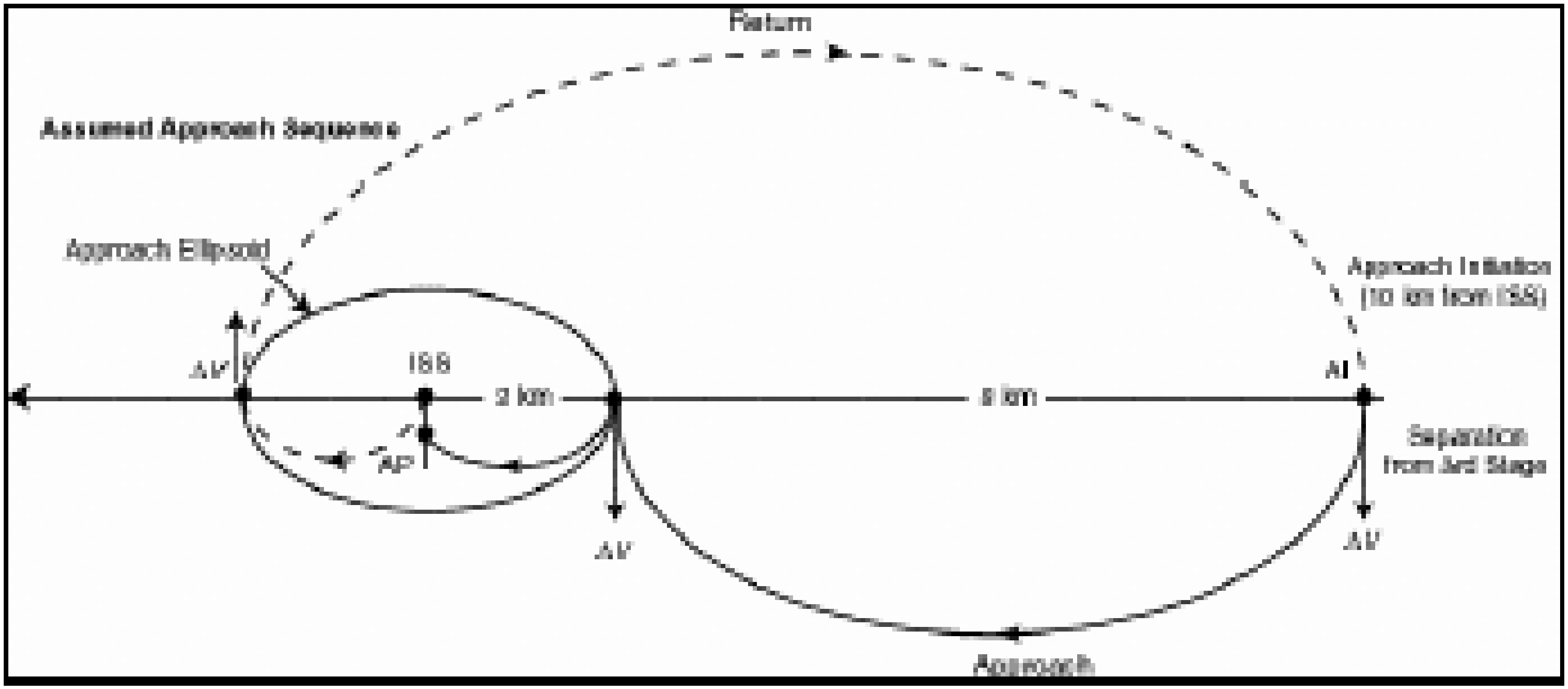
$$x(t) = \left[4 - 3\cos(nt)\right]x_o + \frac{\sin(nt)}{n}\dot{x}_o + \frac{2}{n}\left[1 - \cos(nt)\right]\dot{y}_o$$

$$y(t) = 6\left[\sin(nt) - nt\right]x_o + y_o - \frac{2}{n}\left[1 - \cos(nt)\right]\dot{x}_o + \frac{4\sin(nt) - 3nt}{n}\dot{y}_o$$

$$z(t) = z_o\cos(nt) + \frac{\dot{z}_o}{n}\sin(nt)$$

$$\dot{z}(t) = -z_o n\sin(nt) + \dot{z}_o\sin(nt)$$

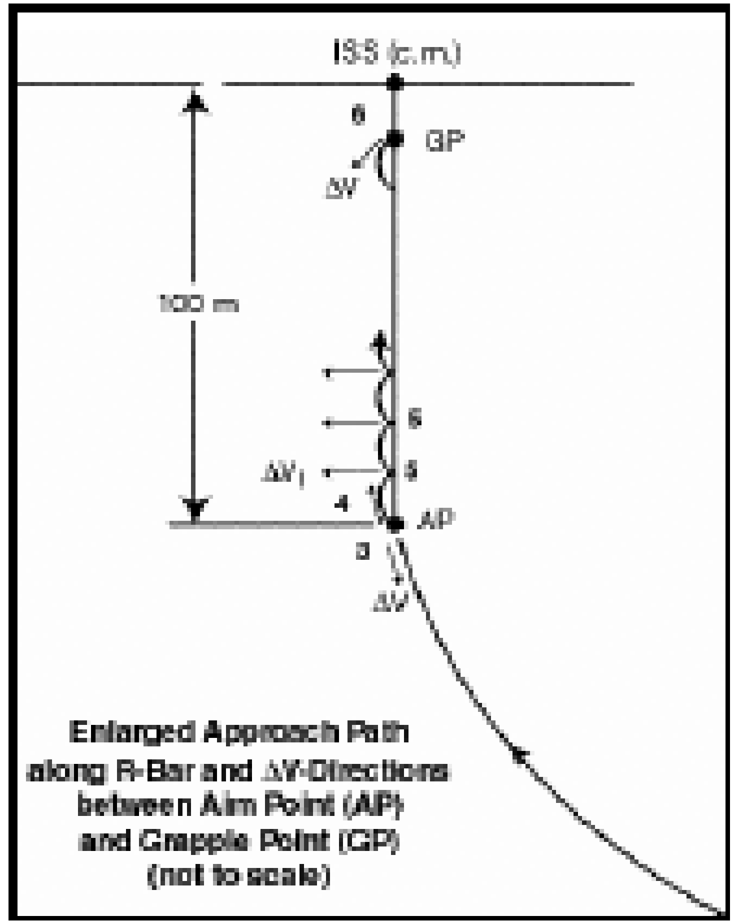
# "V-Bar" Approach




Ref: Collins, Meissinger, and Bell, *Small Orbit Transfer Vehicle (OTV) for On-Orbit Satellite Servicing and Resupply*, 15th USU Small Satellite Conference, 2001

# "R-Bar" Approach

- Approach from along the radius vector ("R-bar")
- Gravity gradients decelerate spacecraft approach velocity - low contamination approach
- Used for Mir, ISS docking approaches




Ref: Collins, Meissinger, and Bell, *Small Orbit Transfer Vehicle (OTV) for On-Orbit Satellite Servicing and Resupply*, 15th USU Small Satellite Conference, 2001

# References for Lecture 3


- Wernher von Braun, *The Mars Project* University of Illinois Press, 1962
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