Terramechanics

- Origin and nature of lunar soil
- Soil mechanics
- Wheel-soil interactions
- Effects of tandem wheels

Lunar Regolith

- Broken down from larger pieces over time
- Major constituents
  - Rock fragments
  - Mineral fragments
  - Glassy particles
- Local environment
  - 10^{-12} torr (= 1.22×10^{-10} Pa = 1.93×10^{-14} psi)
  - Meteorites at velocities >10^{5} m/sec
  - Galactic cosmic rays, solar particles
  - Temperature range +250°F – -250°F

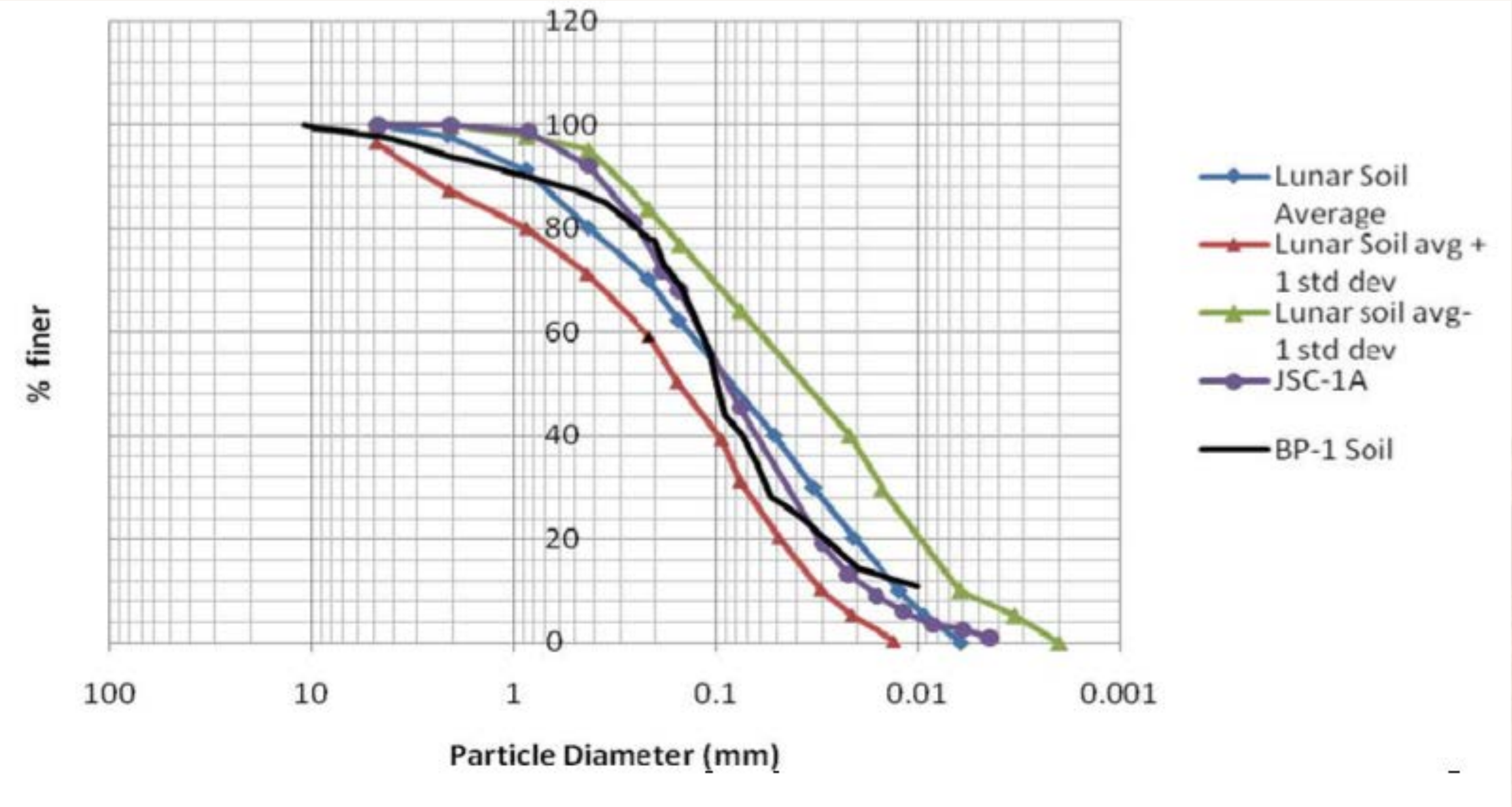
Regolith Creation Process

- Only "weathering" phenomenon on the moon is meteoritic impact!
- Weathering processes
  - Comminution: breaking rocks and minerals into smaller particles
  - Agglutination: welding fragments together with molten glass formed by impact energy
  - Solar wind spallation and implantation (minuscule)
  - Fire fountaining (dormant)

JSC-1 Simulant

- Ash vented from Merriam Crater in San Francisco volcano field near Flagstaff, AZ
- K-Ar dated at 150,000 years old ± 30,000
- Major constituents SiO_2, TiO_2, Al_2O_3, Fe_2O_3, FeO, MgO, CaO, Na_2O, other <1%
- Represents low-Ti regolith from lunar mare
- MLS-1 simulant (U.Minn.) preferred for simulation of highland material
- BP-1 (Flagstaff, AZ) is ground basaltic lava - higher fidelity because of angular grain shapes

Particle Size Distribution in Regolith



Rahmatian and Metzger, “Soil Test Apparatus for Lunar Surfaces” *Earth and Space 2010*, ASCE

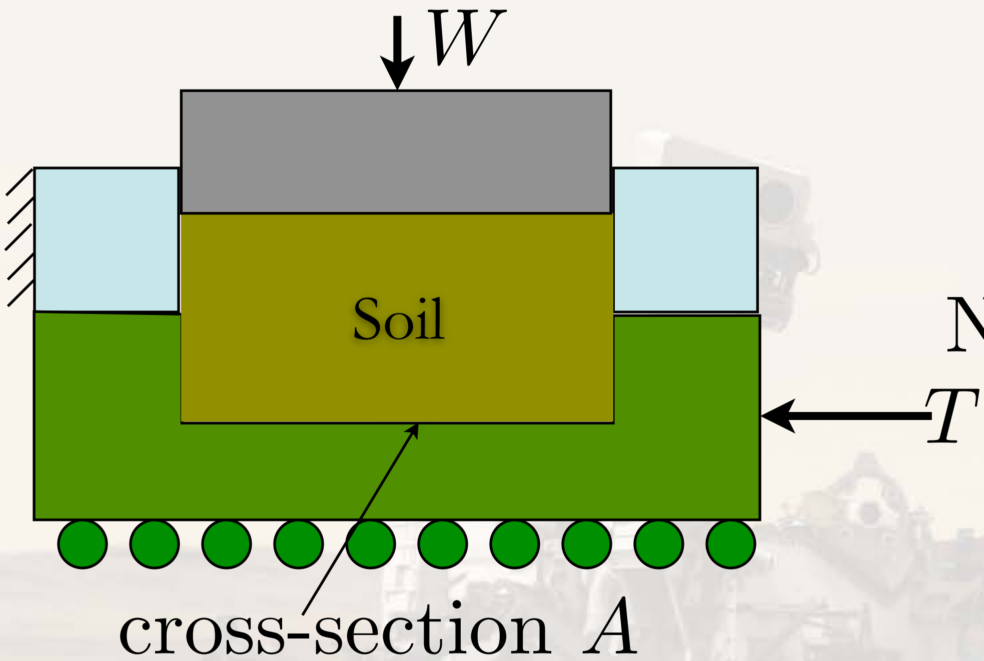
Soil Testing Apparatus

Bevameter (force vs. displacement)

Internal friction angle φ

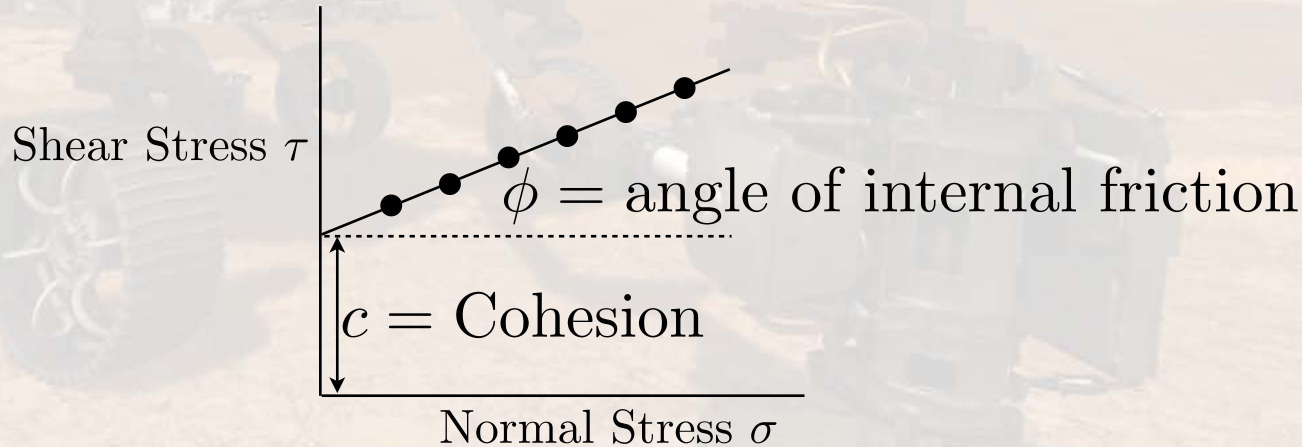
Shear deformation modulus K

Soil Characterization – Direct Shear



$$\text{Shear Stress } \tau = \frac{T}{A}$$

$$\text{Normal Stress } \sigma = \frac{W}{A}$$



Modeling Soil Reaction to a Wheel

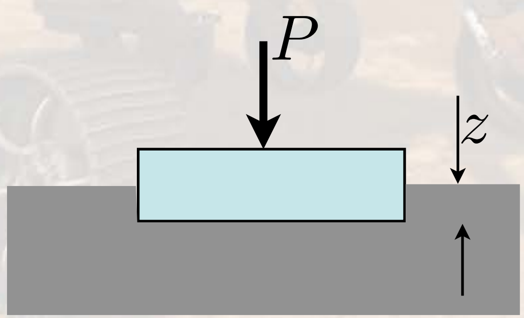
Assume soil reaction is like a (nonlinear) spring

$$P = kz^n$$

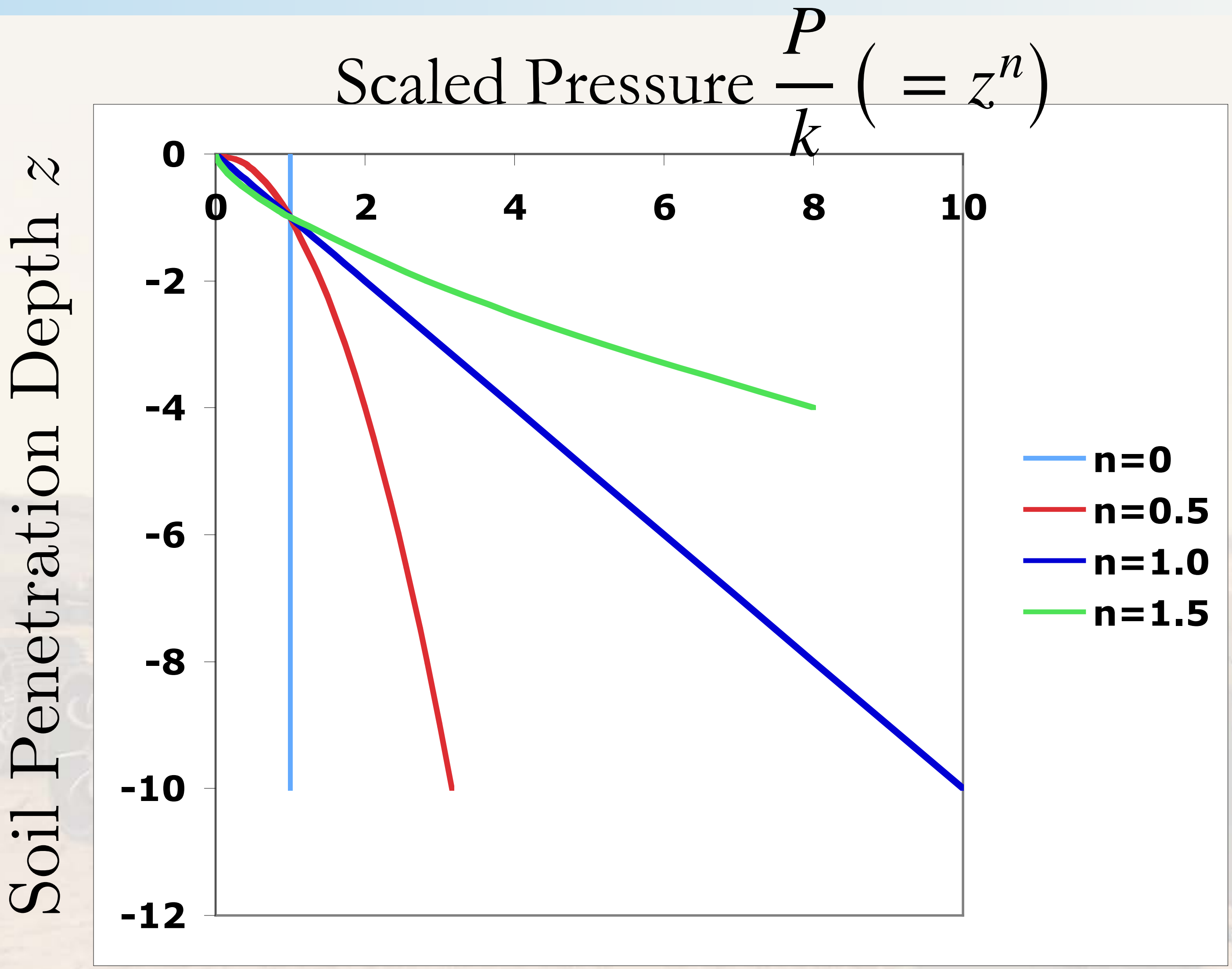
$$P = \text{applied pressure}$$

$$z = \text{compression depth}$$

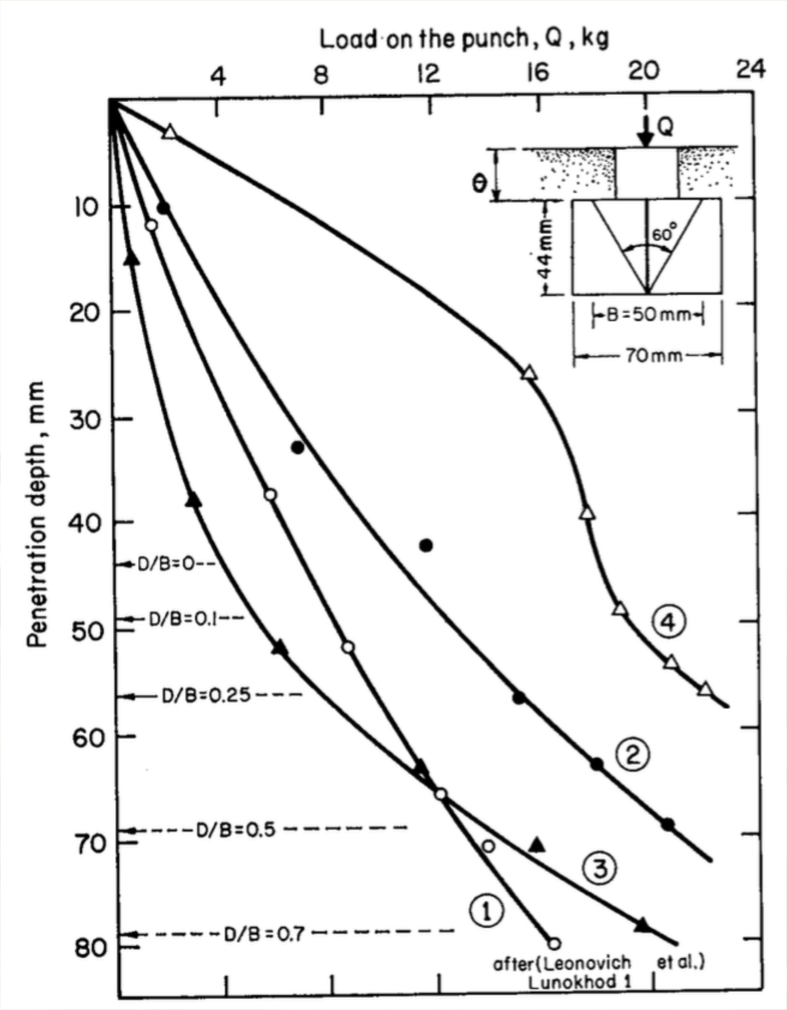
$$k, n = \text{heuristic parameters}$$



Effects of Soil Mechanics



Lunokhod 1 Penetrometer Data

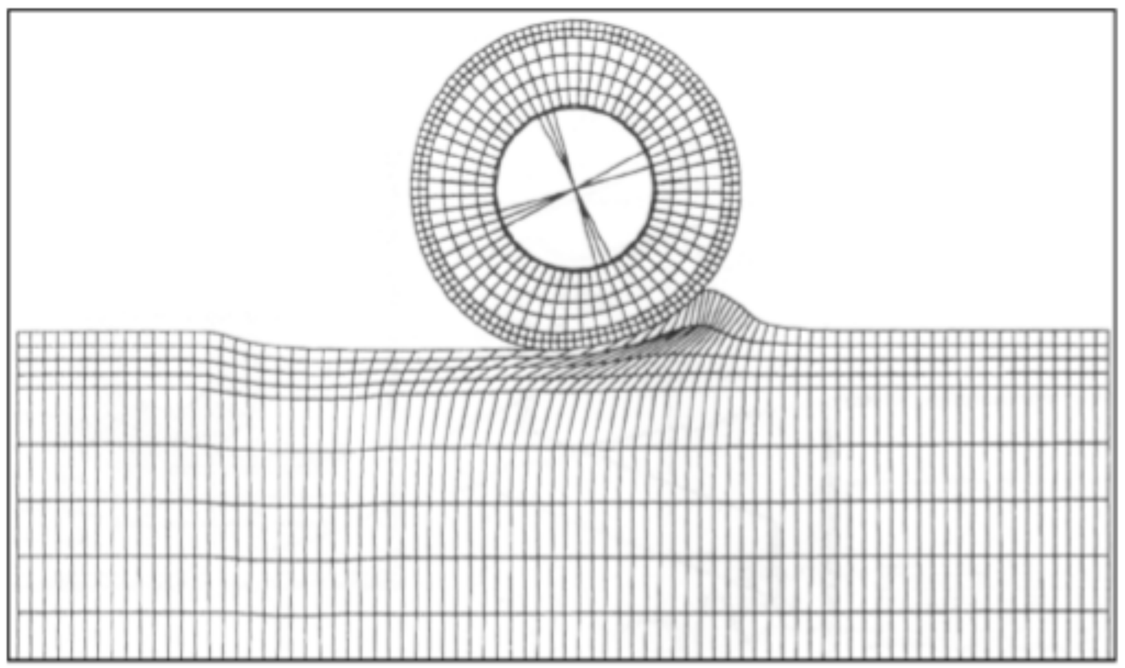


Mitchell et.al., Mechanical Properties of Lunar Soil: Density, Porosity, Cohesion, and Angle of Internal Friction" Proceedings of the Third Lunar Science Conference, Vol. 3, MIT Press, 1972

Wheel-Soil Interaction

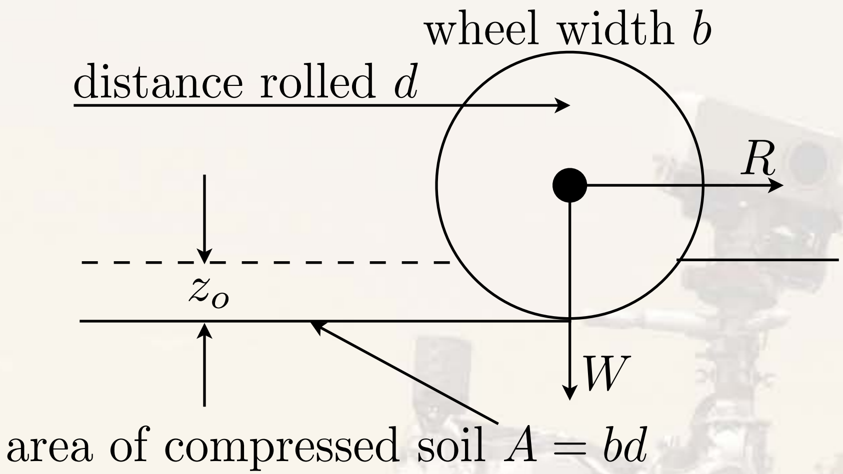
Wheel rolling over soil does physical work

- Compression
- “Bulldozing”



from Gibbesch and Schafer, “Advanced and Simulation Methods of Planetary Rover Mobility on Soft Terrain” *8th ESA Workshop on Advanced Space Technologies for Robotics and Automation*, Noordwijk, The Netherlands, November, 2004

Wheel-Soil Interactions



$$\text{Displacement Energy } \frac{E}{A} = \int \frac{F}{A} dz = \int P dz$$

$$\frac{E}{A} = \int_0^{z_o} P dz = \int_0^{z_o} k z^n dz = k \frac{z_o^{n+1}}{n+1}$$

Rolling Resistance

$$\text{Total Energy } \frac{E}{A}A = \frac{E}{A}bd = k\frac{z_o^{n+1}}{n+1}bd$$

Given a force resisting rolling $\equiv R$,

the energy required to roll a distance d is

$$E_{roll} = Rd$$

$$E_{roll} = E_{displacement} \Rightarrow Rd = \frac{E}{A}bd$$

Rolling Resistance

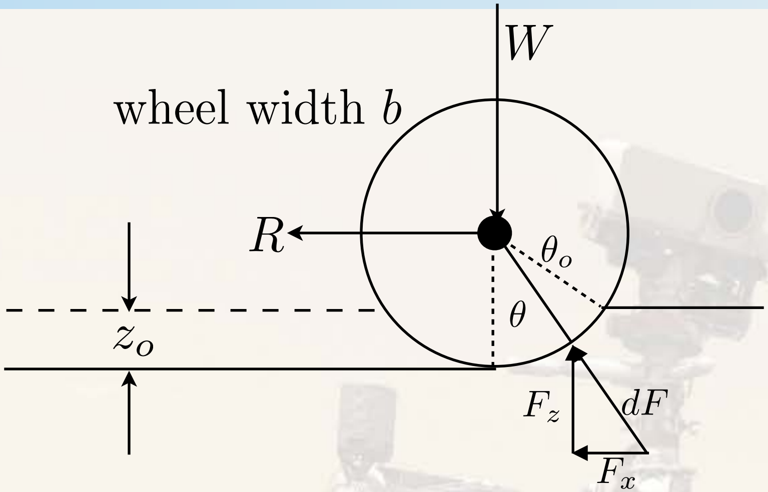
$$\text{For } n = 1 : P = kz; \frac{E}{A} = k\frac{z_o^2}{2}; R = \frac{1}{2}kbz_o^2$$

$$\text{For } n = \frac{1}{2} : P^2 = k^2 z; \frac{E}{A} = \frac{2}{3}kz_o^{\frac{3}{2}}; R = \frac{2}{3}kbz_o^{\frac{3}{2}}$$

$$\text{For } n = 0 : P = k; \frac{E}{A} = kz_o; R = kbz_o$$

$$\text{Generic case: } P = kz^n; \frac{E}{A} = k\frac{z_o^{n+1}}{n+1}; R = kb\frac{z_o^{n+1}}{n+1}$$

Soil Displacement Calculations



$$R - \int_0^{\theta_o} dF \sin\theta = 0$$

$$-W + \int_0^{\theta_o} dF \cos\theta = 0$$

$$dF \cos\theta = -Pb\,dx$$

$$dF = Pbr\,d\theta$$

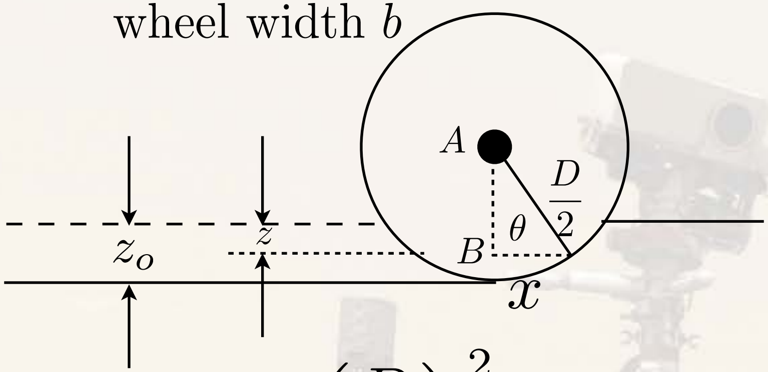
$$dF \sin\theta = Pb\,dz$$

$$R = \int_0^{\theta_o} Pb\,dz \qquad W = -\int_0^{\theta_o} Pb\,dx$$

In general, $P = kx^n$

$$W = -\int_0^{z_o} bkz^n dx$$

Soil Displacement Calculations



$$\bar{AB} = \frac{D}{2} - (z_o - z)$$

$$x^2 = \left(\frac{D}{2}\right)^2 - \bar{AB}^2 = \left(\frac{D}{2}\right)^2 - \left[\frac{D}{2} - (z_o - z)\right]^2$$

$$= \left(\frac{D}{2}\right)^2 - \left(\frac{D}{2}\right)^2 + 2\frac{D}{2}(z_o - z) - (z_o - z)^2$$

$$x^2 = [D - (z_o - z)](z_o - z)$$

Soil Compression Calculations

But $D \gg z_o - z$

$$x^2 \approx D(z_o - z) \Rightarrow 2x\ dx = -D\ dz$$

$$\text{so from } W = -\int_0^{z_o} bkz^n dx \text{ we get } W = -\int_0^{z_o} bkz^n \frac{-D}{2x} dz$$

$$W = -bk \int_0^{z_o} z^n \left(\frac{-D}{2\sqrt{D}\sqrt{z_o - z}} \right) dz$$

$$W = bk \int_0^{z_o} z^n \left(\frac{\sqrt{D}dz}{2\sqrt{z_o - z}} \right) dz$$

Soil Displacement Calculations

$$\text{Define } z_o - z \equiv t^2 \Rightarrow dz = -2t\,dt$$

$$W = bk\sqrt{D}\int_0^{\sqrt{z_o}} (z_o - t^2)^n dt$$

$$\text{Taylor Series expansion } (z_o - t^2)^n \cong z_o^n - n z_o^{n-1} t^2 + \cdots$$

$$W \approx \frac{bk\sqrt{D z_o}}{3} z_o^n (3 - n)$$

$$\text{for } n = 1 \Rightarrow W = \frac{2}{3} bk z_o \sqrt{D z_o}$$

$$\text{for } n = \frac{1}{2} \Rightarrow W = \frac{5}{6} bk z_o \sqrt{D}$$

$$\text{for } n = 0 \Rightarrow W = bk\sqrt{D z_o}$$

Rolling Resistance as f(W)

$$\text{for } n = 0 \Rightarrow W = bk\sqrt{Dz_o} \Rightarrow z_o = \left(\frac{W}{bk}\right)^2 \frac{1}{D}$$

$$R = kbz_o \Rightarrow R = \frac{kb}{(kb)^2}\frac{W^2}{D} \Rightarrow R = \frac{W^2}{kbD}$$

$$\text{for } n = \frac{1}{2} \Rightarrow W = \frac{5}{6}bkz_o\sqrt{D} \Rightarrow z_o = \frac{6}{5}\frac{W}{bk\sqrt{D}}$$

$$R = \frac{2}{3}kbz_o^{\frac{3}{2}} \Rightarrow R = \frac{2}{3}kb\left(\frac{6}{5}\frac{W}{kb\sqrt{D}}\right)^{\frac{3}{2}} = \frac{2}{3}\left(\frac{6}{5}\right)^{\frac{3}{2}}\frac{W^{\frac{3}{2}}}{\sqrt{kb}D^{\frac{3}{4}}}$$

$$R = 0.876\frac{W^{\frac{3}{2}}}{\sqrt{kb}D^{\frac{3}{4}}}$$

Rolling Resistance as f(W)

$$\text{for } n = 1 \Rightarrow W = \frac{2}{3} b k z_o^{\frac{3}{2}} \sqrt{D} \Rightarrow z_o^2 = \left(\frac{3W}{2kb\sqrt{D}} \right)^{\frac{4}{3}}$$

$$R = \frac{1}{2} k b z_o^2 \Rightarrow R = \frac{1}{2} k b \left(\frac{3W}{2kb\sqrt{D}} \right)^{\frac{4}{3}} = \frac{1}{2} \left(\frac{3}{2} \right)^{\frac{4}{3}} \left(\frac{W^4}{kbD^2} \right)^{\frac{1}{3}}$$

$$R = 0.859 \left(\frac{W^4}{kbD^2} \right)^{\frac{1}{3}}$$

Rolling Resistance as f(W) (Generic)

$$W = \frac{bk\sqrt{Dz_o}}{3} z_o^n (3-n) = \frac{bk\sqrt{D}}{3} z_o^{n+\frac{1}{2}} (3-n)$$

$$z_o^{n+\frac{1}{2}} = \frac{3}{(3-n)} \frac{W}{bk\sqrt{D}}$$

$$z_o^{n+1} = \left(\frac{3}{3-n} \frac{W}{bk\sqrt{D}} \right)^{\frac{n+1}{n+\frac{1}{2}}} = \left(\frac{3}{3-n} \frac{W}{bk\sqrt{D}} \right)^{\frac{2(n+1)}{2n+1}}$$

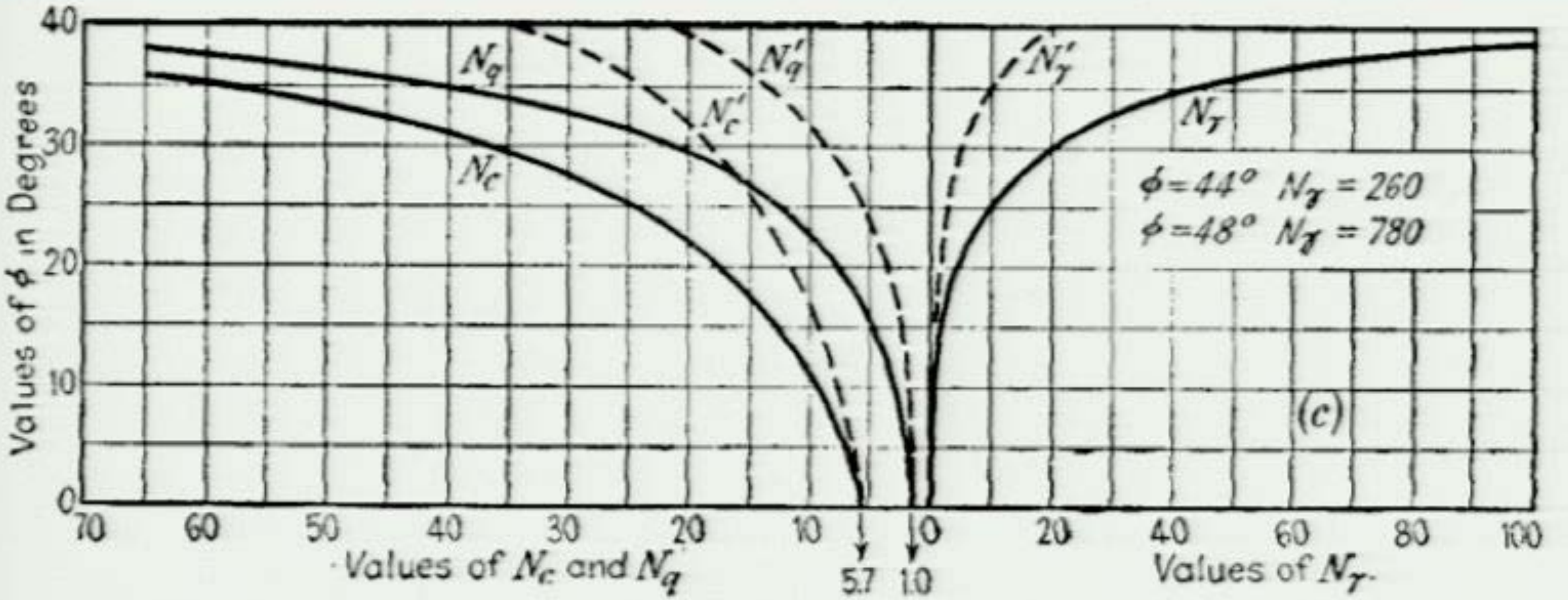
$$R = \frac{bk}{n+1} z_o^{n+1} = \frac{bk}{n+1} \left(\frac{3}{3-n} \frac{W}{bk\sqrt{D}} \right)^{\frac{2(n+1)}{2n+1}}$$

$$R = \frac{1}{n+1} \left(\frac{3}{3-n} \frac{W}{\sqrt{D}} \right)^{\frac{2(n+1)}{2n+1}} \left(\frac{1}{bk} \right)^{\frac{1}{2n+1}}$$

Soil Characteristics

soil type	n	$k_c \ \langle \frac{N}{m^{n+1}} \rangle$	$k_\phi \ \langle \frac{N}{m^{n+2}} \rangle$
Dry Sand	1.1	990	1,528,000
Lunar Regolith	1	1400	820,000
Sandy Loam	0.7	5270	1,515,000
Sandy Loam (MER-B)	1	28,000	7,600,000
Slope Soil (MER-B)	0.8	6800	210,000
Clay (Earth)	0.5	13,190	692,200

Terzaghi Analysis of Soil



General shear failure: $Q_D = 2B(cN_c + \gamma D_f N_q + \gamma B N_\gamma)$ } per unit length of footing

Local shear failure: $Q_D' = 2B(\tfrac{2}{3}cN_c' + \gamma D_f N_q' + \gamma B N_\gamma')$

Circular footing, Diameter 2R,

Total critical load: $Q_{Dr} \cong R^2\pi(1.3cN_c + \gamma D_f N_q + 0.6\gamma R N_\gamma)$

Unit weight of earth = γ

Unit shear resistance $s = c + \sigma \tan\phi$

Terzaghi Soil Bearing Capacity

$$N_q = \frac{\exp\left[\left(\frac{3\pi}{2} - \phi\right)\tan\phi\right]}{2\cos^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right)}$$

$$N_c = \cot\phi\left\{\frac{\exp\left[\left(\frac{3\pi}{2} - \phi\right)\tan\phi\right]}{2\cos^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right)} - 1\right\} = \frac{N_q - 1}{\tan\phi}$$

$$N_\gamma = \frac{2(N_q + 1)\tan(\phi)}{1 + 0.4\sin(4\phi)}$$

$\phi =$ Angle of internal resistance of soil

Parameter Definition

$K_c \equiv$ Cohesive modulus of soil deformation

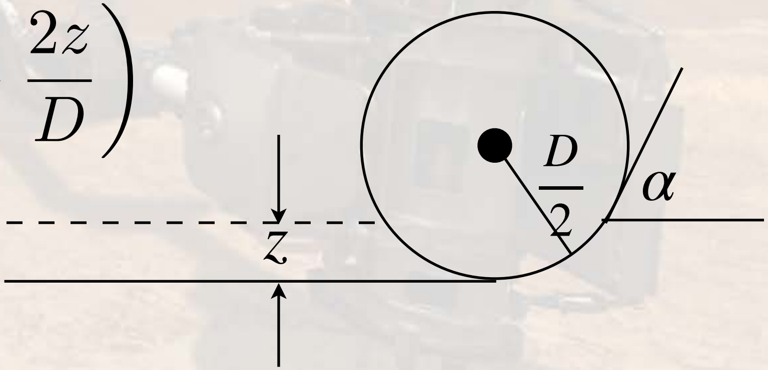
$$K_c = (N_c - \tan\phi)\cos^2\phi$$

$K_\phi \equiv$ Frictional modulus of soil deformation

$$K_\gamma = \left[\frac{2N_\gamma}{\tan\phi} + 1\right]\cos^2\phi$$

$\alpha \equiv$ Angle of approach of wheel to soil

$$\alpha = \cos^{-1}\left(1 - \frac{2z}{D}\right)$$



$\gamma \equiv$ Weight density of soil $\left(\frac{N}{m^3}\right)$

Lunar Soil Canonical Values

$$\text{Soil density: } \rho = 1600 \ \frac{kg}{m^3} \Longrightarrow \gamma = 2470 \ \frac{N}{m^3}$$

$$n = 1$$

$$k_c = 1400 \ N/m^2$$

$$k_\phi = 830{,}000 \ N/m^3$$

$$\phi = 33^{\text{O}} = 0.576 \ rad$$

$$\text{Cohesion } c = 170 \ N/m^2$$

$$N_q = 32.23$$

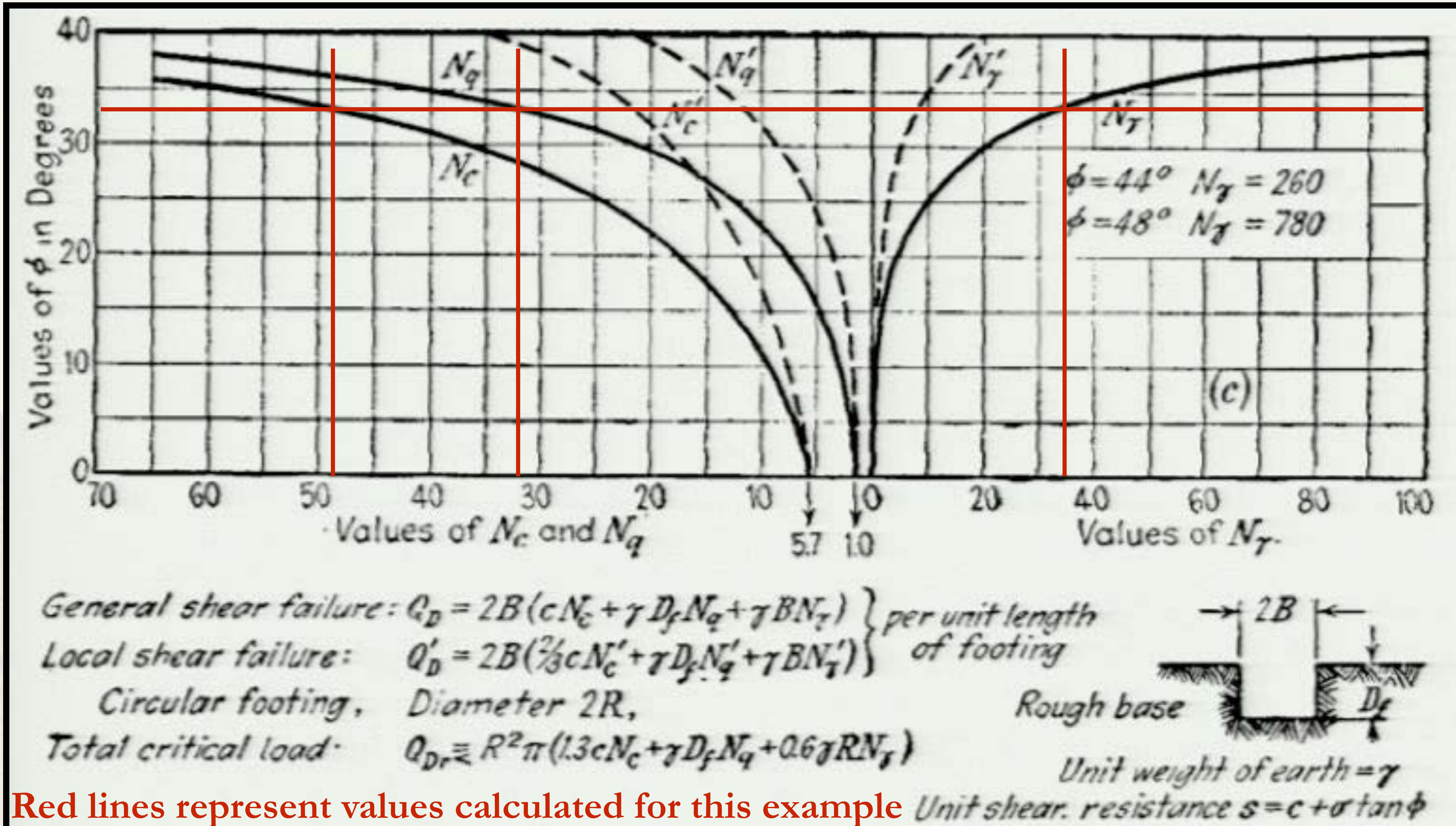
$$N_c = 48.09$$

$$N_\gamma = 33.27$$

$$K_c = 33.37$$

$$K_\phi = 72.77$$

LRV Terzhagi Parameters



Red lines represent values calculated for this example

More Detailed Soil Compression Equation

$$k = \frac{k_c}{b} + k_\phi$$

$$k_c = \text{modulus of cohesion of soil deformation}$$

$$k_c \text{ units} \Rightarrow < N/m^{(n+1)} >$$

$$k_\phi = \text{modulus of friction of soil deformation}$$

$$k_\phi \text{ units} \Rightarrow < N/m^{(n+2)} >$$

$$b = \text{wheel width}$$

$$P = \left(\frac{k_c}{b} + k_\phi \right) z^n$$

Equations for Compression Resistance

From page 21,

$$z = \left(\frac{3}{3-n} \frac{W_w}{bk\sqrt{D}} \right)^{\frac{2}{2n+1}}$$

$$W_w = \text{weight on wheel}$$

$$D = \text{wheel diameter}$$

$$R_c = \left(\frac{bk}{n+1} \right) z^{n+1}$$

$$R_c = \text{compression resistance (per wheel)}$$

$$b = \frac{k_c}{b} + k_\phi \Longrightarrow bk = k_c + bk_\phi$$

Equations for Compression Resistance

$$z = \left(\frac{3}{3-n} \frac{W_w}{(k_c + b k_\phi)\sqrt{D}} \right)^{\frac{2}{2n+1}}$$

$$R_c = \left(\frac{k_c + b k_\phi}{n+1} \right) z^{n+1}$$

$$R_c = \left(\frac{k_c + b k_\phi}{n+1} \right) \left(\frac{3}{3-n} \frac{W_w}{(k_c + b k_\phi)\sqrt{D}} \right)^{\frac{2(n+1)}{2n+1}}$$

Compression Resistance (Lunar Soil)

$$R_c = \frac{1}{n+1}(k_c + bk_\phi)^{\frac{-1}{2n+1}}\left(\frac{3W_w}{(3-n)\sqrt{D}}\right)^{\frac{2(n+1)}{2n+1}}$$

$$n = 1$$

$$z = \left(\frac{3}{2}\frac{W_w}{(k_c + bk_\phi)\sqrt{D}}\right)^{\frac{2}{3}}$$

$$R_c = \frac{1}{2}(k_c + bk_\phi)^{\frac{-1}{3}}\left(\frac{3W_w}{2\sqrt{D}}\right)^{\frac{4}{3}}$$

Relevant LRV Parameters

$$W_w = 370\ lbs = 168\ kg = 259\ N \text{ (on Moon)}$$

$$D = 32\ in = 0.813\ m$$

$$b = 9\ in = 0.229\ m$$

$$z = \left(\frac{3}{2} \frac{259\ N}{\left(1400\ \frac{N}{m^2} + (0.229\ m)830000\ \frac{N}{m^3}\right)\sqrt{0.813\ m}} \right)^{\frac{2}{3}} = 0.0151\ m$$

$$\text{Units check: } \left(\frac{N}{(\frac{N}{m^2} + \frac{Nm}{m^3})\sqrt{m}} \right)^{\frac{2}{3}} = \left(m^{\frac{3}{2}}\right)^{\frac{2}{3}} = m$$

Soil Bearing Limit

Safe weight on the soil

$$W_s = A\left(cN_c + \gamma z N_q + \frac{1}{2}\gamma b N_\gamma\right)$$

Safe soil pressure $P_s = \dfrac{W_s}{A}$

$c \equiv \text{Soil cohesion } (Pa)$

$b \equiv \text{Wheel width } (m)$

For our LRV example case, $P_s = 18{,}790\ Pa$

For $W_v = 1004\ N$, the minimum contact area is $534\ cm^2$ or $\ell > 5.8\ cm$

LRV Compression Resistance

$$R_c = \frac{1}{2}\left(1400\ \frac{N}{m^2} + (0.229\ m)820000\ \frac{N}{m^3}\right)^{\frac{-1}{3}}\left(\frac{3(259\ N)}{2\sqrt{0.813\ m}}\right)^{\frac{4}{3}} = 28.3\ N$$

$$\text{Check units: } \left(\frac{N}{m^2} + \frac{Nm}{m^3}\right)^{\frac{-1}{3}}\left(\frac{N}{\sqrt{m}}\right)^{\frac{4}{3}} = \left(\frac{m^{2/3}}{N^{1/3}}\right)\left(\frac{N^{4/3}}{m^{2/3}}\right) = N$$

Rolling and Gravitation Resistance

- Rolling resistance (tires, bearings, etc.)

$$R_r = W_v c_f$$

$$W_v = \text{weight of vehicle}$$

$$c_f = \text{coefficient of friction (typ. 0.05)}$$

- Gravitational resistance

$$R_g = W_v \sin \theta_{slope}$$

- LRV examples (15° slope, W_v=1004 N)

$$R_r = 50\ N \qquad R_g = 260\ N$$

Bulldozing Resistance

General case:

All angles in radians!

$$R_b = \frac{b \sin(\alpha + \phi)}{2 \sin \alpha \cos \phi} \left(2zcK_c + \gamma z^2 K_\gamma\right) + \frac{\ell_o^3 \gamma}{3}\left(\frac{\pi}{2} - \phi\right) + c\ell_o^2 \left[1 + \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right]$$

$$\alpha = \text{angle of attack of wheel in soil} \equiv \cos^{-1}\left(1 - \frac{2z}{D}\right)$$

$$\gamma = \text{density of soil} \left\langle \frac{N}{m^3} \right\rangle \qquad \ell_o = z \tan^2\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$$

For tracked vehicles, only the first term applies:

$$R_b = \frac{b \sin(\alpha + \phi)}{2 \sin \alpha \cos \phi} \left(2zcK_c + \gamma z^2 K_\gamma\right)$$

LRV Bulldozing Example (1)

$$\phi = 33^o = 0.576\ rad$$

$$\alpha = \cos^{-1}\left(1 - \frac{2z}{d}\right) = \cos^{-1}\left(1 - \frac{2(1.812)}{81.2}\right) = 17.18^o = 0.2999\ rad$$

$$\ell_o = z_1 \tan^2\left(\frac{\pi}{2} - \frac{\phi}{4}\right) = 0.5341$$

ℓ_o is length of soil ruptured by compression

Soil parameter values calculated on page 26

$$\ell = \frac{d}{2}\cos^{-1}\left(1 - \frac{2z}{d}\right) = 12.18\ cm$$

ℓ is the length of the wheel's circumference in contact with the soil

(Apologies: I started using *d* for wheel diameter instead of *D*)

LRV Bulldozing Example (3)

$$R_b = \frac{b \sin(\alpha + \phi)}{2 \sin \alpha \cos \phi} \left(2zcK_c + \gamma z^2 K_\gamma\right)$$

$$+ \frac{\ell_o^3 \gamma}{3} \left(\frac{\pi}{2} - \phi\right) + c\ell_o^2 \left[1 + \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right]$$

$$\langle R_b \rangle = cm \left(cm \frac{N}{cm^2} + \frac{N}{cm^3} cm^2 \right) + cm^3 \frac{N}{cm^3} + \frac{N}{cm^2} cm^2$$

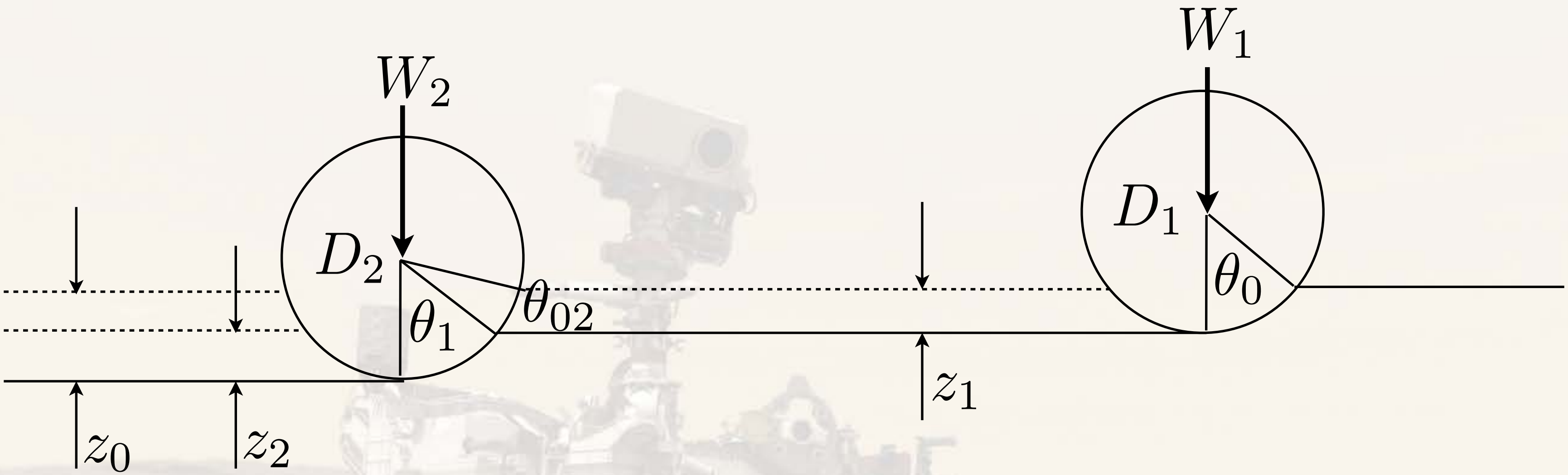
$$R_b = 94.98 + 0.000131 + 0.014 = 95.00 \ N \text{ per leading wheel}$$

$$R_{b,total} = 190.0 \ N$$

Total LRV Rolling Resistance Estimate

- Compression resistance R_c (4 wheels) 113 N
- Rolling resistance R_r 50 N
- Bulldozing resistance R_b (2 front wheels) 190 N
- *Total resistance, flat ground* *352 N*
- Gravitational resistance, 15° slope 260 N
- *Total resistance, 15° slope* *613 N*

Tandem Wheels



$$z_0 = z_1 + z_2$$

$$\text{Assume } n = \frac{1}{2} \Longrightarrow P = k\sqrt{z}$$

$$P_1 = k\sqrt{z_1} \qquad P_2 = k\sqrt{z_1 + z_2}$$

Soil Weight Bearing Analysis

In general,

$$W = \int_0^{\theta_0} dF \cos\theta = \int_0^{\theta_0} Pb \; ds \cos\theta$$

$$W = \int_0^{\theta_0} bk\sqrt{z} \cos\theta \; ds$$

$$ds = r \; d\theta \qquad\qquad z = r(\cos\theta - \cos\theta_0)$$

$$W = \int_0^{\theta_0} bkr\sqrt{r(\cos\theta - \cos\theta_0)} \cos\theta \; d\theta$$

Generic Wheel Soil Suspension

Assuming small sinkage,

$$z \to \text{ small}, \ \theta \to \text{ small},$$

$$\cos\theta \approx 1 \qquad \cos\theta \ d\theta \approx d\theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2} + (\text{higher order terms})$$

$$W = \frac{bkr^{3/2}}{\sqrt{2}} \int_0^{\theta_0} \sqrt{\theta_0^2 - \theta^2} \ d\theta$$

$$W = \frac{bkr^{3/2}}{\sqrt{2}} \frac{1}{2} \left[\theta_0^2 \sin^{-1}\left(\frac{\theta}{\theta_0}\right) + \theta\sqrt{\theta_0^2 - \theta^2} \right]_0^{\theta_0}$$

Weight on the Front Wheel

$$W = \frac{\pi b k r^{3/2}}{4\sqrt{2}} \theta_0^2$$

$$\text{Front wheel: } z_1 = r_1 - r_1 \cos\theta_0$$

$$z_1 = r_1 - r_1 \left(1 - \frac{\theta_0^2}{2} + \cdots\right) \implies \theta_0^2 \cong 2\frac{z_1}{r_1}$$

$$\boxed{W_1 \cong \frac{\pi b k z_1 \sqrt{r_1}}{2\sqrt{2}}}$$

Weight on Back Wheel

Change to limits of integration:

$$0 \longrightarrow \theta_0, \quad r \longrightarrow r_2$$

$$W_2 = \frac{bkr_2^{3/2}}{\sqrt{2}} \int_0^{\theta_0} \sqrt{\theta_{02}^2 - \theta^2}\; d\theta$$

$$\sqrt{\theta_{02}^2 - \theta^2} \cong \theta_{02}\left(1 - \frac{1}{2}\frac{\theta^2}{\theta_{02}^2} + \cdots\right)$$

$$\int_0^{\theta_0} \sqrt{\theta_{02}^2 - \theta^2}\; d\theta \cong \theta_{02}\theta_1 - \frac{1}{6}\frac{\theta_1^3}{\theta_{02}} = \theta_{02}\theta_1\left(1 - \frac{1}{6}\frac{\theta_1^2}{\theta_{02}^2}\right) \cong \theta_{02}\theta_1$$

Weight on Back Wheel

$$W_2 = \frac{bkr^{3/2}}{\sqrt{2}}\theta_{02}\theta_1$$

$$z_0 \cong r_2\frac{\theta_{02}^2}{2} \qquad z_2 \cong r_2\frac{\theta_1^2}{2}$$

$$\theta_{02} \cong \sqrt{2\frac{z_0}{r_2}} \qquad \theta_1 \cong \sqrt{2\frac{z_2}{r_2}}$$

$$\boxed{W_2 = bk\sqrt{2r_2}\sqrt{z_0 z_2}}$$

Track Depth of Tandem Wheels

$$\text{Front: } z_1 = \frac{2\sqrt{2}W_1}{\pi b k \sqrt{r_1}}$$

$$\text{Back: } z_2 = \left[\frac{W_2}{bk\sqrt{2r_2}}\frac{1}{\sqrt{z_0}}\right]^2$$

$$z_0 = z_1 + z_2 = \frac{2\sqrt{2}W_1}{\pi b k \sqrt{r_1}} + \frac{W_2^2}{(bk)^2 2r_2}\frac{1}{z_0}$$

Much algebra then ensues...

$$z_0^2 - \frac{2\sqrt{2}W_1}{\pi b k \sqrt{r_1}} z_0 + \frac{W_2^2}{(bk)^2 2r_2} = 0$$

Rolling Resistance of Tandem Wheels

Solve the quadratic equation to get

$$z_0 = \frac{1}{bk}\left(\frac{\sqrt{2}W_1}{\pi\sqrt{r_1}} + \sqrt{\frac{2W_1^2}{\pi^2 r_1} + \frac{W_2^2}{2r_2}}\right)$$

This was all done for $n = \frac{1}{2} \implies R = \frac{2}{3}bkz_0^{3/2}$

$$R = \frac{2}{3}\frac{1}{\sqrt{bk}}\left(\frac{\sqrt{2}W_1}{\pi\sqrt{r_1}} + \sqrt{\frac{2W_1^2}{\pi^2 r_1} + \frac{W_2^2}{2r_2}}\right)^{3/2}$$

$$R = \frac{2}{3}\frac{1}{\sqrt{bk}}\left(\frac{2W_1}{\pi\sqrt{D_1}} + \sqrt{\frac{4W_1^2}{\pi^2 D_1} + \frac{W_2^2}{D_2}}\right)^{3/2}$$

Nondimensional Forms

Total wheel load $W = W_1 + W_2$ Wheel weight ratio $a \equiv \dfrac{W_1}{W_2}$

$$\text{For } W_1 = W_2 = \frac{W}{2} \implies a = 1$$

$$W_1 = \frac{a}{1+a} W \qquad W_2 = \frac{1}{1+a} W$$

$$R = \frac{2}{3} \frac{1}{(a+1)^{3/2}} \frac{W^{3/2}}{\sqrt{bk}} \left(\frac{2a}{\pi\sqrt{D_1}} + \sqrt{\frac{4a^2}{\pi^2 D_1} + \frac{1}{D_2}} \right)^{3/2}$$

Define wheel diameter ratio $\rho \equiv \dfrac{D_1}{D_2}$

Nondimensional Forms

$$R = \frac{2}{3}\frac{1}{(a+1)^{3/2}}\frac{W^{3/2}}{D_2^{3/4}\sqrt{bk}}\left(\frac{2a}{\pi\sqrt{\rho}}+\sqrt{1+\frac{4a^2}{\pi^2\rho}}\right)^{3/2}$$

$$\text{Let } \xi \equiv \frac{2}{3}\frac{1}{(a+1)^{3/2}}\left(\frac{2a}{\pi\sqrt{\rho}}+\sqrt{1+\frac{4a^2}{\pi^2\rho}}\right)^{3/2}$$

$$\boxed{R = \frac{\xi}{\sqrt{bk}}\frac{W^{3/2}}{D_2^{3/4}}}$$

Simple Example Case

$$\text{Consider } \rho = 1 \quad (D_1 = D_2 = D)$$

$$a = 1 \quad \left(W_1 = W_2 = \frac{W}{2} \right)$$

$$\text{For tandem wheels, } R = \frac{0.580}{\sqrt{bk}} \frac{W^{3/2}}{D_2^{3/4}}$$

$$\text{For single wheel (n=1/2), } R = \frac{0.876}{\sqrt{bk}} \frac{W^{3/2}}{D_2^{3/4}}$$

Tandem wheels reduce rolling resistance by 34%

Dual Wheels

Equivalent to single wheel case twice as wide $b \Longrightarrow 2b$

$$R = \frac{0.876}{\sqrt{2}} \frac{1}{\sqrt{bk}} \frac{W^{3/2}}{D_2^{3/4}}$$

$$R_{dual} = \frac{0.619}{\sqrt{bk}} \frac{W^{3/2}}{D_2^{3/4}}$$

Dual wheel rolling resistance 29% less than single,

7% higher than tandem