Rocket Performance

- The rest of orbital mechanics
- The rocket equation
- Mass ratio and performance
- Structural and payload mass fractions
- Regression analysis
- Multistaging
- Optimal ΔV distribution between stages
- Trade-off ratios
- Parallel staging
- Modular staging
Rocket Equation (First Look)

![Graph showing the rocket equation with mass ratio (M_{final}/M_{initial}) on the y-axis and velocity ratio (\Delta V/V_e) on the x-axis. The typical range for launch to Low Earth Orbit is highlighted.](image)
Sources and Categories of Vehicle Mass

Payload
Propellants
Structure
Propulsion
Avionics
Power
Mechanisms
Thermal
Etc.
Sources and Categories of Vehicle Mass

- Payload
- Propellants
- Inert Mass
- Structure
- Propulsion
- Avionics
- Power
- Mechanisms
- Thermal
- Etc.
Basic Vehicle Parameters

- Basic mass summary
  \[ m_o = m_{pl} + m_{pr} + m_{in} \]
- Inert mass fraction
  \[ \delta \equiv \frac{m_{in}}{m_o} = \frac{m_{in}}{m_{pl} + m_{pr} + m_{in}} \]
- Payload fraction
  \[ \lambda \equiv \frac{m_{pl}}{m_o} = \frac{m_{pl}}{m_{pl} + m_{pr} + m_{in}} \]
- Parametric mass ratio
  \[ r = \lambda + \delta \]

\( m_o = \) initial mass
\( m_{pl} = \) payload mass
\( m_{pr} = \) propellant mass
\( m_{in} = \) inert mass
Rocket Equation (including Inert Mass)

Payload Fraction, \( \left( \frac{M_{\text{payload}}}{M_{\text{initial}}} \right) \)

Velocity Ratio, \( (\Delta V/V_e) \)

Typical Range for Launch to Low Earth Orbit

Inert Mass Fraction \( \delta \)
- 0
- 0.05
- 0.1
- 0.15
- 0.2
Limiting Performance Based on Inert Mass

Asymptotic Velocity Ratio, \((\Delta V/V_e)\)

Inert Mass Fraction, \((M_{\text{inert}}/M_{\text{initial}})\)

\begin{align*}
0 & \quad 0.05 & \quad 0.1 & \quad 0.15 & \quad 0.2 & \quad 0.25 & \quad 0.3 & \quad 0.35 \\
0.5 & \quad 1 & \quad 1.5 & \quad 2 & \quad 2.5 & \quad 3 & \quad 3.5 & \quad 4 & \quad 4.5 & \quad 5
\end{align*}
Limiting Performance Based on Inert Mass

![Graph showing the relationship between Asymptotic Velocity Ratio, $\Delta V/V_e$, and Inert Mass Fraction, $M_{\text{inert}}/M_{\text{initial}}$. The graph illustrates the typical feasible design range for inert mass fraction.](image)
Limiting Performance Based on Inert Mass

Asymptotic Velocity Ratio, \( (\Delta V/V_e) \)

Inert Mass Fraction, \( (M_{\text{inert}}/M_{\text{initial}}) \)

Typical Feasible Design Range for Inert Mass Fraction
### Regression Analysis of Existing Vehicles

<table>
<thead>
<tr>
<th>Veh/Stage</th>
<th>prop mass (lbs)</th>
<th>gross mass (lbs)</th>
<th>Type</th>
<th>Propellants</th>
<th>isp vac (sec)</th>
<th>isp sl (sec)</th>
<th>sigma</th>
<th>eps</th>
<th>delta</th>
</tr>
</thead>
<tbody>
<tr>
<td>Delta 6925 Stage 2</td>
<td>13,367</td>
<td>15,394</td>
<td>Storable</td>
<td>N2O4-A50</td>
<td>319.4</td>
<td></td>
<td>0.152</td>
<td>0.132</td>
<td>0.070</td>
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<td>0.132</td>
<td>0.065</td>
</tr>
<tr>
<td>Titan II Stage 2</td>
<td>59,000</td>
<td>65,000</td>
<td>Storable</td>
<td>N2O4-A50</td>
<td>316.0</td>
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<td>0.102</td>
<td>0.092</td>
<td>0.087</td>
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<tr>
<td>Titan III Stage 2</td>
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<td>83,600</td>
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<td>N2O4-A50</td>
<td>316.0</td>
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<td>0.083</td>
<td>0.077</td>
<td>0.055</td>
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<tr>
<td>Titan IV Stage 2</td>
<td>77,200</td>
<td>87,000</td>
<td>Storable</td>
<td>N2O4-A50</td>
<td>316.0</td>
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<td>0.127</td>
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<td>0.078</td>
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<tr>
<td>Proton Stage 3</td>
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<td>315.0</td>
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<td>0.118</td>
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<td>0.078</td>
</tr>
<tr>
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<td>260,000</td>
<td>269,000</td>
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<td>N2O4-A50</td>
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<td></td>
<td>0.035</td>
<td>0.033</td>
<td>0.027</td>
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<tr>
<td>Titan III Stage 1</td>
<td>294,000</td>
<td>310,000</td>
<td>Storable</td>
<td>N2O4-A50</td>
<td>302.0</td>
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<td>0.054</td>
<td>0.052</td>
<td>0.038</td>
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<td>359,000</td>
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<td>N2O4-A50</td>
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<td>0.056</td>
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<td>N2O4-A50</td>
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<td>0.106</td>
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<td>0.066</td>
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<td>285.0</td>
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<td><strong>average</strong></td>
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<td><strong>285.0</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>0.100</strong></td>
<td><strong>0.089</strong></td>
<td><strong>0.061</strong></td>
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<tr>
<td><strong>standard deviation</strong></td>
<td><strong>8.1</strong></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td><strong>0.039</strong></td>
<td><strong>0.033</strong></td>
<td><strong>0.019</strong></td>
</tr>
</tbody>
</table>
Inert Mass Fraction Data for Existing LVs
Regression Analysis

- Given a set of N data points \((x_i, y_i)\)
- Linear curve fit: \(y = Ax + B\)
  - find A and B to minimize sum squared error
    \[
    \text{error} = \sum_{i=1}^{N} (Ax_i + B - y_i)^2
    \]
  - Analytical solutions exist, or use Solver in Excel
- Power law fit: \(y = Bx^A\)
  \[
  \text{error} = \sum_{i=1}^{N} [A \log(x_i) + B - \log(y_i)]^2
  \]
- Polynomial, exponential, many other fits possible
Solution of Least-Squares Linear Regression

\[ \frac{\partial (\text{error})}{\partial A} = 2 \sum_{i=1}^{N} (Ax_i + B - y_i)x_i = 0 \]

\[ \frac{\partial (\text{error})}{\partial B} = 2 \sum_{i=1}^{N} (Ax_i + B - y_i) = 0 \]

\[ A \sum_{i=1}^{N} x_i^2 + B \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} x_i y_i = 0 \]

\[ A \sum_{i=1}^{N} x_i + NB - \sum_{i=1}^{N} y_i = 0 \]

\[ A = \frac{N \sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{N \sum_{i=1}^{N} x_i^2 - (\sum_{i=1}^{N} x_i)^2} \]

\[ B = \frac{\sum_{i=1}^{N} y_i \sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i}{N \sum_{i=1}^{N} x_i^2 - (\sum_{i=1}^{N} x_i)^2} \]
Regression Analysis - Storables

![Graph showing the relationship between gross mass and inert mass fraction with an R² value of 0.0541.](image)

**Gross Mass (MT)**

**Inert Mass Fraction**

$R^2 = 0.0541$
## Regression Values for Design Parameters

<table>
<thead>
<tr>
<th></th>
<th>Vacuum Ve (m/sec)</th>
<th>Inert Mass Fraction</th>
<th>Max ΔV (m/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LOX/LH2</td>
<td>4273</td>
<td>0.075</td>
<td>11,070</td>
</tr>
<tr>
<td>LOX/RP-1</td>
<td>3136</td>
<td>0.063</td>
<td>8664</td>
</tr>
<tr>
<td>Storables</td>
<td>3058</td>
<td>0.061</td>
<td>8575</td>
</tr>
<tr>
<td>Solids</td>
<td>2773</td>
<td>0.087</td>
<td>6783</td>
</tr>
</tbody>
</table>
Economy of Scale for Stage Size

\[ \epsilon = \frac{M_{inert}}{M_{inert} + M_{prop}} \]

\[ PMF = 1 - \epsilon \]
Stage Inert Mass Fraction Estimation

$$\epsilon_{LOX/LH_2} = 0.987 \left( M_{stage} \langle kg \rangle \right)^{-0.183}$$

$$\epsilon_{storables} = 1.6062 \left( M_{stage} \langle kg \rangle \right)^{-0.275}$$
The Rocket Equation for Multiple Stages

- Assume two stages

\[ \Delta V_1 = -V_{e1} \ln \left( \frac{m_{final1}}{m_{initial1}} \right) = -V_{e1} \ln(r_1) \]

\[ \Delta V_2 = -V_{e2} \ln \left( \frac{m_{final2}}{m_{initial2}} \right) = -V_{e2} \ln(r_2) \]

- Assume \( V_{e1} = V_{e2} = V_e \)

\[ \Delta V_1 + \Delta V_2 = -V_e \ln(r_1) - V_e \ln(r_2) = -V_e \ln(r_1 r_2) \]
Continued Look at Multistaging

- There’s a historical tendency to define $r_0 = r_1 r_2$
  \[ \Delta V_1 + \Delta V_2 = -V_e \ln (r_1 r_2) = -V_e \ln (r_0) \]

- But it’s important to remember that it’s really
  \[ \Delta V_1 + \Delta V_2 = -V_e \ln (r_1 r_2) = -V_e \ln \left( \frac{m_{\text{final}1}}{m_{\text{initial}1}} \frac{m_{\text{final}2}}{m_{\text{initial}2}} \right) \]

- And that $r_0$ has no physical significance, since
  \[ m_{\text{final}1} \neq m_{\text{initial}2} \Rightarrow r_0 \neq \frac{m_{\text{final}2}}{m_{\text{initial}1}} \]
Multistage Inert Mass Fraction

- Total inert mass fraction

\[
\delta_0 = \frac{m_{in,1} + m_{in,2} + m_{in,3}}{m_0} = \frac{m_{in,1}}{m_0} + \frac{m_{in,2}}{m_0} + \frac{m_{in,3}}{m_0}
\]

\[
\delta_0 = \frac{m_{in,1}}{m_0} + \frac{m_{in,2}}{m_{0,2}} \frac{m_{0,2}}{m_0} + \frac{m_{in,3}}{m_{0,3}} \frac{m_{0,3}}{m_{0,2}} \frac{m_{0,2}}{m_0}
\]

- Convert to dimensionless parameters

\[
\delta_0 = \delta_1 + \delta_2 \lambda_1 + \delta_3 \lambda_2 \lambda_1
\]

- General form of the equation

\[
\delta_0 = \sum_{j=1}^{n \text{ stages}} \left[ \delta_j \prod_{\ell=1}^{j-1} \lambda_\ell \right]
\]
Multistage Payload Fraction

- Total payload fraction (3 stage example)
  \[ \lambda_0 = \frac{m_{pl}}{m_0} = \frac{m_{pl}}{m_{0,3}} \frac{m_{0,3}}{m_{0,2}} \frac{m_{0,2}}{m_0} \]

- Convert to dimensionless parameters
  \[ \lambda_0 = \lambda_3 \lambda_2 \lambda_1 \]

- Generic form of the equation
  \[ \lambda_0 = \prod_{j=1}^{n \text{ stages}} \lambda_j \]
Effect of $\delta$ and $\Delta V/V_e$ on Payload

\[
\frac{\Delta V}{V_e} = 0.25
\]
Effect of $\delta$ and $\Delta V/V_e$ on Payload

\[
\frac{\Delta V}{V_e} = 4.0
\]

Inert Mass Fraction vs. Total Payload Fraction for different stages:
- 1 stage
- 2 stages
- 3 stages
- 4 stages
Effect of Staging

Inert Mass Fraction $\delta = 0.15$

Payload Fraction

Velocity Ratio ($\Delta V/V_e$)

1 stage
2 stage
3 stage
4 stage

Single Stage
Two Stage
Three Stage
Four Stage
Trade Space for Number of Stages

The graph illustrates the velocity ratio $DV/Ve$ as a function of the inert mass fraction for different stages of a launch vehicle. The stages are categorized as Single Stage, Two Stage, Three Stage, and Four Stage. The graph shows that as the number of stages increases, the velocity ratio decreases for a given inert mass fraction.
Trade Space for Number of Stages

[Graph showing the relationship between Velocity Ratio (\(\Delta V/V_e\)) and Inert Mass Fraction for different stages and propellants: LOX/RP-1, LOX/LH2, Solids, and Storables. The graph illustrates how the velocity ratio decreases as the inert mass fraction increases for each stage configuration.]
Effect of $\Delta V$ Distribution

1st Stage: Solids  2nd Stage: LOX/LH2

Normalized Mass (kg/kg of payload)

Stage 2 $\Delta V$ (m/sec)

- Total mass
- Stage 2 mass
- Stage 1 mass
\[ \Delta V \text{ Distribution and Design Parameters} \]

![Graph showing \( \Delta V \) distribution for 1st Stage: Solids and 2nd Stage: LOX/LH2.](image)
Lagrange Multipliers

- Given an objective function
  \[ y = f(x) \]
  subject to constraint function
  \[ z = g(x) \]
- Create a new objective function
  \[ z = f(x) + \lambda[g(x) - z] \]
- Solve simultaneous equations
  \[ \frac{\partial y}{\partial x} = 0 \quad \frac{\partial y}{\partial \lambda} = 0 \]
Optimum $\Delta V$ Distribution Between Stages

- Maximize payload fraction (2 stage case)
  \[
  \lambda_0 = \lambda_1 \lambda_2 = (r_1 - \delta_1)(r_2 - \delta_2)
  \]
  subject to constraint function
  \[
  \Delta V_{total} = \Delta V_1 + \Delta V_2
  \]
- Create a new objective function
  
  \[
  \lambda_o = \left( e^{-\frac{\Delta V_1}{V_{e,1}}} - \delta_1 \right) \left( e^{-\frac{\Delta V_2}{V_{e,2}}} - \delta_2 \right) + K \left[ \Delta V_1 + \Delta V_2 - \Delta V_{total} \right]
  \]
  ➡️Very messy for partial derivatives!
Optimum $\Delta V$ Distribution (continued)

- Use substitute objective function

$$\max (\lambda_o) \iff \max [\ln (\lambda_o)]$$

- Create a new constrained objective function

$$\ln (\lambda_o) = \ln (r_1 - \delta_1) + \ln (r_2 - \delta_2) + K [\Delta V_1 + \Delta V_2 - \Delta V_{total}]$$

- Take partials and set equal to zero

$$\frac{\partial [\ln (\lambda_o)]}{\partial r_1} = 0 \quad \frac{\partial [\ln (\lambda_o)]}{\partial r_2} = 0 \quad \frac{\partial [\ln (\lambda_o)]}{\partial K} = 0$$
Optimum $\Delta V$ Special Cases

- “Generic” partial of objective function

$$\frac{\partial [\ln (\lambda_o)]}{\partial r_i} = \frac{1}{r_i - \delta_i} + K \frac{V_{e,i}}{r_i} = 0$$

- Special case: $\delta_1 = \delta_2$ $V_{e,1} = V_{e,2}$

$$r_1 = r_2 \implies \Delta V_1 = \Delta V_2 = \frac{\Delta V_{total}}{2}$$

- Same principle holds for $n$ stages

$$r_1 = r_2 = \cdots = r_n \implies$$

$$\Delta V_1 = \Delta V_2 = \cdots = \Delta V_n = \frac{\Delta V_{total}}{n}$$
Sensitivity to Inert Mass

\[ \Delta V \text{ for multistaged rocket} \]

\[ \Delta V_{\text{tot}} = \sum_{k=1}^{n \text{ stages}} \Delta V_k = \sum_{k=1}^{n} V_{e,k} \ln \left( \frac{m_{o,k}}{m_{f,k}} \right) \]

where

\[ m_{o,k} = m_{pl} + m_{pr,k} + m_{in,k} + \sum_{j=k+1}^{n} (m_{pr,j} + m_{in,j}) \]

\[ m_{f,k} = m_{pl} + m_{in,k} + \sum_{j=k+1}^{n} (m_{pr,j} + m_{in,j}) \]
Finding Payload Sensitivity to Inert Mass

- Given the equation linking mass to $\Delta V$, take

$$\frac{\partial (\Delta V_{tot})}{\partial m_{pl}} dm_{pl} + \frac{\partial (\Delta V_{tot})}{\partial m_{in,j}} dm_{in,j} = 0$$

and solve to find

$$\left. \frac{dm_{pl}}{dm_{in,k}} \right|_{\partial (\Delta V_{tot})=0} = - \sum_{j=1}^{k} V_{e,j} \left( \frac{1}{m_{o,j}} - \frac{1}{m_{f,j}} \right) \frac{1}{\sum_{\ell=1}^{N} V_{e,\ell} \left( \frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}} \right)}$$

- This equation shows the “trade-off ratio” - $\Delta$payload resulting from a change in inert mass for stage k (for a vehicle with N total stages)
### Trade-off Ratio Example: Gemini-Titan II

<table>
<thead>
<tr>
<th></th>
<th>Stage 1</th>
<th>Stage 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Initial Mass (kg)</strong></td>
<td>150,500</td>
<td>32,630</td>
</tr>
<tr>
<td><strong>Final Mass (kg)</strong></td>
<td>39,370</td>
<td>6099</td>
</tr>
<tr>
<td><strong>$Ve$ (m/sec)</strong></td>
<td>2900</td>
<td>3097</td>
</tr>
<tr>
<td><strong>$dm_{pl}/dm_{in,k}$</strong></td>
<td>-0.1164</td>
<td>-1</td>
</tr>
</tbody>
</table>
Payload Sensitivity to Propellant Mass

- In a similar manner, solve to find

\[
\left. \frac{dm_{pl}}{dm_{pr,k}} \right|_{\partial(\Delta V_{tot})=0} = \frac{- \sum_{j=1}^{k} V_{e,j} \left( \frac{1}{m_{o,j}} \right)}{\sum_{\ell=1}^{N} V_{e,\ell} \left( \frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}} \right)}
\]

- This equation gives the change in payload mass as a function of additional propellant mass (all other parameters held constant)
# Trade-off Ratio Example: Gemini-Titan II

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<td>3097</td>
</tr>
<tr>
<td>$\frac{dm_{pl}}{dm_{in,k}}$</td>
<td>-0.1164</td>
<td>-1</td>
</tr>
<tr>
<td>$\frac{dm_{pl}}{dm_{pr,k}}$</td>
<td>0.04124</td>
<td>0.2443</td>
</tr>
</tbody>
</table>
Payload Sensitivity to Exhaust Velocity

- Use the same technique to find the change in payload resulting from additional exhaust velocity for stage k

\[
\left. \frac{d m_{pl}}{d V_{e,k}} \right| \partial (\Delta V_{tot}) = 0 = \sum_{j=1}^{k} \ln \left( \frac{m_{o,j}}{m_{f,j}} \right) \frac{1}{\sum_{\ell=1}^{N} V_{e,\ell} \left( \frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}} \right)}
\]

- This trade-off ratio (unlike the ones for inert and propellant masses) has units - kg/(m/sec)
### Trade-off Ratio Example: Gemini-Titan II

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<td>-0.1164</td>
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<td>(\frac{dm_{pl}}{dm_{pr,k}})</td>
<td>0.04124</td>
<td>0.2443</td>
</tr>
<tr>
<td>(\frac{dm_{pl}}{dV_{e,k}}) (kg/m/sec)</td>
<td>2.870</td>
<td>6.459</td>
</tr>
</tbody>
</table>
Parallel Staging

- Multiple dissimilar engines burning simultaneously
- Frequently a result of upgrades to operational systems
- General case requires “brute force” numerical performance analysis
Parallel-Staging Rocket Equation

- Momentum at time $t$:
  $$M = m v$$

- Momentum at time $t+\Delta t$:
  (subscript “b”=boosters; “c”=core vehicle)
  $$M = (m - \Delta m_b - \Delta m_c)(v + \Delta v) + \Delta m_b(v - V_{e,b}) + \Delta m_c(v - V_{e,c})$$

- Assume thrust (and mass flow rates) constant
Parallel-Staging Rocket Equation

- Rocket equation during booster burn

\[ \Delta V = -\bar{V}_e \ln \left( \frac{m_{\text{final}}}{m_{\text{initial}}} \right) = -\bar{V}_e \ln \left( \frac{m_{\text{in},b} + m_{\text{in},c} + \chi m_{pr,c} + m_0,2}{m_{\text{in},b} + m_{pr,b} + m_{\text{in},c} + m_{pr,c} + m_0,2} \right) \]

where \( \chi \) = fraction of core propellant remaining after booster burnout, and where

\[ \bar{V}_e = \frac{V_{e,b} \dot{m}_b + V_{e,c} \dot{m}_c}{\dot{m}_b + \dot{m}_c} = \frac{V_{e,b} m_{pr,b} + V_{e,c} (1-\chi) m_{pr,c}}{m_{pr,b} + (1-\chi) m_{pr,c}} \]
Analyzing Parallel-Staging Performance

Parallel stages break down into pseudo-serial stages:

- **Stage “0”** (boosters and core)

\[ \Delta V_0 = -\bar{V}_e \ln \left( \frac{m_{\text{in},b} + m_{\text{in},c} + \chi m_{\text{pr},c} + m_{0,2}}{m_{\text{in},b} + m_{\text{pr},b} + m_{\text{in},c} + m_{\text{pr},c} + m_{0,2}} \right) \]

- **Stage “1”** (core alone)

\[ \Delta V_1 = -V_{e,c} \ln \left( \frac{m_{\text{in},c} + m_{0,2}}{m_{\text{in},c} + \chi m_{\text{pr},c} + m_{0,2}} \right) \]

- Subsequent stages are as before
Parallel Staging Example: Space Shuttle

• 2 x solid rocket boosters (data below for single SRB)
  – Gross mass 589,670 kg
  – Empty mass 86,183 kg
  – Ve 2636 m/sec
  – Burn time 124 sec

• External tank (space shuttle main engines)
  – Gross mass 750,975 kg
  – Empty mass 29,930 kg
  – Ve 4459 m/sec
  – Burn time 480 sec

• “Payload” (orbiter + P/L) 125,000 kg
Shuttle Parallel Staging Example

\[ V_{e,b} = 2636 \frac{m}{sec} \]
\[ V_{e,c} = 4459 \frac{m}{sec} \]

\[ \chi = \frac{480 - 124}{480} = 0.7417 \]

\[ \bar{V}_e = \frac{2636(1,007,000) + 4459(721,000)(1 - 0.7417)}{1,007,000 + 721,000(1 - 0.7417)} = 2921 \frac{m}{sec} \]

\[ \Delta V_0 = -2921 \ln \frac{862,000}{3,062,000} = 3702 \frac{m}{sec} \]
\[ \Delta V_1 = -4459 \ln \frac{154,900}{689,700} = 6659 \frac{m}{sec} \]

\[ \Delta V_{tot} = 10,360 \frac{m}{sec} \]
Modular Staging

- Use identical modules to form multiple stages
- Have to cluster modules on lower stages to make up for nonideal ΔV distributions
- Advantageous from production and development cost standpoints
Module Analysis

- All modules have the same inert mass and propellant mass
- Because $\delta$ varies with payload mass, not all modules have the same $\delta$!
- Use module-oriented parameters

$$
\varepsilon \equiv \frac{m_{in}}{m_{in} + m_{pr}} \quad \sigma \equiv \frac{m_{in}}{m_{pr}}
$$

- Conversions

$$
\varepsilon = \frac{\delta}{1 - \lambda} \quad \sigma = \frac{\delta}{1 - \delta - \lambda}
$$
Rocket Equation for Modular Boosters

• Assuming n modules in stage 1,

\[ r_1 = \frac{n(m_{in}) + m_{o2}}{n(m_{in} + m_{pr}) + m_{o2}} = \frac{n\varepsilon + \frac{m_{o2}}{m_{mod}}}{n + \frac{m_{o2}}{m_{mod}}} \]

• If all 3 stages use same modules, \( n_j \) for stage j,

\[ r_1 = \frac{n_1\varepsilon + n_2 + n_3 + \rho_{pl}}{n_1 + n_2 + n_3 + \rho_{pl}} \]

where \( \rho_{pl} \equiv \frac{m_{pl}}{m_{mod}}; \ m_{mod} = m_{in} + m_{pr} \)
Example: Conestoga 1620 (EER)

- Small launch vehicle (1 flight, 1 failure)
- Payload 900 kg
- Module gross mass 11,400 kg
- Module empty mass 1,400 kg
- Exhaust velocity 2754 m/sec
- Staging pattern
  - 1st stage - 4 modules
  - 2nd stage - 2 modules
  - 3rd stage - 1 module
  - 4th stage - Star 48V (gross mass 2200 kg, empty mass 140 kg, $V_e 2842$ m/sec)
Conestoga 1620 Performance

- 4th stage $\Delta V$

$$\Delta V_4 = -V_{e4} \ln \frac{m_{f4}}{m_{o4}} = -2842 \ln \frac{900 + 140}{900 + 2200} = 3104 \text{ m/sec}$$

- Treat like three-stage modular vehicle; $M_{pl}=3100 \text{ kg}$

$$\epsilon = \frac{m_{in}}{m_{mod}} = \frac{1400}{11400} = 0.1228$$

$$\rho_{pl} = \frac{m_{pl}}{m_{mod}} = \frac{3100}{11400} = 0.2719$$

$n_1 = 4; \ n_2 = 2; \ n_3 = 1$
\[ r_1 = \frac{n_1\epsilon + n_2 + n_3 + \rho_{pl}}{n_1 + n_2 + n_3 + \rho_{pl}} = \frac{4 \times 0.1228 + 2 + 1 + 0.2719}{4 + 2 + 1 + 0.2719} = 0.5175 \]

\[ r_2 = \frac{n_2\epsilon + n_3 + \rho_{pl}}{n_2 + n_3 + \rho_{pl}} = \frac{2 \times 0.1228 + 1 + 0.2719}{2 + 1 + 0.2719} = 0.4638 \]

\[ r_3 = \frac{n_3\epsilon + \rho_{pl}}{n_3 + \rho_{pl}} = \frac{1 \times 0.1228 + 0.2719}{1 + 0.2719} = 0.3103 \]

\[ V_1 = 1814 \, \frac{m}{sec}; \quad V_2 = 2116 \, \frac{m}{sec} \]

\[ V_3 = 3223 \, \frac{m}{sec}; \quad V_4 = 3104 \, \frac{m}{sec} \]

\[ V_{total} = 10,257 \, \frac{m}{sec} \]
Discussion about Modular Vehicles

- Modularity has several advantages
  - Saves money (smaller modules cost less to develop)
  - Saves money (larger production run = lower cost/module)
  - Allows resizing launch vehicles to match payloads
- Trick is to optimize number of stages, number of modules/stage to minimize total number of modules
- Generally close to optimum by doubling number of modules at each lower stage
- Have to worry about packing factors, complexity
OTRAG - 1977-1983
Modular Example

- Let’s build a launch vehicle out of seven Space Shuttle Solid Rocket Boosters
  - \( M_{\text{in}} = 86,180 \text{ kg} \)
  - \( M_{\text{pr}} = 503,500 \text{ kg} \)

\[
\varepsilon \equiv \frac{m_{\text{in}}}{m_{\text{in}} + m_{\text{pr}}} = 0.1461 \quad \sigma \equiv \frac{m_{\text{in}}}{m_{\text{pr}}} = 0.1711
\]

- Look at possible approaches to sequential firing
Modular Sequencing - SRB Example

- Assume no payload
- All seven firing at once - $\Delta V_{\text{tot}}=5138$ m/sec
- 3-3-1 sequence - $\Delta V_{\text{tot}}=9087$ m/sec
- 4-2-1 sequence - $\Delta V_{\text{tot}}=9175$ m/sec
- 2-2-2-1 sequence - $\Delta V_{\text{tot}}=9250$ m/sec
- 2-1-1-1-1-1 sequence - $\Delta V_{\text{tot}}=9408$ m/sec
- 1-1-1-1-1-1-1 sequence - $\Delta V_{\text{tot}}=9418$ m/sec
- Sequence limited by need to balance thrust laterally