Rocket Performance

- The rocket equation
- Mass ratio and performance
- Structural and payload mass fractions
- Regression analysis
- Multistaging
- Optimal $\Delta V$ distribution between stages
- Trade-off ratios
- Parallel staging
- Modular staging
Derivation of the Rocket Equation

• Momentum at time $t$:
  \[ M = mv \]

• Momentum at time $t + \Delta t$:
  \[ M = (m - \Delta m)(V + \Delta v) + \Delta m (v - V_e) \]

• Some algebraic manipulation gives:
  \[ m \Delta v = -\Delta m V_e \]

• Take to limits and integrate:
  \[
  \int_{m_{\text{initial}}}^{m_{\text{final}}} \frac{dm}{m} = - \int_{V_{\text{initial}}}^{V_{\text{final}}} \frac{dv}{V_e}
  \]
The Rocket Equation

• Alternate forms

\[ r \equiv \frac{m_{\text{final}}}{m_{\text{initial}}} = e^{-\frac{\Delta V}{V_e}} \]

\[ \Delta v = -V_e \ln \left( \frac{m_{\text{final}}}{m_{\text{initial}}} \right) = -V_e \ln r \]

• Basic definitions/concepts

  – Mass ratio

\[ r \equiv \frac{m_{\text{final}}}{m_{\text{initial}}} \quad \text{or} \quad \mathcal{R} \equiv \frac{m_{\text{initial}}}{m_{\text{final}}} \]

  – Nondimensional velocity change

“Velocity ratio”

\[ \Delta \frac{V}{V_e} \]
Rocket Equation (First Look)

Mass Ratio, \( \frac{M_{\text{final}}}{M_{\text{initial}}} \)

Velocity Ratio, \( \frac{\Delta V}{V_e} \)

Typical Range for Launch to Low Earth Orbit
Sources and Categories of Vehicle Mass

Payload
Propellants
Structure
Propulsion
Avionics
Power
Mechanisms
Thermal
Etc.
Etc.
Basic Vehicle Parameters

- Basic mass summary
  \[ m_o = m_{pl} + m_{pr} + m_{in} \]

- Inert mass fraction
  \[ \delta \equiv \frac{m_{in}}{m_o} = \frac{m_{in}}{m_{pl} + m_{pr} + m_{in}} \]

- Payload fraction
  \[ \lambda \equiv \frac{m_{pl}}{m_o} = \frac{m_{pl}}{m_{pl} + m_{pr} + m_{in}} \]

- Parametric mass ratio
  \[ r = \lambda + \delta \]

\[ m_o = \text{initial mass} \]
\[ m_{pl} = \text{payload mass} \]
\[ m_{pr} = \text{propellant mass} \]
\[ m_{in} = \text{inert mass} \]
Rocket Equation (including Inert Mass)

Payload Fraction, \( \frac{M_{\text{payload}}}{M_{\text{initial}}} \)

Velocity Ratio, \( \frac{\Delta V}{V_e} \)

Typical Range for Launch to Low Earth Orbit

Inert Mass Fraction \( \delta \)
- 0
- 0.05
- 0.1
- 0.15
- 0.2

Rocket Performance
ENAE 483/788D - Principles of Space Systems Design
Limiting Performance Based on Inert

Asymptotic Velocity Ratio, $(\Delta V/V_e)$ vs. Inert Mass Fraction, $(M_{\text{inert}}/M_{\text{initial}})$

Typical Feasible Design Range for Inert Mass Fraction
## Regression Analysis of Existing Vehicles

<table>
<thead>
<tr>
<th>Veh/Stage</th>
<th>prop mass (lbs)</th>
<th>gross mass (lbs)</th>
<th>Type</th>
<th>Propellants</th>
<th>lsp vac (sec)</th>
<th>isp sl (sec)</th>
<th>sigma</th>
<th>eps</th>
<th>delta</th>
</tr>
</thead>
<tbody>
<tr>
<td>Delta 6925 Stage 2</td>
<td>13,367</td>
<td>15,394</td>
<td>Storable</td>
<td>N2O4-A50</td>
<td>319.4</td>
<td>0.152</td>
<td>0.132</td>
<td>0.070</td>
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<td>0.152</td>
<td>0.132</td>
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<tr>
<td>Titan II Stage 2</td>
<td>59,000</td>
<td>65,000</td>
<td>Storable</td>
<td>N2O4-A50</td>
<td>316.0</td>
<td>0.102</td>
<td>0.092</td>
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<tr>
<td>Titan III Stage 2</td>
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<td>83,600</td>
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<tr>
<td>Titan IV Stage 2</td>
<td>77,200</td>
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<td>N2O4-A50</td>
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<td>Proton Stage 3</td>
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<td>N2O4-A50</td>
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<tr>
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<td>Titan III Stage 1</td>
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<td>Storable</td>
<td>N2O4-A50</td>
<td>302.0</td>
<td>0.054</td>
<td>0.052</td>
<td>0.038</td>
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<td>Titan IV Stage 1</td>
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<td>0.056</td>
<td>0.053</td>
<td>0.039</td>
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<tr>
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<td>312.2</td>
<td>285.0</td>
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<td>0.089</td>
<td>0.061</td>
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<tr>
<td><strong>standard deviation</strong></td>
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<td></td>
<td></td>
<td></td>
<td>8.1</td>
<td>0.039</td>
<td>0.033</td>
<td>0.019</td>
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</tr>
</tbody>
</table>
Inert Mass Fractions for Existing LVs

![Graph showing inert mass fractions vs. gross mass for different fuel types: LOX/LH2, LOX/RP-1, Solid, and Storable.](image)

- **LOX/LH2**
- **LOX/RP-1**
- **Solid**
- **Storable**

**Inert Mass Fraction** vs. **Gross Mass (MT)**

**Legend:**
- **LOX/LH2**
- **LOX/RP-1**
- **Solid**
- **Storable**
Regression Analysis

• Given a set of \( N \) data points \((x_i, y_i)\)

• Linear curve fit: \( y = Ax + B \)
  – find \( A \) and \( B \) to minimize sum squared error

\[
\text{error} = \sum_{i=1}^{N} (Ax_i + B - y_i)^2
\]

  – Analytical solutions exist, or use Solver in Excel

• Power law fit: \( y = Bx^A \)

\[
\text{error} = \sum_{i=1}^{N} [A \log(x_i) + B - \log(y_i)]^2
\]

• Polynomial, exponential, many other fits possible
Solution of Least-Squares Linear

\[
\frac{\partial (\text{error})}{\partial A} = 2 \sum_{i=1}^{N} (Ax_i + B - y_i)x_i = 0
\]

\[
\frac{\partial (\text{error})}{\partial B} = 2 \sum_{i=1}^{N} (Ax_i + B - y_i) = 0
\]

\[
A \sum_{i=1}^{N} x_i^2 + B \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} x_i y_i = 0
\]

\[
B = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{N \sum x_i^2 - (\sum x_i)^2}
\]

\[
A = \frac{N \sum x_i y_i - \sum x_i \sum y_i}{N \sum x_i^2 - (\sum x_i)^2}
\]
Regression Analysis - Storables

\[ R^2 = 0.0541 \]
## Regression Values for Design Parameters

<table>
<thead>
<tr>
<th></th>
<th>Vacuum $V_e$ $\text{(m/sec)}$</th>
<th>Inert Mass Fraction</th>
<th>Max $\Delta V$ $\text{(m/sec)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LOX/LH2</td>
<td>4273</td>
<td>0.075</td>
<td>11,070</td>
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<tr>
<td>LOX/RP-1</td>
<td>3136</td>
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<td>Storables</td>
<td>3058</td>
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<tr>
<td>Solids</td>
<td>2773</td>
<td>0.087</td>
<td>6783</td>
</tr>
</tbody>
</table>
The Rocket Equation for Multiple Stages

- Assume two stages

\[ \Delta V_1 = -V_{e1} \ln \left( \frac{m_{\text{final}1}}{m_{\text{initial}1}} \right) = -V_{e1} \ln(r_1) \]

\[ \Delta V_2 = -V_{e2} \ln \left( \frac{m_{\text{final}2}}{m_{\text{initial}2}} \right) = -V_{e2} \ln(r_2) \]

- Assume \( V_{e1} = V_{e2} = V_e \)

\[ \Delta V_1 + \Delta V_2 = -V_e \ln(r_1) - V_e \ln(r_2) = -V_e \ln(r_1 r_2) \]
Continued Look at Multistaging

• There’s a historical tendency to define $r_0 = r_1 r_2$

$$\Delta V_1 + \Delta V_2 = -V_e \ln (r_1 r_2) = -V_e \ln (r_0)$$

• But it’s important to remember that it’s really

$$\Delta V_1 + \Delta V_2 = -V_e \ln (r_1 r_2) = -V_e \ln \left( \frac{m_{\text{final}1}}{m_{\text{initial}1}} \frac{m_{\text{final}2}}{m_{\text{initial}2}} \right)$$

• And that $r_0$ has no physical significance, since

$$m_{\text{final}1} \neq m_{\text{initial}2} \Rightarrow r_0 \neq \frac{m_{\text{final}2}}{m_{\text{initial}1}}$$
Multistage Inert Mass Fraction

• Total inert mass fraction
\[
\delta_0 = \frac{m_{in,1} + m_{in,2} + m_{in,3}}{m_0} = \frac{m_{in,1}}{m_0} + \frac{m_{in,2}}{m_0} + \frac{m_{in,3}}{m_0}
\]
\[
\delta_0 = \frac{m_{in,1}}{m_0} + \frac{m_{in,2}}{m_0} \frac{m_{0,2}}{m_0} + \frac{m_{in,3}}{m_0} \frac{m_{0,3}}{m_{0,2}} \frac{m_{0,2}}{m_0}
\]

• Convert to dimensionless parameters
\[
\delta_0 = \delta_1 + \delta_2 \lambda_1 + \delta_3 \lambda_2 \lambda_1
\]

• General form of the equation
\[
\delta_0 = \sum_{j=1}^{n \text{ stages}} \left( \delta_j \prod_{\ell=1}^{j-1} \lambda_\ell \right)
\]
Multistage Payload Fraction

• Total payload fraction (3 stage example)

\[ \lambda_0 = \frac{m_{pl}}{m_0} = \frac{m_{pl}}{m_{0,3}} \frac{m_{0,3}}{m_{0,2}} \frac{m_{0,2}}{m_0} \]

• Convert to dimensionless parameters

\[ \lambda_0 = \lambda_3 \lambda_2 \lambda_1 \]

• Generic form of the equation

\[ \lambda_0 = \prod_{j=1}^{n \text{ stages}} \lambda_j \]
Effect of Staging

Inert Mass Fraction $\delta = 0.15$

Payload Fraction vs. Velocity Ratio ($\Delta V/V_e$)

1 stage
2 stage
3 stage
4 stage

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Rocket Performance

ENAE 483/788D - Principles of Space Systems Design
Effect of $\Delta V$ Distribution

Stage 2

Normalized Mass (kg/kg of payload)

- Total mass
- Stage 2 mass
- Stage 1 mass

1st Stage: Solids  2nd Stage: LOX/LH2

Stage 2 $\Delta V$ (m/sec)
ΔV Distribution and Design Parameters

1st Stage: Solids  2nd Stage: LOX/LH2

- delta_0
- lambda_0
- lambda/delta

Stage 2 ΔV (m/sec)
Lagrange Multipliers

• Given an objective function
  \[ y = f(x) \]
  subject to constraint function
  \[ z = g(x) \]

• Create a new objective function
  \[ z = f(x) + \lambda[g(x) - z] \]

• Solve simultaneous equations
  \[ \frac{\partial y}{\partial x} = 0 \quad \frac{\partial y}{\partial \lambda} = 0 \]
Optimum $\Delta V$ Distribution Between Stages

- Maximize payload fraction (2 stage case)
  $$\lambda_0 = \lambda_1 \lambda_2 = (r_1 - \delta_1)(r_2 - \delta_2)$$
  subject to constraint function
  $$\Delta V_{total} = \Delta V_1 + \Delta V_2$$

- Create a new objective function
  $$\lambda_o = \left( e^{-\Delta V_1 / V_{e,1}} - \delta_1 \right) \left( e^{-\Delta V_2 / V_{e,2}} - \delta_2 \right) + K \left[ \Delta V_1 + \Delta V_2 - \Delta V_{total} \right]$$

$\Rightarrow$ Very messy for partial derivatives!
Optimum $\Delta V$ Distribution (continued)

- Use substitute objective function
  
  $\max (\lambda_o) \iff \max [\ln (\lambda_o)]$

- Create a new constrained objective function
  
  $\ln (\lambda_o) = \ln (r_1 - \delta_1) + \ln (r_2 - \delta_2) + K [\Delta V_1 + \Delta V_2 - \Delta V_{total}]$

- Take partials and set equal to zero
  
  $\frac{\partial [\ln (\lambda_o)]}{\partial r_1} = 0$  
  $\frac{\partial [\ln (\lambda_o)]}{\partial r_2} = 0$  
  $\frac{\partial [\ln (\lambda_o)]}{\partial K} = 0$
Optimum $\Delta V$ Special Cases

- “Generic” partial of objective function
  \[
  \frac{\partial [\ln (\lambda_o)]}{\partial r_i} = \frac{1}{r_i - \delta_i} + K \frac{V_{e,i}}{r_i} = 0
  \]

- Case 1: $\delta_1 = \delta_2 \quad V_{e,1} = V_{e,2}$
  \[
  r_1 = r_2 \implies \Delta V_1 = \Delta V_2 = \frac{\Delta V_{total}}{2}
  \]

- Same principle holds for $n$ stages
  \[
  r_1 = r_2 = \cdots = r_n \implies \Delta V_1 = \Delta V_2 = \cdots = \Delta V_n = \frac{\Delta V_{total}}{n}
  \]
Sensitivity to Inert Mass

\[ \Delta V \text{ for multistaged rocket} \]

\[ \Delta V_{tot} = \sum_{k=1}^{n \text{ stages}} \Delta V_k = \sum_{k=1}^{n} V_{e,k} \ln \left( \frac{m_{o,k}}{m_{f,k}} \right) \]

where

\[ m_{o,k} = m_{pl} + m_{pr,k} + m_{in,k} + \sum_{j=k+1}^{n} (m_{pr,j} + m_{in,j}) \]

\[ m_{f,k} = m_{pl} + m_{in,k} + \sum_{j=k+1}^{n} (m_{pr,j} + m_{in,j}) \]
Finding Payload Sensitivity to Inert Mass

- Given the equation linking mass to $\Delta V$, take

$$\frac{\partial (\Delta V_{tot})}{\partial m_{pl}} dm_{pl} + \frac{\partial (\Delta V_{tot})}{\partial m_{in,j}} dm_{in,j} = 0$$

and solve to find

$$\left. \frac{\partial m_{pl}}{\partial m_{in,k}} \right|_{\partial (\Delta V_{tot})=0} = -\sum_{j=1}^{k} V_{e,j} \left( \frac{1}{m_{o,j}} - \frac{1}{m_{f,j}} \right) \frac{1}{\sum_{\ell=1}^{N} V_{e,\ell} \left( \frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}} \right)}$$

- This equation shows the "trade-off ratio" - $\Delta$payload resulting from a change in inert mass for stage $k$ (for a vehicle with $N$ total stages)
# Trade-off Ratio Example: Gemini-Titan II

<table>
<thead>
<tr>
<th></th>
<th>Stage 1</th>
<th>Stage 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Mass (kg)</td>
<td>150,500</td>
<td>32,630</td>
</tr>
<tr>
<td>Final Mass (kg)</td>
<td>39,370</td>
<td>6099</td>
</tr>
<tr>
<td>Ve (m/sec)</td>
<td>2900</td>
<td>3097</td>
</tr>
<tr>
<td>$\frac{dm_{pl}}{dm_{in,k}}$</td>
<td>-0.1164</td>
<td>-1</td>
</tr>
</tbody>
</table>
Payload Sensitivity to Propellant Mass

• In a similar manner, solve to find

\[
\frac{\partial m_{pl}}{\partial m_{pr,k}} \bigg|_{\partial(\Delta V_{tot})=0} = \frac{-\sum_{j=1}^{k} V_{e,j} \left( \frac{1}{m_{o,j}} \right)}{\sum_{\ell=1}^{N} V_{e,\ell} \left( \frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}} \right)}
\]

• This equation gives the change in payload mass as a function of additional propellant mass (all other parameters held constant)
## Trade-off Ratio Example: Gemini-Titan II

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<td>-0.1164</td>
<td>-1</td>
</tr>
<tr>
<td>$\frac{dm_{pl}}{dm_{pr,k}}$</td>
<td>0.04124</td>
<td>0.2443</td>
</tr>
</tbody>
</table>
Payload Sensitivity to Exhaust Velocity

- Use the same technique to find the change in payload resulting from additional exhaust velocity for stage k

\[
\left. \frac{\partial m_{pl}}{\partial V_{e,k}} \right|_{\partial(\Delta V_{tot})=0} = \frac{\sum_{j=1}^{k} \ln \left( \frac{m_{o,j}}{m_{f,j}} \right)}{\sum_{\ell=1}^{N} V_{e,\ell} \left( \frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}} \right)}
\]

- This trade-off ratio (unlike the ones for inert and propellant masses) has units - kg/(m/sec)
### Trade-off Ratio Example: Gemini-Titan II

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<td>$dm_{pl}/dm_{in,k}$</td>
<td>-0.1164</td>
<td>-1</td>
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<tr>
<td>$dm_{pl}/dm_{pr,k}$</td>
<td>0.04124</td>
<td>0.2443</td>
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<tr>
<td>$dm_{pl}/dV_{e,k}$ (kg/m/sec)</td>
<td>2.870</td>
<td>6.459</td>
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</table>
Parallel Staging

- Multiple dissimilar engines burning simultaneously
- Frequently a result of upgrades to operational systems
- General case requires “brute force” numerical performance analysis
Parallel-Staging Rocket Equation

- Momentum at time $t$:
  \[ M = mv \]

- Momentum at time $t + \Delta t$:
  (subscript “b”=boosters; “c”=core vehicle)
  \[ M = (m - \Delta m_b - \Delta m_c)(v + \Delta v) \]
  \[ + \Delta m_b (v - V_{e,b}) + \Delta m_c (v - V_{e,c}) \]

- Assume thrust (and mass flow rates) constant
Parallel-Staging Rocket Equation

- Rocket equation during booster burn

\[ \Delta V = -\bar{V}_e \ln \left( \frac{m_{\text{final}}}{m_{\text{initial}}} \right) = -\bar{V}_e \ln \left( \frac{\frac{m_{\text{in, b}} + m_{\text{in, c}} + \chi m_{\text{pr, c}} + m_0}{2}}{m_{\text{in, b}} + m_{\text{pr, b}} + m_{\text{in, c}} + m_{\text{pr, c}} + m_0} \right) \]

where \( \chi \) = fraction of core propellant remaining after booster burnout, and where

\[ \bar{V}_e = \frac{V_{e, b} \dot{m}_b + V_{e, c} \dot{m}_c}{\dot{m}_b + \dot{m}_c} = \frac{V_{e, b} m_{\text{pr, b}} + V_{e, c} (1-\chi) m_{\text{pr, c}}}{m_{\text{pr, b}} + (1-\chi) m_{\text{pr, c}}} \]
Analyzing Parallel-Staging Performance

Parallel stages break down into pseudo-serial stages:

- **Stage “0”** (boosters and core)

  \[
  \Delta V_0 = -\bar{V}_e \ln \left( \frac{m_{in,b} + m_{in,c} + \chi m_{pr,c} + m_{0,2}}{m_{in,b} + m_{pr,b} + m_{in,c} + m_{pr,c} + m_{0,2}} \right)
  \]

- **Stage “1”** (core alone)

  \[
  \Delta V_1 = -V_{e,c} \ln \left( \frac{m_{in,c} + m_{0,2}}{m_{in,c} + \chi m_{pr,c} + m_{0,2}} \right)
  \]

- **Subsequent stages** are as before
Parallel Staging Example: Space Shuttle

• 2 x solid rocket boosters (data below for single SRB)
  – Gross mass 589,670 kg
  – Empty mass 86,183 kg
  – Isp 269 sec
  – Burn time 124 sec

• External tank (space shuttle main engines)
  – Gross mass 750,975 kg
  – Empty mass 29,930 kg
  – Isp 455 sec
  – Burn time 480 sec

• “Payload” (orbiter + P/L) 125,000 kg
Shuttle Parallel Staging Example

\[ V_{e,b} = g I_{sp,e} = (9.8)(269) = 2636 \text{ m/sec} \]

\[ V_{e,c} = 4459 \text{ m/sec} \]

\[ \chi = \frac{480 - 124}{480} = 0.7417 \]

\[ \bar{V}_e = \frac{2636(1,007,000) + 4459(721,000)(1 - .7417)}{1,007,000 + 721,000(1 - .7417)} = 2921 \text{ m/sec} \]

\[ \Delta V_0 = -2921 \ln \frac{862,000}{3,062,000} = 3702 \text{ m/sec} \]

\[ \Delta V_1 = -4459 \ln \frac{154,900}{689,700} = 6659 \text{ m/sec} \]

\[ \Delta V_{tot} = 10,360 \text{ m/sec} \]
Modular Staging

- Use identical modules to form multiple stages
- Have to cluster modules on lower stages to make up for nonideal $\Delta V$ distributions
- Advantageous from production and development cost standpoints
Module Analysis

• All modules have the same inert mass and propellant mass
• Because $\delta$ varies with payload mass, not all modules have the same $\delta$!
• Introduce two new parameters

$$\varepsilon \equiv \frac{m_{in}}{m_{in} + m_{pr}} = \frac{m_{in}}{m_{mod}}$$

$$\sigma \equiv \frac{m_{in}}{m_{pr}}$$

• Conversions

$$\varepsilon = \frac{\delta}{1 - \lambda}$$

$$\sigma = \frac{\delta}{1 - \delta - \lambda}$$
Rocket Equation for Modular Boosters

• Assuming \( n \) modules in stage 1,

\[
\frac{r_1}{n} = \frac{n(m_{in}) + m_{o2}}{n(m_{in} + m_{pr}) + m_{o2}} = \frac{n\varepsilon + \frac{m_{o2}}{m_{mod}}}{n + \frac{m_{o2}}{m_{mod}}}
\]

• If all 3 stages use same modules, \( n_j \) for stage \( j \),

\[
r_1 = \frac{n_1\varepsilon + n_2 + n_3 + \rho_{pl}}{n_1 + n_2 + n_3 + \rho_{pl}}
\]

where

\[
\rho_{pl} \equiv \frac{m_{pl}}{m_{mod}}; \quad m_{mod} = m_{in} + m_{pr}
\]
Example: Conestoga 1620 (EER)

• Small launch vehicle (1 flight, 1 failure)
• Payload 900 kg
• Module gross mass 11,400 kg
• Module empty mass 1,400 kg
• Exhaust velocity 2754 m/sec
• Staging pattern
  – 1st stage - 4 modules
  – 2nd stage - 2 modules
  – 3rd stage - 1 module
  – 4th stage - Star 48V (gross mass 2200 kg, empty mass 140 kg, $V_e$ 2842 m/sec)
Conestoga 1620 Performance

• 4th stage $\Delta V$

$$\Delta V_4 = -V_{e4} \ln \frac{m_{f4}}{m_{o4}} = -2842 \ln \frac{900 + 140}{900 + 2200} = 3104 \text{ m/sec}$$

• Treat like three-stage modular vehicle; $M_{pl}$=3100 kg

$$\epsilon = \frac{m_{in}}{m_{mod}} = \frac{1400}{11400} = 0.1228$$

$$\rho_{pl} = \frac{m_{pl}}{m_{mod}} = \frac{3100}{11400} = 0.2719$$

$$n_1 = 4; \quad n_2 = 2; \quad n_3 = 1$$
\[ r_1 = \frac{n_1 \epsilon + n_2 + n_3 + \rho_{pl}}{n_1 + n_2 + n_3 + \rho_{pl}} = \frac{4 \times 0.1228 + 2 + 1 + 0.2719}{4 + 2 + 1 + 0.2719} = 0.5175 \]
\[ r_2 = \frac{n_2 \epsilon + n_3 + \rho_{pl}}{n_2 + n_3 + \rho_{pl}} = \frac{2 \times 0.1228 + 1 + 0.2719}{2 + 1 + 0.2719} = 0.4638 \]
\[ r_3 = \frac{n_3 \epsilon + \rho_{pl}}{n_3 + \rho_{pl}} = \frac{1 \times 0.1228 + 0.2719}{1 + 0.2719} = 0.3103 \]

\[ V_1 = 1814 \ \text{m/s}; \quad V_2 = 2116 \ \text{m/s}; \quad V_3 = 3223 \ \text{m/s}; \quad V_4 = 3104 \ \text{m/s} \]

\[ V_{total} = 10,257 \ \text{m/s} \]
Discussion about Modular Vehicles

• Modularity has several advantages
  – Saves money (smaller modules cost less to develop)
  – Saves money (larger production run = lower cost/module)
  – Allows resizing launch vehicles to match payloads

• Trick is to optimize number of stages, number of modules/stage to minimize total number of modules

• Generally close to optimum by doubling number of modules at each lower stage

• Have to worry about packing factors, complexity