ENAE 483/788D LECTURE #02
(ROCKET PERFORMANCE) PROBLEMS – FALL, 2024

Although it has been eclipsed by the success of the Falcon 9 and is being phased out in favor of Vulcan-Centaur, the Atlas V is still in operation as one of only two active human-rated launch vehicles. The mass and propellant properties of the Atlas V (in its 501 configuration) are shown in the following table. In this configuration, the rocket is capable of placing 8,123 kg of payload into low Earth orbit (LEO).

Stage	empty mass (kg)	propellant mass (kg)	exhaust velocity (m/sec)	nominal burn time (sec)
First stage	21,050	284,100	3,313	253
Second stage	2,316	20,830	4,418	842

(1) Calculate

(a) Gross mass of the entire vehicle m_o

$$m_o = m_{in,1}+m_{pr,1}+m_{in,2}+m_{pr,2}+m_{pl} = 21,050+284,100+2,316+20,830+8,123 = \boxed{336{,}400 \text{ kg}}$$

(b) Inert mass fraction for the first stage δ_1

$$m_{o,2} = m_{in,2} + m_{pr,2} + m_{pl} = 2,316 + 20,830 + 8,123 = 31,270\ kg$$

$$\delta_1 = \frac{m_{in,1}}{m_{in,1} + m_{pr,1} + m_{o,2}} = \frac{21,050}{21,050 + 284,100 + 31,269} = \boxed{0.06257}$$

(c) Inert mass fraction for the second stage δ_2

$$\delta_2 = \frac{m_{in,2}}{m_{o,2}} = \frac{2,316}{31,270} = \boxed{0.07407}$$

(d) Stage inert mass fraction for the first stage ϵ_1

$$\epsilon_1 = \frac{m_{in,1}}{m_{in,1} + m_{pr,1}} = \frac{21,050}{21,050 + 20,830} = \boxed{0.06898}$$

(e) Stage inert mass fraction for the second stage ϵ_2

$$\epsilon_2 = \frac{m_{in,2}}{m_{in,2} + m_{pr,2}} = \frac{2,316}{2,316 + 20,830} = \boxed{0.1000}$$

(f) Predicted value for ϵ_1, based on the heuristic equation from lecture for storable (i.e., non-LH_2) propellants

From page 22 in the lecture slides,

$$\epsilon_{\text{storables}} = 1.6062 \left(M_{\text{stage}}\langle kg\rangle\right)^{-0.275}$$

$$\epsilon_1 = 1.6062 \left(21,050 + 284,100\right)^{-0.275} = \boxed{0.04984}$$

(g) Predicted value for ϵ_2, based on the heuristic equation from lecture for LOX-LH$_2$) propellants
From page 22 in the lecture slides,

$$\epsilon_{(\text{LOX-LH}_2)} = 0.987\left(M_{\text{stage}}\langle kg\rangle\right)^{-0.183}$$
$$\epsilon_2 = 0.987\,(2,316 + 20,830)^{-0.183} = \boxed{0.1569}$$

(h) Velocity change provided by the first stage Δv_1

$$m_{f,1} = m_{in,1} + m_{o,2} = 21,050 + 31,270 = 52,319\ kg$$

From page 14 in the lecture slides,

$$V_e = g_o * I_sp$$
$$\Delta v_1 = -g_o I_{sp} \ln\frac{m_{f,1}}{m_o} = -9.8(312)\ln\frac{52,319}{336,419} = \boxed{\text{6,165 m/sec}}$$

(i) Velocity change provided by the second stage Δv_2

$$m_{f,2} = m_{in,2} + m_{pl} = 2,316 + 8,123 = 10,439\ kg$$
$$\Delta v_2 = -g_o I_{sp} \ln\frac{m_{f,2}}{m_{o,2}} = -9.8(348)\ln\frac{10,439}{31,270} = \boxed{\text{4,847 m/sec}}$$

(j) Total velocity change provided by the entire launch vehicle Δv_{total}

$$\Delta v_{total} = \Delta v_1 + \Delta v_2 = 6,165 + 4,847 = \boxed{\text{11,012 m/sec}}$$

(k) Trade-off ratio for change in payload mass due to a change in first stage inert mass $\frac{\partial m_{pl}}{\partial m_{in,1}}$

$$\frac{\partial m_{pl}}{\partial m_{in,1}} = \frac{-gI_{sp,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right)}{gI_{sp,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + gI_{sp,2}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)}$$

$m_{o,1} = m_o$, *and* $gI_{sp} = v_e$

$$\frac{\partial m_{pl}}{\partial m_{in,1}} = \frac{-3,313\left(\frac{1}{336{,}400} - \frac{1}{52{,}320}\right)}{3,313\left(\frac{1}{336{,}400} - \frac{1}{31{,}270}\right) + 4,418\left(\frac{1}{31{,}270} - \frac{1}{10{,}440}\right)} = \boxed{-0.1594}$$

Note: this is a dimensionless number, so it is metric tons of payload lost per metric ton of inert mass, or kg of payload per kg of inert mass

(l) Trade-off ratio for change in payload mass due to a change in first stage propellant mass $\frac{\partial m_{pl}}{\partial m_{pr,1}}$

$$\frac{\partial m_{pl}}{\partial m_{pr,1}} = \frac{-gI_{sp,1}\left(\frac{1}{m_{o,1}}\right)}{gI_{sp,1}\left(\frac{1}{m_{o,1}} - \frac{1}{m_{f,1}}\right) + gI_{sp,2}\left(\frac{1}{m_{o,2}} - \frac{1}{m_{f,2}}\right)}$$

$$\frac{\partial m_{pl}}{\partial m_{pr,1}} = \frac{-3,313\left(\frac{1}{336{,}400}\right)}{3,313\left(\frac{1}{336{,}400} - \frac{1}{52{,}320}\right) + 4,418\left(\frac{1}{31{,}270} - \frac{1}{10{,}440}\right)} = \boxed{0.02936}$$

(2) The 551 version of the Atlas V is the system above with the addition of five solid rocket motors to increase payload. Assuming you have to reach the same Δv_{total} you calculated in (1)(j), find the new payload to LEO assuming the following parameters for *each* solid rocket motor, following the assumptions for parallel staging from the lecture:

Stage	empty mass (kg)	propellant mass (kg)	exhaust velocity (m/sec)	nominal burn time (sec)
GEM 63	5,100	44,200	2,739	94

Here we have a parallel staging problem. The five solid outer boosters burn together along with the center core, followed by the center core burning by itself after solid burnout and ejection. (Effectively a three-stage rocket)

This means that the center core burn time is split among the first "two" stages. The core burns part of the time alongside the solid boosters ("stage 0"), and part of the time the core burns alone ("stage 1"). Stage 2 follows as before.

Additionally, we have a mixed 0^{th} stage so we cannot ignore the difference for this "pseudo" stage and therefore must calculate the weighted average for the parallel stage. Treating the period where the boosters and core "first" stage modules are firing as the aforementioned stage "0", we have:

$$\Delta V_0 = -\bar{V}_e \ln\left(\frac{m_{\text{final}}}{m_{\text{initial}}}\right) = -\bar{V}_e \ln\left(\frac{m_{in,b} + m_{in,c} + \chi m_{pr,c} + m_{0,2}}{m_{in,b} + m_{pr,b} + m_{inn,c} + m_{pr,c} + m_{0,2}}\right)$$

and

$$\bar{V}_e = \frac{V_{e,b}\dot{m}_b + V_{e,c}\dot{m}_c}{\dot{m}_b + \dot{m}_c} = \frac{V_{e,b}m_{pr,b} + V_{e,c}(1-\chi)m_{pr,c}}{m_{pr,b} + (1-\chi)m_{pr,c}}$$

$\bar{V}_e$ *is the effective exit velocity of the* 0^{th} *stage* (NOTE: The above equations for parallel staging come from page 44 of the lecture slides.)

The center core is has a burn time of 253 sec with a booster burnout at 94 sec. The amount remaining of the center core remaining after stage 0 is denoted as χ:

$$\chi = \frac{253 - 94}{253} = 0.6285$$

We have 5 boosters firing at the same time as the center core which we will feed into the amount of mass change therefore:

$$\dot{m}_b = 5 * 44,200 \quad \text{and} \quad \dot{m}_c = (1 - 0.6285) * 284,100$$

$$\bar{V}_e = \frac{2,739 * 5 * 44,200 + 3,313(1 - 0.6285) * 284,100}{5 * 44,200 + (1 - 0.6285) * 284,100} = \boxed{\text{2,925 m/sec}}$$

$$\begin{aligned} m_o &= 5 * m_{in,0} + 5 * m_{pr,0} + m_{in,1} + m_{pr,1} + m_{in,2} + m_{pr,2} + m_{pl} \\ &= 5(5,100) + 5(44,200) + 21,050 + 284,100 + 2,316 + 20,830 + m_{pl} \\ &= 574,796 + m_{pl} \end{aligned}$$

$$\begin{aligned} m_{f,0} &= 5m_{in,0} + m_{in,1} + \chi m_{pr,1} + m_{in,2} + m_{pr,2} + m_{pl} \\ &= 5(5,100) + 21,050 + 0.6285 * 284,100 + 2,316 + 20,830 + m_{pl} \\ &= 248,200 + m_{pl} \end{aligned}$$

$$\begin{aligned} m_{o,1} &= m_{in,1} + \chi * m_{pr,1} + m_{in,2} + m_{pr,2} + m_{pl} \\ &= 21,050 + 0.6285 * 284,100 + 2,316 + 20,830 + m_{pl} \\ &= 222,800 + m_{pl} \end{aligned}$$

$$\begin{aligned} m_{f,1} &= m_{in,1} + m_{in,2} + m_{pr,2} + m_{pl} \\ &= 21,050 + 2,316 + 20,830 + m_{pl} \\ &= 44,200 + m_{pl} \end{aligned}$$

$$\begin{aligned} m_{o,2} &= m_{in,2} + m_{pr,2} + m_{pl} \\ &= 2,316 + 20,830 + m_{pl} \\ &= 23,150 + m_{pl} \end{aligned}$$

$$\begin{aligned} m_{f,2} &= m_{in,2} + m_{pl} \\ &= 2,316 + m_{pl} \end{aligned}$$

$$\Delta v_0 = -\bar{V}_e \ln \frac{m_{f,0}}{m_o} = -2,925 \ln \frac{248,200 + m_{pl}}{574,800 + m_{pl}}$$

$$\Delta v_1 = -v_e \ln \frac{m_{f,1}}{m_o} = -3,313 \ln \frac{44,200 + m_{pl}}{222,800 + m_{pl}}$$

Here it is important to note that $m_{f,2}$ and $m_{o,2}$ here are the same as question 1 but with a different unknown payload

$$\Delta v_2 = -v_e \ln \frac{m_{f,2}}{m_{o,2}} = -4,418 \ln \frac{2,316 + m_{pl}}{23,150 + m_{pl}}$$

$$\Delta v_{tot} = \Delta v_0 + \Delta v_1 + \Delta v_2 = 11,012 \quad m/sec$$

Solving the system of equations numerically, the answer is $m_{pl} = \boxed{12{,}300\ kg}$