Problem 4 Solutions
Fall, 2002

The masses given are for each stage. You need to manipulate them to find the masses in the forms we used them in the lecture analysis. The initial mass for the second stage m_{o2} is the stage 2 gross mass plus the payload = 28,900 + 3500 = 32,400 kg. The inert mass for the second stage m_{i2} is the stage gross mass minus the propellant mass = 28,900 – 26,500 = 2400 kg. Similarly, $m_{o1} = m_{o2} + m_{sg1}$ = 150,300 kg; $m_{i1} = m_{sg1} - m_{p1}$ = 6700 kg.

a) $$\delta_1 = \frac{m_{i1}}{m_{o1}} = \frac{6700}{150{,}300} = \boxed{0.04458}$$

$$\delta_2 = \frac{m_{i2}}{m_{o2}} = \frac{2400}{32{,}400} = \boxed{0.07407}$$

b) $$\Delta v_1 = -gI_{sp}\ln\left(\frac{m_{i1} + m_{o2}}{m_{o1}}\right) = (9.8)(296)\ln\left(\frac{6700 + 32{,}400}{150{,}300}\right) = \boxed{3905 \text{ m/sec}}$$

$$\Delta v_2 = -gI_{sp2}\ln\left(\frac{m_{i2} + m_L}{m_{o2}}\right) = (9.8)(316)\ln\left(\frac{2400 + 3500}{32{,}400}\right) = \boxed{5274 \text{ m/sec}}$$

Note that "g" in the rocket equation is only a conversion between mass and force units, and has nothing to do with the local gravitation in whatever part of space you find yourself. Anywhere in the universe, "g" in this application is 9.8 N/kg=m/sec^2.

c) When I ask about trading inert mass for payload, I'm clearly doing trade-off ratios. I'm going to need to compute the final mass for each stage = initial mass – propellant mass. This gives m_{f1}=39,100kg and m_{f2}=5900 kg. For this case,

$$\left.\frac{\partial m_L}{\partial m_{i1}}\right|_{\partial \Delta V_{Total}=0} = \frac{-\sum_{j=1}^{1} V_{e,j}\left(\frac{1}{m_{o,j}} - \frac{1}{m_{f,j}}\right)}{\sum_{\ell=1}^{2} V_{e,\ell}\left(\frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}}\right)} = \frac{-V_{e,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right)}{V_{e,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + V_{e,j}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)}$$

$$= \frac{-9.8 \times 296\left(\frac{1}{150{,}300} - \frac{1}{39{,}100}\right)}{9.8 \times 296\left(\frac{1}{150{,}300} - \frac{1}{39{,}100}\right) + 9.8 \times 316\left(\frac{1}{32{,}400} - \frac{1}{5900}\right)} = -0.1134$$

So to gain 100 kg of payload, I have to lose 100/(-0.1134)=$\boxed{882.1 \text{ kg}}$ of first stage inert mass.

d) Similarly to the previous question,

$$\left.\frac{\partial m_L}{\partial m_{p2}}\right|_{\partial\Delta V_{Total}=0} = \frac{-\sum_{j=1}^{2} V_{e,j}\left(\frac{1}{m_{o,j}}\right)}{\sum_{\ell=1}^{2} V_{e,\ell}\left(\frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}}\right)} = \frac{-V_{e,1}\left(\frac{1}{m_{o,1}}\right) - V_{e,2}\left(\frac{1}{m_{o,2}}\right)}{V_{e,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + V_{e,j}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)}$$

$$= \frac{-9.8 \times 296\left(\frac{1}{150{,}300}\right) - 9.8 \times 316\left(\frac{1}{32{,}400}\right)}{9.8 \times 296\left(\frac{1}{150{,}300} - \frac{1}{39{,}100}\right) + 9.8 \times 316\left(\frac{1}{32{,}400} - \frac{1}{5900}\right)} = 0.2373$$

So to gain 100 kg of payload, I have to load an additional 100/(0.2373)=**421.5 kg** of second stage propellant.

e) $\Delta v_{tot} = \Delta v_1 + \Delta v_2 = -gI_{sp1} \ln\left(\frac{m_{i1} + m_{o2}}{m_{o1}}\right) - gI_{sp2} \ln\left(\frac{m_{i2} + m_L}{m_{o2}}\right)$ - Of course, m_{o1} and m_{o2} are also functions of m_L. The most straightforward way to do this is to code the equation in Excel and use Goal Seek or Solver to find the value of m_L which will produce the desired 10,500 m/sec, which is **1342 kg**.

f) This is adding a modular first stage onto the existing vehicle. Since the vehicle after booster separation is unchanged, the Δv for that unit will remain 9180 m/sec. That means the boosters will have to supply 12,500 - 9180 = 3320 m/sec. Due to the 500 kg aerodynamic fairing, each booster will have an initial mass of 118,400 kg and an inert mass of 7200 kg. The mass ratio for the booster stage, given N booster units, is therefore $r_B = \frac{Nm_{iB} + m_{o1}}{Nm_{oB} + m_{o1}}$. Plugging this into the rocket equation gives you Δv_B = 2479 m/sec for N=2, 3129 m/sec for N=3, and 3620 m/sec for N=4. Since we need 3320 m/sec, we will choose **N=4** (and have 300 m/sec reserve margin…)