Orbital Mechanics

- Energy and velocity in orbit
- Elliptical orbit parameters
- Orbital elements
- Coplanar orbital transfers
- Noncoplanar transfers
- Time and flight path angle as a function of orbital position
- Relative orbital motion ("proximity operations")
Energy in Orbit

- **Kinetic Energy**

  \[ K.E. = \frac{1}{2}mv^2 \Rightarrow \frac{K.E.}{m} = \frac{v^2}{2} \]

- **Potential Energy**

  \[ P.E. = -\frac{m\mu}{r} \Rightarrow \frac{P.E.}{m} = -\frac{\mu}{r} \]

- **Total Energy**

  \[ Const. = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \quad \text{-- Vis-Viva Equation} \]
Implications of Vis-Viva

- Circular orbit (r=a)
  \[ v_{\text{circular}} = \sqrt{\frac{\mu}{r}} \]

- Parabolic escape orbit (a tends to infinity)
  \[ v_{\text{escape}} = \sqrt{\frac{2\mu}{r}} \]

- Relationship between circular and parabolic orbits
  \[ v_{\text{escape}} = \sqrt{2} v_{\text{circular}} \]
Some Useful Constants

• Gravitation constant $\mu = GM$
  - Earth: 398,604 km$^3$/sec$^2$
  - Moon: 4667.9 km$^3$/sec$^2$
  - Mars: 42,970 km$^3$/sec$^2$
  - Sun: $1.327 \times 10^{11}$ km$^3$/sec$^2$

• Planetary radii
  - $r_{\text{Earth}} = 6378$ km
  - $r_{\text{Moon}} = 1738$ km
  - $r_{\text{Mars}} = 3393$ km
Classical Parameters of Elliptical Orbits

- $a$
- $b$
- $c (= ae)$
- $p$
- $rp$
- $ra$
- $r$
- $f$ or $\theta$
Basic Orbital Parameters

- Semi-latus rectum (or parameter)
  \[ p = a(1 - e^2) \]

- Radial distance as function of orbital position
  \[ r = \frac{p}{1 + e \cos \theta} \]

- Periapse and apoapse distances
  \[ r_p = a(1 - e) \quad r_a = a(1 + e) \]

- Angular momentum
  \[ \vec{h} = \vec{r} \times \vec{v} \quad h = \sqrt{\mu p} \]
The Classical Orbital Elements

Reference plane

first point
in Aries

ascending
node

periapsis

Ω : longitude of the ascending node
ω : argument of periapsis
\( \bar{\omega} = \Omega + \omega \) : longitude of periapsis
f : true anomaly
L = \( \bar{\omega} + f \) : true longitude

The Hohmann Transfer

\[ V_{\text{perigee}} \rightarrow V_{1} \rightarrow V_{\text{apogee}} \rightarrow V_{2} \]
First Maneuver Velocities

- Initial vehicle velocity
  \[ v_1 = \sqrt{\frac{\mu}{r_1}} \]

- Needed final velocity at perigee
  \[ v_{\text{perigee}} = \sqrt{\frac{\mu}{r_1}} \sqrt{\frac{2r_2}{r_1 + r_2}} \]

- Delta-V
  \[ \Delta v_1 = \sqrt{\frac{\mu}{r_1}} \left( \sqrt{\frac{2r_2}{r_1 + r_2}} - 1 \right) \]
Second Maneuver Velocities

- Initial vehicle velocity
  \[ v_{\text{apogee}} = \sqrt{\frac{\mu}{r_2}} \sqrt{\frac{2r_1}{r_1 + r_2}} \]

- Needed final velocity
  \[ v_2 = \sqrt{\frac{\mu}{r_2}} \]

- Delta-V
  \[ \Delta v_2 = \sqrt{\frac{\mu}{r_2}} \left( 1 - \sqrt{\frac{2r_1}{r_1 + r_2}} \right) \]
Limitations on Launch Inclinations
Differences in Inclination
Choosing the Wrong Line of Apsides
Simple Plane Change

$v_{\text{perigee}}$  \[ \Delta v_2 \]

$v_{\text{apogee}}$  \[ v_1 \]  \[ v_2 \]
Optimal Plane Change

\[ v_{perigee} \quad v_1 \quad \Delta v_1 \quad v_{apogee} \quad \Delta v_2 \quad v_2 \]
First Maneuver with Plane Change $\Delta i_1$

- Initial vehicle velocity
  $$v_1 = \sqrt{\frac{\mu}{r_1}}$$

- Needed final velocity
  $$v_p = \sqrt{\frac{\mu}{r_1}} \sqrt{\frac{2r_2}{r_1 + r_2}}$$

- Delta-V
  $$\Delta v_1 = \sqrt{v_1^2 + v_p^2 - 2v_1v_p \cos(\Delta i_1)}$$
Second Maneuver with Plane Change $\Delta i_2$

- Initial vehicle velocity\[ v_a = \sqrt{\frac{\mu}{r_2}} \sqrt{\frac{2r_1}{r_1 + r_2}} \]
- Needed final velocity\[ v_2 = \sqrt{\frac{\mu}{r_2}} \]
- Delta-V\[ \Delta v_2 = \sqrt{v_2^2 + v_a^2 - 2v_2 v_a \cos(\Delta i_2)} \]
Sample Plane Change Maneuver

Optimum initial plane change = 2.20°
Bielliptic Transfer
Coplanar Transfer Velocity Requirements

Noncoplanar Bielliptic Transfers
Calculating Time in Orbit

\[ t \text{ or } \theta \]

\[ r \]

\[ a \]
Time in Orbit

- Period of an orbit
  \[ P = 2\pi \sqrt{\frac{a^3}{\mu}} \]

- Mean motion (average angular velocity)
  \[ n = \sqrt{\frac{\mu}{a^3}} \]

- Time since pericenter passage
  \[ M = nt = E - e \sin E \]

\( \Rightarrow M = \text{mean anomaly} \)
Dealing with the Eccentric Anomaly

- Relationship to orbit

\[ r = a(1 - e \cos E) \]

- Relationship to true anomaly

\[ \tan \frac{\theta}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2} \]

- Calculating \( M \) from time interval: iterate

\[ E_{i+1} = nt + e \sin E_i \]

until it converges
Patched Conics

• Simple approximation to multi-body motion (e.g., traveling from Earth orbit through solar orbit into Martian orbit)

• Treats multibody problem as “hand-offs” between gravitating bodies --> reduces analysis to sequential two-body problems

• Caveat Emptor: There are a number of formal methods to perform patched conic analysis. The approach presented here is a very simple, convenient, and not altogether accurate method for performing this calculation. Results will be accurate to a few percent, which is adequate at this level of design analysis.
Example: Lunar Orbit Insertion

- $v_2$ is velocity of moon around Earth
- Moon overtakes spacecraft with velocity of $(v_2 - v_{\text{apogee}})$
- This is the velocity of the spacecraft relative to the moon while it is effectively “infinitely” far away (before lunar gravity accelerates it) = “hyperbolic excess velocity”
Planetary Approach Analysis

- Spacecraft has $v_h$ hyperbolic excess velocity, which fixes total energy of approach orbit
  \[ \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} = \frac{v_h^2}{2} \]

- Vis-viva provides velocity of approach
  \[ v = 2 \sqrt{\frac{v_h^2}{2} + \frac{\mu}{r}} \]

- Choose transfer orbit such that approach is tangent to desired final orbit at periapse
  \[ \Delta v = 2 \sqrt{\frac{v_h^2}{2} + \frac{\mu}{r_{orbit}}} - \sqrt{\frac{\mu}{r_{orbit}}} \]
## ΔV Requirements for Lunar Missions

<table>
<thead>
<tr>
<th>From:</th>
<th>To:</th>
<th>Low Earth Orbit</th>
<th>Lunar Transfer Orbit</th>
<th>Low Lunar Orbit</th>
<th>Lunar Descent Orbit</th>
<th>Lunar Landing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low Earth Orbit</td>
<td></td>
<td>3.107 km/sec</td>
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<tr>
<td>Lunar Transfer Orbit</td>
<td></td>
<td>3.107 km/sec</td>
<td>0.837 km/sec</td>
<td>0.022 km/sec</td>
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<td>3.140 km/sec</td>
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<tr>
<td>Low Lunar Orbit</td>
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<td>0.837 km/sec</td>
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<td>0.022 km/sec</td>
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<tr>
<td>Lunar Descent Orbit</td>
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<td>0.022 km/sec</td>
<td>2.684 km/sec</td>
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<tr>
<td>Lunar Landing</td>
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<td>2.890 km/sec</td>
<td>2.312 km/sec</td>
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</table>
Hill’s Equations (Proximity Operations)

\[
\begin{align*}
\ddot{x} &= 3n^2 x + 2ny \dot{y} + a_{dx} \\
\ddot{y} &= -2n\dot{x} + a_{dy} \\
\ddot{z} &= -n^2 z + a_{dz}
\end{align*}
\]

Ref: J. E. Prussing and B. A. Conway, Orbital Mechanics
Oxford University Press, 1993
Clohessy-Wiltshire ("CW") Equations

\[ x(t) = \left[4 - 3\cos(nt)\right]x_o + \frac{\sin(nt)}{n}\dot{x}_o + \frac{2}{n}[1 - \cos(nt)]\dot{y}_o \]

\[ y(t) = 6[\sin(nt) - nt]x_o + y_o - \frac{2}{n}[1 - \cos(nt)]\dot{x}_o + \frac{4\sin(nt) - 3nt}{n}\dot{y}_o \]

\[ z(t) = z_o \cos(nt) + \frac{\dot{z}_o}{n} \sin(nt) \]

\[ \dot{z}(t) = -z_o n \sin(nt) + \dot{z}_o \sin(nt) \]
“V-Bar” Approach

Ref: Collins, Meissinger, and Bell, Small Orbit Transfer Vehicle (OTV) for On-Orbit Satellite Servicing and Resupply, 15th USU Small Satellite Conference, 2001
“R-Bar” Approach

- Approach from along the radius vector (“R-bar”)
- Gravity gradients decelerate spacecraft approach velocity - low contamination approach
- Used for Mir, ISS docking approaches

References for Lecture 3

• Francis J. Hale, *Introduction to Space Flight* Prentice-Hall, 1994