Orbital Mechanics

• Planetary launch and entry overview
• Energy and velocity in orbit
• Elliptical orbit parameters
• Orbital elements
• Coplanar orbital transfers
• Noncoplanar transfers
• Time in orbit
• Interplanetary trajectories
• Relative orbital motion (“proximity operations”)
Space Launch - The Physics

• Minimum orbital altitude is $\sim 200 \text{ km}$

$$\frac{\text{Potential Energy}}{\text{kg in orbit}} = -\frac{\mu}{r_{\text{orbit}}} + \frac{\mu}{r_E} = 1.9 \times 10^6 \frac{J}{\text{kg}}$$

• Circular orbital velocity there is $7784 \text{ m/sec}$

$$\frac{\text{Kinetic Energy}}{\text{kg in orbit}} = \frac{1}{2} \frac{\mu}{r_{\text{orbit}}^2} = 30 \times 10^6 \frac{J}{\text{kg}}$$

• Total energy per kg in orbit

$$\frac{\text{Total Energy}}{\text{kg in orbit}} = KE + PE = 32 \times 10^6 \frac{J}{\text{kg}}$$
Theoretical Cost to Orbit

• Convert to usual energy units

\[ \frac{Total \ Energy}{kg \ in \ orbit} = 32 \times 10^6 \ \frac{J}{kg} = 8.888 \ \frac{kW\ hrs}{kg} \]

• Domestic energy costs are \(~$0.05/kWhr\)

\[ \text{Theoretical cost to orbit} \ \$0.44/kg \]
Actual Cost to Orbit

Delta IV Heavy
- 23,000 kg to LEO
- $250 M per flight

$10,900/kg of payload
Factor of 25,000x higher than theoretical energy costs!
What About Airplanes?

- For an aircraft in level flight,
  \[
  \frac{\text{Weight}}{\text{Thrust}} = \frac{\text{Lift}}{\text{Drag}}, \text{ or } \frac{mg}{T} = \frac{L}{D}
  \]

- Energy = force x distance, so
  \[
  \frac{\text{Total Energy}}{\text{kg}} = \frac{\text{thrust} \times \text{distance}}{\text{mass}} = \frac{Td}{m} = \frac{gd}{L/D}
  \]

- For an airliner (L/D=25) to equal orbital energy, 
d=81,000 km (2 roundtrips NY-Sydney)
Equivalent Airline Costs?

• Average economy ticket NY-Sydney round-round-trip (Travelocity 9/3/09) ~$1300
• Average passenger (+ luggage) ~100 kg
• Two round trips = $26/kg
  – Factor of 60x more than electrical energy costs
  – Factor of 420x less than current launch costs
• But…
  you get to refuel at each stop!
Equivalence to Air Transport

• 81,000 km ~ twice around the world
• Voyager - one of only two aircraft to ever circle the world non-stop, non-refueled - once!
Orbital Entry - The Physics

- 32 MJ/kg dissipated by friction with atmosphere over ~8 min = 66kW/kg
- Pure graphite (carbon) high-temperature material: $c_p = 709 \text{ J/kg}^\circ\text{K}$
- Orbital energy would cause temperature gain of 45,000$^\circ\text{K}$!
- (If you’re interested in how this works out, you can take ENAE 791 Launch and Entry Vehicle Design next term...)}
Newton’s Law of Gravitation

• Inverse square law

\[ F = \frac{GMm}{r^2} \]

• Since it’s easier to remember one number,

\[ \mu = GM \]

• If you’re looking for local gravitational acceleration,

\[ g = \frac{\mu}{r^2} \]
Some Useful Constants

• Gravitation constant $\mu = GM$
  – Earth: $398,604 \text{ km}^3/\text{sec}^2$
  – Moon: $4667.9 \text{ km}^3/\text{sec}^2$
  – Mars: $42,970 \text{ km}^3/\text{sec}^2$
  – Sun: $1.327 \times 10^{11} \text{ km}^3/\text{sec}^2$

• Planetary radii
  – $r_{\text{Earth}} = 6378 \text{ km}$
  – $r_{\text{Moon}} = 1738 \text{ km}$
  – $r_{\text{Mars}} = 3393 \text{ km}$
Energy in Orbit

- **Kinetic Energy**
  \[
  K.E. = \frac{1}{2}mv^2 \quad \Rightarrow \quad \frac{K.E.}{m} = \frac{v^2}{2}
  \]

- **Potential Energy**
  \[
  P.E. = -\frac{\mu m}{r} \quad \Rightarrow \quad \frac{P.E.}{m} = -\frac{\mu}{r}
  \]

- **Total Energy**
  \[
  \text{Constant} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}
  \]

  \[
  v^2 = \mu \left(\frac{2}{r} - \frac{1}{a}\right)
  \]

<--Vis-Viva Equation
Classical Parameters of Elliptical Orbits
The Classical Orbital Elements


\[ \Omega : \text{longitude of the ascending node} \]
\[ \omega : \text{argument of periapsis} \]
\[ \bar{\Omega} = \Omega + \omega : \text{longitude of periapsis} \]
\[ f : \text{true anomaly} \]
\[ L = \bar{\Omega} + f : \text{true longitude} \]
Implications of Vis-Viva

- Circular orbit \((r=a)\)
  \[
  v_{\text{circular}} = \sqrt{\frac{\mu}{r}}
  \]

- Parabolic escape orbit \((a \text{ tends to infinity})\)
  \[
  v_{\text{escape}} = \sqrt{\frac{2\mu}{r}}
  \]

- Relationship between circular and parabolic orbits
  \[
  v_{\text{escape}} = \sqrt{2}v_{\text{circular}}
  \]
The Hohmann Transfer

\[ r_1, r_2, v_1, v_2, v_{\text{perigee}}, v_{\text{apogee}} \]
First Maneuver Velocities

• Initial vehicle velocity

\[ v_1 = \sqrt{\frac{\mu}{r_1}} \]

• Needed final velocity

\[ v_{perigee} = \sqrt{\frac{\mu}{r_1}} \sqrt{\frac{2r_2}{r_1 + r_2}} \]

• Delta-V

\[ \Delta v_1 = \sqrt{\frac{\mu}{r_1}} \left( \sqrt{\frac{2r_2}{r_1 + r_2}} - 1 \right) \]
Second Maneuver Velocities

• Initial vehicle velocity

\[ v_{\text{apogee}} = \sqrt{\frac{\mu}{r_2}} \sqrt{\frac{2r_1}{r_1 + r_2}} \]

• Needed final velocity

\[ v_2 = \sqrt{\frac{\mu}{r_2}} \]

• Delta-V

\[ \Delta v_2 = \sqrt{\frac{\mu}{r_2}} \left( 1 - \sqrt{\frac{2r_1}{r_1 + r_2}} \right) \]
Limitations on Launch Inclinations
Differences in Inclination

Line of Nodes
Choosing the Wrong Line of Apsides
Simple Plane Change
Optimal Plane Change

\[ \Delta v_1 \]

\[ \Delta v_2 \]

\[ v_{\text{perigee}} \]

\[ v_{\text{apogee}} \]
First Maneuver with Plane Change $\Delta i_1$

- Initial vehicle velocity
  \[ v_1 = \sqrt{\frac{\mu}{r_1}} \]

- Needed final velocity
  \[ v_p = \sqrt{\frac{\mu}{r_1}} \sqrt{\frac{2r_2}{r_1 + r_2}} \]

- Delta-V
  \[ \Delta v_1 = \sqrt{v_1^2 + v_p^2 - 2v_1 v_p \cos \Delta i_1} \]
Second Maneuver with Plane Change $\Delta i_2$

- Initial vehicle velocity
  
  \[ v_a = \sqrt{\frac{\mu}{r_2}} \sqrt{\frac{2r_1}{r_1 + r_2}} \]

- Needed final velocity
  
  \[ v_2 = \sqrt{\frac{\mu}{r_2}} \]

- Delta-V
  
  \[ \Delta v_2 = \sqrt{v_2^2 + v_a^2 - 2v_2v_a \cos \Delta i_2} \]
Sample Plane Change Maneuver

Optimum initial plane change = 2.20°
Bielliptic Transfer

\[ \Delta v_3 \quad \Delta v_1 \quad \Delta v_2 \]
Coplanar Transfer Velocity

Noncoplanar Bielliptic Transfers

\[ \Delta v_3 \]

\[ \Delta v_1 \]

\[ \Delta v_2 \]
Calculating Time in Orbit
Time in Orbit

• Period of an orbit

\[ P = 2\pi \sqrt{\frac{a^3}{\mu}} \]

• Mean motion (average angular velocity)

\[ n = \sqrt{\frac{\mu}{a^3}} \]

• Time since pericenter passage

\[ M = nt = E - e \sin E \]

⇒ \( M \) = mean anomaly
Dealing with the Eccentric Anomaly

- Relationship to orbit

\[ r = a (1 - e \cos E) \]

- Relationship to true anomaly

\[ \tan \frac{\theta}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2} \]

- Calculating M from time interval: iterate

\[ E_{i+1} = nt + e \sin E_i \]

until it converges
Example: Time in Orbit

- Hohmann transfer from LEO to GEO
  - $h_1 = 300 \text{ km} \rightarrow r_1 = 6378 + 300 = 6678 \text{ km}$
  - $r_2 = 42240 \text{ km}$

- Time of transit (1/2 orbital period)

  \[
a = \frac{1}{2} (r_1 + r_2) = 24,459 \text{ km}
  \]

  \[
t_{\text{transit}} = \frac{P}{2} = \pi \sqrt{\frac{a^3}{\mu}} = 19,034 \text{ sec} = 5h17m14s
  \]
Example: Time-based Position

Find the spacecraft position 3 hours after perigee

\[ n = \sqrt{\frac{\mu}{a^3}} = 1.650 \times 10^{-4} \text{ rad/sec} \]

\[ e = 1 - \frac{r_p}{a} = 0.7270 \]

\[ E_{j+1} = nt + e \sin E_j = 1.783 + 0.7270 \sin E_j \]

E = 0; 1.783; 2.494; 2.222; 2.361; 2.294; 2.328; 2.311; 2.320; 2.316; 2.318; 2.317; 2.317; 2.317
Example: Time-based Position (cont.)

\[ E = 2.317 \]

\[ r = a(1 - e \cos E) = 12,387 \text{ km} \]

\[ \tan \frac{\theta}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2} \implies \theta = 160 \text{ deg} \]

Have to be sure to get the position in the proper quadrant - since the time is less than 1/2 the period, the spacecraft has yet to reach apogee \( --> \) \( 0° < \theta < 180° \)
Basic Orbital Parameters

- Semi-latus rectum (or parameter)
  \[ p = a(1 - e^2) \]

- Radial distance as function of orbital position
  \[ r = \frac{p}{1 + e \cos \theta} \]

- Periapse and apoapse distances
  \[ r_p = a(1 - e) \quad r_a = a(1 + e) \]

- Angular momentum
  \[ \vec{h} = \vec{r} \times \vec{v} \quad h = \sqrt{\mu p} \]
Velocity Components in Orbit

\[ r = \frac{p}{1 + e \cos \theta} \]

\[ v_r = \frac{dr}{dt} = \frac{d}{dt} \left( \frac{p}{1 + e \cos \theta} \right) = \frac{-p(-e \sin \theta \frac{d\theta}{dt})}{(1 + e \cos \theta)^2} \]

\[ v_r = \frac{pe \sin \theta}{(1 + e \cos \theta)^2} \frac{d\theta}{dt} \]

\[ 1 + e \cos \theta = \frac{p}{r} \Rightarrow v_r = \frac{r^2 \frac{d\theta}{dt} e \sin \theta}{p} \]

\[ \vec{h} = \vec{r} \times \vec{v} \]
Velocity Components in Orbit (cont.)

\[ \mathbf{h} = \mathbf{r} \times \mathbf{\dot{r}} \]

\[ h = rv \cos \gamma = r \left( r \frac{d\theta}{dt} \right) = r^2 \frac{d\theta}{dt} \]

\[ v_r = \frac{r^2 \frac{d\theta}{dt} e \sin \theta}{p} = \frac{he \sin \theta}{p} = \frac{\sqrt{p\mu}}{p} e \sin \theta \]

\[ v_r = \sqrt{\frac{\mu}{p}} e \sin \theta \]

\[ v_\theta = r \frac{d\theta}{dt} = r \frac{h}{r^2} = \frac{h}{r} = \frac{\sqrt{p\mu}}{r} \]

\[ v_\theta = \sqrt{\frac{\mu}{p}} \left( 1 + e \cos \theta \right) \]

\[ \tan \gamma = \frac{v_r}{v_\theta} = \frac{e \sin \theta}{1 + e \cos \theta} \]
Patched Conics

• Simple approximation to multi-body motion (e.g., traveling from Earth orbit through solar orbit into Martian orbit)
• Treats multibody problem as “hand-offs” between gravitating bodies --> reduces analysis to sequential two-body problems
• Caveat Emptor: There are a number of formal methods to perform patched conic analysis. The approach presented here is a very simple, convenient, and not altogether accurate method for performing this calculation. Results will be accurate to a few percent, which is adequate at this level of design analysis.
Example: Lunar Orbit Insertion

- $v_2$ is velocity of moon around Earth
- Moon overtakes spacecraft with velocity of $(v_2 - v_{\text{apogee}})$
- This is the velocity of the spacecraft relative to the moon while it is effectively “infinitely” far away (before lunar gravity accelerates it) = “hyperbolic excess velocity”
Planetary Approach Analysis

• Spacecraft has $v_h$ hyperbolic excess velocity, which fixes total energy of approach orbit

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} = \frac{v_h^2}{2}$$

• Vis-viva provides velocity of approach

$$v = \sqrt{v_h^2 + \frac{2\mu}{r}}$$

• Choose transfer orbit such that approach is tangent to desired final orbit at periapse

$$\Delta v = \sqrt{v_h^2 + \frac{2\mu}{r}} - \sqrt{\frac{\mu}{r}}$$
Patched Conic - Lunar Approach

- Lunar orbital velocity around the Earth
  \[ v_m = \sqrt{\frac{\mu}{r_m}} = \sqrt{\frac{398,604}{384,400}} = 1.018 \ \frac{km}{sec} \]

- Apogee velocity of Earth transfer orbit from initial 400 km low Earth orbit
  \[ v_a = v_m \sqrt{\frac{2r_1}{r_1 + r_m}} = 1.018 \sqrt{\frac{6778}{6778 + 384,400}} = 0.134 \ \frac{km}{sec} \]

- Velocity difference between spacecraft “infinitely” far away and moon (hyperbolic excess velocity)
  \[ v_h = v_m - v_a = v_m = 1.018 - 0.134 = 0.884 \ \frac{km}{sec} \]
Patched Conic - Lunar Orbit Insertion

- The spacecraft is now in a hyperbolic orbit of the moon. The velocity it will have at the perilune point tangent to the desired 100 km low lunar orbit is
  \[ v_{pm} = \sqrt{v_h^2 + \frac{2\mu_m}{r_{LLO}}} = \sqrt{1.018^2 + \frac{2(4667.9)}{1878}} = 2.451 \frac{km}{sec} \]
- The required delta-V to slow down into low lunar orbit is
  \[ \Delta v = v_{pm} - v_{cm} = 2.451 - \sqrt{\frac{4667.9}{1878}} = 0.874 \frac{km}{sec} \]
### ΔV Requirements for Lunar Missions

<table>
<thead>
<tr>
<th>From:</th>
<th>To:</th>
<th>Low Earth Orbit</th>
<th>Lunar Transfer Orbit</th>
<th>Low Lunar Orbit</th>
<th>Lunar Descent Orbit</th>
<th>Lunar Landing</th>
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<tbody>
<tr>
<td>Low Earth Orbit</td>
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<td>3.107 km/sec</td>
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<td>3.140 km/sec</td>
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<tr>
<td>Low Lunar Orbit</td>
<td>Lunar Transfer Orbit</td>
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<td>2.684 km/sec</td>
<td>2.312 km/sec</td>
<td></td>
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<tr>
<td>Lunar Landing</td>
<td>Lunar Descent Orbit</td>
<td>2.890 km/sec</td>
<td>2.312 km/sec</td>
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</table>

- ΔV stands for change in velocity.
- These values represent the ΔV required for different stages of a lunar mission.
- Units are km/sec.
LOI $\Delta V$ Based on Landing Site
LOI \( \Delta V \) Including Loiter Effects
Interplanetary “Pork Chop” Plots

• Summarize a number of critical parameters
  – Date of departure
  – Date of arrival
  – Hyperbolic energy (“C3”)
  – Transfer geometry

• Launch vehicle determines available C3 based on window, payload mass
Hill’s Equations (Proximity Operations)

\[
\begin{align*}
\ddot{x} &= 3n^2 x + 2n\dot{y} + a_{dx} \\
\ddot{y} &= -2n\dot{x} + a_{dy} \\
\ddot{z} &= -n^2 z + a_{dz}
\end{align*}
\]

Ref: J. E. Prussing and B. A. Conway, Orbital Mechanics
Oxford University Press, 1993
Clohessy-Wiltshire ("CW") Equations

\[ x(t) = [4 - 3 \cos(nt)]x_o + \frac{\sin(nt)}{n} \dot{x}_o + \frac{2}{n}[1 - \cos(nt)]\dot{y}_o \]

\[ y(t) = 6[\sin(nt) - nt]x_o + y_o - \frac{2}{n}[1 - \cos(nt)]\dot{x}_o + \frac{4\sin(nt) - 3nt}{n}\dot{y}_o \]

\[ z(t) = z_o \cos(nt) + \frac{\dot{z}_o}{n}\sin(nt) \]

\[ \dot{z}(t) = -z_o n \sin(nt) + \dot{z}_o \sin(nt) \]
“V-Bar” Approach

“R-Bar” Approach

• Approach from along the radius vector (“R-bar”)
• Gravity gradients decelerate spacecraft approach velocity - low contamination approach
• Used for Mir, ISS docking approaches

References for This Lecture

• Francis J. Hale, *Introduction to Space Flight* Prentice-Hall, 1994