Rocket Performance

- The rest of orbital mechanics
- The rocket equation
- Mass ratio and performance
- Structural and payload mass fractions
- Regression analysis
- Multistaging
- Optimal $\Delta V$ distribution between stages
- Trade-off ratios
- Parallel staging
- Modular staging
Classical Parameters of Elliptical Orbits
Basic Orbital Parameters

- Semi-latus rectum (or parameter)
  \[ p = a(1 - e^2) \]

- Radial distance as function of orbital position
  \[ r = \frac{p}{1 + e \cos \theta} \]

- Periapse and apoapse distances
  \[ r_p = a(1 - e) \quad r_a = a(1 + e) \]

- Angular momentum
  \[ \vec{h} = \vec{r} \times \vec{v} \quad h = \sqrt{\mu p} \]
The Classical Orbital Elements

Ref: J. E. Prussing and B. A. Conway, Orbital Mechanics Oxford University Press, 1993
The Hohmann Transfer
First Maneuver Velocities

- Initial vehicle velocity
  \[ v_1 = \sqrt{\frac{\mu}{r_1}} \]

- Needed final velocity
  \[ v_{\text{perigee}} = \sqrt{\frac{\mu}{r_1}} \sqrt{\frac{2r_2}{r_1 + r_2}} \]

- Required \( \Delta V \)
  \[ \Delta v_1 = \sqrt{\frac{\mu}{r_1}} \left( \sqrt{\frac{2r_2}{r_1 + r_2}} - 1 \right) \]
Second Maneuver Velocities

- Initial vehicle velocity
  \[ v_{\text{apogee}} = \sqrt{\frac{\mu}{r_2}} \sqrt{\frac{2r_1}{r_1 + r_2}} \]

- Needed final velocity
  \[ v_2 = \sqrt{\frac{\mu}{r_2}} \]

- Required \( \Delta V \)
  \[ \Delta v_2 = \sqrt{\frac{\mu}{r_2}} \left( 1 - \sqrt{\frac{2r_1}{r_1 + r_2}} \right) \]
Limitations on Launch Inclinations
Simple Plane Change

$v_{perigee}$

$v_1$

$v_{apogee}$

$\Delta v_2$

$v_2$
Optimal Plane Change

\[ v_{\text{perigee}} \quad \Delta v_1 \quad v_1 \quad \Delta v_2 \quad v_{\text{apogee}} \quad v_2 \]
First Maneuver with Plane Change $\Delta i_1$

- Initial vehicle velocity
  \[ v_1 = \sqrt{\frac{\mu}{r_1}} \]

- Needed final velocity
  \[ v_p = \sqrt{\frac{\mu}{r_1}} \sqrt{\frac{2r_2}{r_1 + r_2}} \]

- Required $\Delta V$
  \[ \Delta v_1 = \sqrt{v_1^2 + v_p^2 - 2v_1 v_p \cos \Delta i_1} \]
Second Maneuver with Plane Change $\Delta i_2$

- Initial vehicle velocity
  \[ v_a = \sqrt{\frac{\mu}{r_2}} \sqrt{\frac{2r_1}{r_1 + r_2}} \]

- Needed final velocity
  \[ v_2 = \sqrt{\frac{\mu}{r_2}} \]

- Required $\Delta V$
  \[ \Delta v_2 = \sqrt{v_2^2 + v_a^2 - 2v_2 v_a \cos \Delta i_2} \]
Sample Plane Change Maneuver

Optimum initial plane change = 2.20°
Calculating Time in Orbit
Time in Orbit

- Period of an orbit

\[ P = 2\pi \sqrt{\frac{a^3}{\mu}} \]

- Mean motion (average angular velocity)

\[ n = \sqrt{\frac{\mu}{a^3}} \]

- Time since pericenter passage

\[ M = nt = E - e \sin E \]

\( \Rightarrow M = \text{mean anomaly} \)
Dealing with the Eccentric Anomaly

• Relationship to orbit

\[ r = a \left(1 - e \cos E\right) \]

• Relationship to true anomaly

\[ \tan \frac{\theta}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2} \]

• Calculating M from time interval: iterate

\[ E_{i+1} = nt + e \sin E_i \]

until it converges
Example: Time in Orbit

- Hohmann transfer from LEO to GEO
  - \( h_1 = 300 \text{ km} \Rightarrow r_1 = 6378 + 300 = 6678 \text{ km} \)
  - \( r_2 = 42240 \text{ km} \)

- Time of transit (1/2 orbital period)

\[
a = \frac{1}{2} (r_1 + r_2) = 24,459 \text{ km}
\]

\[
t_{\text{transit}} = \frac{P}{2} = \pi \sqrt{\frac{a^3}{\mu}} = 19,034 \text{ sec} = 5h17m14s
\]
Example: Time-based Position

Find the spacecraft position 3 hours after perigee

\[
\begin{align*}
n &= \sqrt{\frac{\mu}{a^3}} = 1.650 \times 10^{-4} \text{ rad sec}^{-1} \\
e &= 1 - \frac{r_p}{a} = 0.7270 \\
E_{j+1} &= nt + e \sin E_j = 1.783 + 0.7270 \sin E_j
\end{align*}
\]

E=0; 1.783; 2.494; 2.222; 2.361; 2.294; 2.328; 2.311; 2.320; 2.316; 2.318; 2.317; 2.317; 2.317
Example: Time-based Position (cont.)

\[ E = 2.317 \]

\[ r = a(1 - e \cos E) = 12,387 \text{ km} \]

\[ \tan \frac{\theta}{2} = \sqrt{\frac{1 + e}{1 - e} \tan \frac{E}{2}} \implies \theta = 160 \text{ deg} \]

Have to be sure to get the position in the proper quadrant - since the time is less than 1/2 the period, the spacecraft has yet to reach apogee \( --> 0^\circ < \theta < 180^\circ \)
Velocity Components in Orbit

\[ r = \frac{p}{1 + e \cos \theta} \]

\[ v_r = \frac{dr}{dt} = \frac{d}{dt} \left( \frac{p}{1 + e \cos \theta} \right) = \frac{-p(-e \sin \theta \frac{d\theta}{dt})}{(1 + e \cos \theta)^2} \]

\[ v_r = \frac{pe \sin \theta}{(1 + e \cos \theta)^2} \frac{d\theta}{dt} \]

\[ 1 + e \cos \theta = \frac{p}{r} \Rightarrow v_r = \frac{r^2 \frac{d\theta}{dt} e \sin \theta}{p} \]

\[ \vec{h} = \vec{r} \times \vec{v} \]
Velocity Components in Orbit (cont.)

\[ \vec{h} = \vec{r} \times \vec{v} \]

\[ h = rv \cos \gamma = r \left( r \frac{d\theta}{dt} \right) = r^2 \frac{d\theta}{dt} \]

\[ v_r = \frac{r^2 \frac{d\theta}{dt} e \sin \theta}{p} = \frac{h e \sin \theta}{p} = \frac{\sqrt{p\mu}}{p} e \sin \theta \]

\[ v_r = \sqrt{\frac{\mu}{p}} e \sin \theta \]

\[ v_\theta = r \frac{d\theta}{dt} = r \frac{h}{r^2} = \frac{h}{r} = \frac{\sqrt{p\mu}}{r} \]

\[ v_\theta = \sqrt{\frac{\mu}{p}} (1 + e \cos \theta) \]

\[ \tan \gamma = \frac{v_r}{v_\theta} = \frac{e \sin \theta}{1 + e \cos \theta} \]
Earth-Moon Potential Field
The Earth-Moon System

Note: Earth and Moon are in scale with size of orbits

Photograph of Earth and Moon taken by Mars Odyssey April 19, 2001 from a distance of 3,564,000 km
The Earth-Moon System

Note: Earth and Moon are in scale with size of orbits

Photograph of Earth and Moon taken by Mars Odyssey April 19, 2001 from a distance of 3,564,000 km
Free-Body Diagram with Spherical Planet

\[ \begin{align*}
\text{Vector } r & \quad \text{representing the distance from the center of the planet.} \\
\text{Vector } g & \quad \text{representing the gravitational force.} \\
\text{Vector } \theta & \quad \text{representing the angle.} \\
\text{Vector } v & \quad \text{representing the velocity.} \\
\text{Vector } \omega & \quad \text{representing the angular velocity.} \\
\text{Vector } \gamma & \quad \text{representing another vector.}
\end{align*} \]
Orbital Planar State Equations

Inertial angular velocity

\[ \omega = \dot{\gamma} - \dot{\theta} \]

Sum of accelerations normal to velocity vector

\[ -g \cos \gamma = \omega v \]

Sum of accelerations perpendicular to velocity vector

\[ -g \sin \gamma = \ddot{v} \]
Orbital Planar State Equations (2)

\[
\begin{align*}
\dot{r} &= v \sin \gamma \\
r \dot{\theta} &= v \cos \gamma \\
\omega &= \dot{\gamma} - \dot{\theta} = \dot{\gamma} - \frac{v}{r} \cos \gamma \\
-g \cos \gamma &= \left( \dot{\gamma} - \frac{v}{r} \cos \gamma \right) v \\
- \left( g - \frac{v^2}{r} \right) \cos \gamma &= \dot{\gamma} v \\
- \left( 1 - \frac{v^2}{r g} \right) g \cos \gamma &= \dot{\gamma} v
\end{align*}
\]
Canonical Orbital Planar State Equations

\[ \dot{\gamma} = -\frac{1}{v} \left( 1 - \frac{v^2}{v_c^2} \right) g \cos \gamma \]

\[ \dot{v} = -g \sin \gamma \]

\[ \dot{r} = v \sin \gamma \]

\[ \dot{\theta} = \frac{v}{r} \cos \gamma \]

Coupled first-order ODEs

\[ g = g_o \left( \frac{r_o}{r} \right)^2 \]
Numerical Integration - 4th Order R-K

Given a series of equations
\[
\dot{y} = \bar{f}(t, \bar{x})
\]
\[
\bar{k}_1 = \Delta t \ \bar{f} (t_n, y_n)
\]
\[
\bar{k}_2 = \Delta t \ \bar{f} \left( t_n + \frac{\Delta t}{2}, y_n + \frac{\bar{k}_1}{2} \right)
\]
\[
\bar{k}_3 = \Delta t \ \bar{f} \left( t_n + \frac{\Delta t}{2}, y_n + \frac{\bar{k}_2}{2} \right)
\]
\[
\bar{k}_4 = \Delta t \ \bar{f} \left( t_n + \Delta t, y_n + \bar{k}_3 \right)
\]
\[
y_{n+1} = y_n + \frac{\bar{k}_1}{6} + \frac{\bar{k}_2}{3} + \frac{\bar{k}_3}{3} + \frac{\bar{k}_4}{6} + O(\Delta t^5)
\]
Derivation of the Rocket Equation

- Momentum at time $t$:
  \[ M = mv \]

- Momentum at time $t + \Delta t$:
  \[ M = (m - \Delta m)(V + \Delta v) + \Delta m(v - V_e) \]

- Some algebraic manipulation gives:
  \[ m\Delta v = -\Delta mV_e \]

- Take to limits and integrate:
  \[
  \int_{m_{initial}}^{m_{final}} \frac{dm}{m} = -\int_{V_{initial}}^{V_{final}} \frac{dv}{V_e}
  \]
Derivation of the Rocket Equation

- Momentum at time $t$:
  \[ M = mv \]

- Momentum at time $t+\Delta t$:
  \[ M = (m - \Delta m)(V + \Delta v) + \Delta m(v - V_e) \]

- Some algebraic manipulation gives:
  \[ m\Delta v = -\Delta m V_e \]

- Take to limits and integrate:
  \[ \int_{m_{initial}}^{m_{final}} \frac{dm}{m} = - \int_{V_{initial}}^{V_{final}} \frac{dv}{V_e} \]
The Rocket Equation

- Alternate forms
  \[ r \equiv \frac{m_{\text{final}}}{m_{\text{initial}}} = e^{-\frac{\Delta V}{V_e}} \]

  \[ \Delta v = -V_e \ln \left( \frac{m_{\text{final}}}{m_{\text{initial}}} \right) = -V_e \ln r \]

- Basic definitions/concepts
  - Mass ratio
    \[ r \equiv \frac{m_{\text{final}}}{m_{\text{initial}}} \quad \text{or} \quad R \equiv \frac{m_{\text{initial}}}{m_{\text{final}}} \]
  - Nondimensional velocity change
    “Velocity ratio”
    \[ \nu \equiv \frac{\Delta V}{V_e} \]
Rocket Equation (First Look)

Typical Range for Launch to Low Earth Orbit