Problem 3 Solutions
Fall, 2002

a) For an "ideally expanded" (i.e., expanded to vacuum) rocket engine, the exhaust velocity is $V_e = \sqrt{\frac{2\gamma}{\gamma-1}\frac{\Re T_0}{\overline{M}}}$. Some algebra gives $T_0 = \frac{\gamma-1}{2\gamma}\frac{\overline{M}}{\Re}V_e^2$ and, plugging in numbers, $T_0 = \frac{0.4}{2\cdot 1.4}\frac{2}{8314}\left[(900)(9.8)\right]^2$ = $\boxed{2673°K}$

b) All other things being equal, $V_{e1} = V_{e2}\sqrt{\frac{\overline{M}_2}{\overline{M}_1}}$, and since the H_2O/H_2 molecular weight ratio is 18/2=9, the I_{sp} for a water-propelled NERVA engine would be 900/3=$\boxed{300 \text{ sec}}$. Actually, the ratio of specific heats for a triatomic molecule like water would be more like 1.33, and using $V_e = \sqrt{\frac{2\cdot 1.33}{0.33}\frac{8314\cdot 2673}{18}}$ =3154 m/sec gives I_{sp}=$\boxed{322 \text{ sec}}$ (either answer will be counted as correct.)

c) The density of LOX is 1140 kg/m^3 (lecture 10, page 4), so the LOX tank volume V_{LOX}=8.772 m^3. Assuming adiabatic expansion,

$P_{He,initial}V_{He}^{\gamma} = P_{He,final}V_{He}^{\gamma} + P_{LOX}V_{LOX}^{\gamma}$ or $V_{He} = \left(\frac{P_{LOX}}{P_{He,initial} - P_{He,final}}\right)^{1/\gamma} V_{LOX}$. We know that P_{LOX}=300 psi and $P_{He,final}$=P_{LOX}+100=400 psi, and we don't have to convert to metric because it's a straight ratio, and that gamma for a monatomic gas like helium will be 1.67, so $V_{He} = \left(\frac{300}{3000-400}\right)^{1/1.67}$8.772=$\boxed{2.407 \text{ m}^3}$. The density of helium can be found from the ideal gas law $P = \rho RT = \frac{\rho \Re T}{M}$ or $\rho = \frac{PM}{\Re T}$. This is unfortunately one place where we do have to convert to a real metric pressure, so 3000 lbs/in^2=20.68 MPa. So the density is $\rho = \frac{20{,}680{,}000\cdot 4}{8314\cdot 300}$=33.16 kg/m^3, so the mass of helium required is $\boxed{79.82 \text{ kg}}$.

d) Using the rocket equation, you set up an expression for total ΔV based on the masses, including the mass of the nozzle, and using the I_{sp} from the nozzle equation. The issue here is that a longer, more fully expanded nozzle increases I_{sp} and therefore ΔV, but it also has a greater mass which reduces ΔV. There is an optimum point, which is shown in this table:

Pe/P0	Pexit	P0/pe	Ve	At/Ae	Ae/At	Nozzle area	Nozzle mass	r	Delta-V
100	10.0	0.010000	3360	0.08424	11.87	7.205	72.05	0.06670	9096
150	6.7	0.006667	3453	0.06176	16.19	9.827	98.27	0.06822	9272
200	5.0	0.005000	3515	0.04946	20.22	12.270	122.70	0.06964	9365

250	4.0	0.004000	3560	0.04159	24.04	14.592	145.92	0.07097	9417
300	3.3	0.003333	3595	0.03608	27.71	16.820	168.20	0.07225	9445
350	2.9	0.002857	3623	0.03199	31.26	18.974	189.74	0.07349	9459
399.2	2.505	0.002505	3647	0.02885	34.66	21.035	210.35	0.07467	9463
450	2.2	0.002222	3668	0.02626	38.08	23.110	231.10	0.07585	9460
500	2.0	0.002000	3686	0.02417	41.37	25.108	251.08	0.07698	9451

The optimum exit pressure is right at 2.5 psi, which results in the maximum possible ΔV for the vehicle.