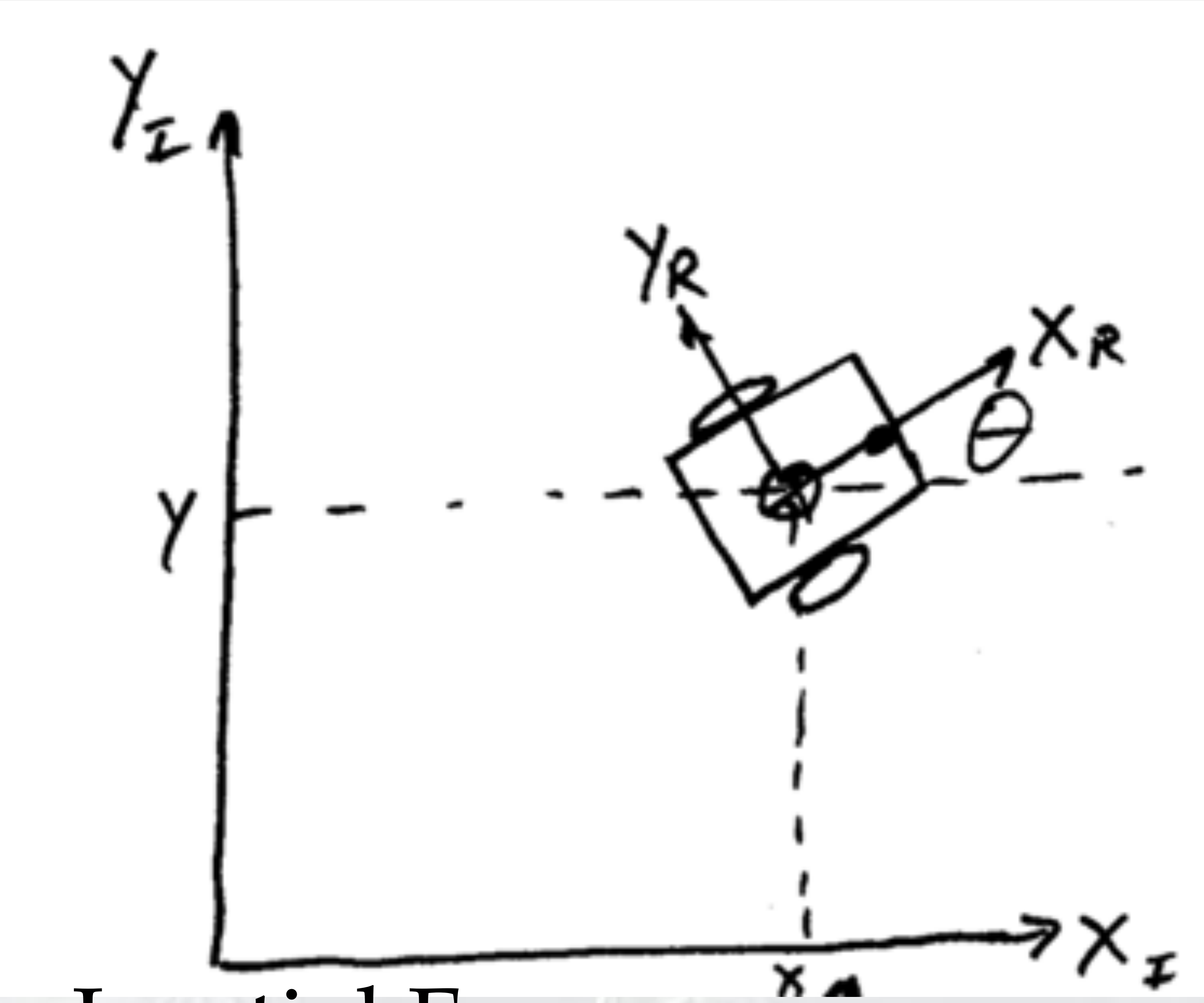
Mobile Robot Kinematics

- Description of vehicle and control axes
- Kinematic descriptions of wheel types
- Maneuverability
- Odometry

Mobile Robot Kinematics



Inertial Frame

State vector in inertial frame

$$\xi_I = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

Rotational transform

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mapping motion $\rightarrow \dot{\xi}_R = R(\theta)\dot{\xi}_I$

Example of Mapping Motion through Transforms

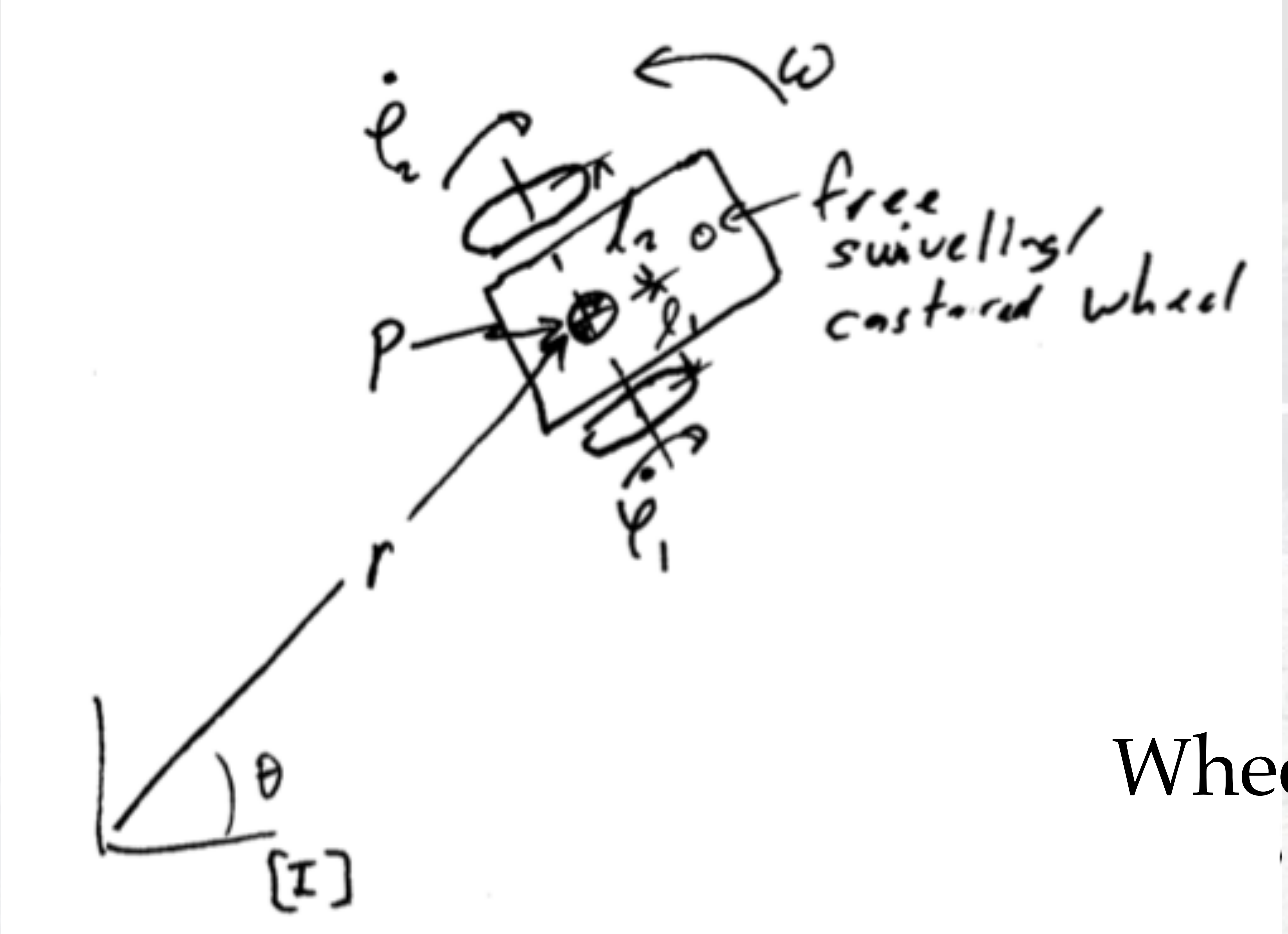
Example: $\theta = \frac{\pi}{2}$ $$R\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Given a desired inertial velocity $\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$

In rover coordinates $\dot{\xi}_R = R\left(\frac{\pi}{2}\right)\dot{\xi}_I = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -\dot{x} \\ \dot{\theta} \end{bmatrix}$

Forward Kinematics

Differential drive rover - control point between two driven wheels



$$\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f\left(l, r, \theta, \dot{\varphi}_1, \dot{\varphi}_2\right)$$

(assumes $l_1 = l_2 = l$)

So in the inertial frame

$$\dot{\xi}_I = R(\theta)^{-1}\dot{\xi}_R$$

Wheels rotate with angular velocities $\dot{\varphi}_1$ and $\dot{\varphi}_2$

If only wheel 1 rotates, vehicle pivots about wheel 2

Resultant base rotation rate $\omega_1 = \dfrac{r\dot{\varphi}_1}{2l}$

$\Longleftarrow$ Velocity of wheel rim

$\Longleftarrow$ Radius to pivot point

Similarly, $\omega_2 = \dfrac{-r\dot{\varphi}_2}{2l}$

$$\dot{\xi}_I = R(\theta)^{-1} \begin{bmatrix} \dfrac{r\dot{\varphi}_1}{2} + \dfrac{r\dot{\varphi}_2}{2} \\ 0 \\ \dfrac{r\dot{\varphi}_1}{2l} - \dfrac{r\dot{\varphi}_2}{2l} \end{bmatrix} \begin{array}{l} \Longleftarrow \dot{x} \\ \Longleftarrow \dot{y} \\ \Longleftarrow \dot{\theta} \end{array}$$

$$R(\theta)^{-1} = R(\theta)^\top = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Previous example: $\theta = \dfrac{\pi}{2} \quad r = 1 \quad l = 1$

Assume $\dot{\varphi}_1 = 4$ and $\dot{\varphi}_2 = 2$

$$\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \Longleftarrow$$

Moves in direction y_I with speed 3 and rotation rate 1

Units? $l_1 r = \langle m \rangle \quad \varphi_1, \varphi_2 = \langle rad/sec \rangle \Longrightarrow \dot{y}_I = 3\ m/sec, \quad \dot{\theta}_I = 1\ rad/sec$

Kinematic Constraints for Various Wheel Types

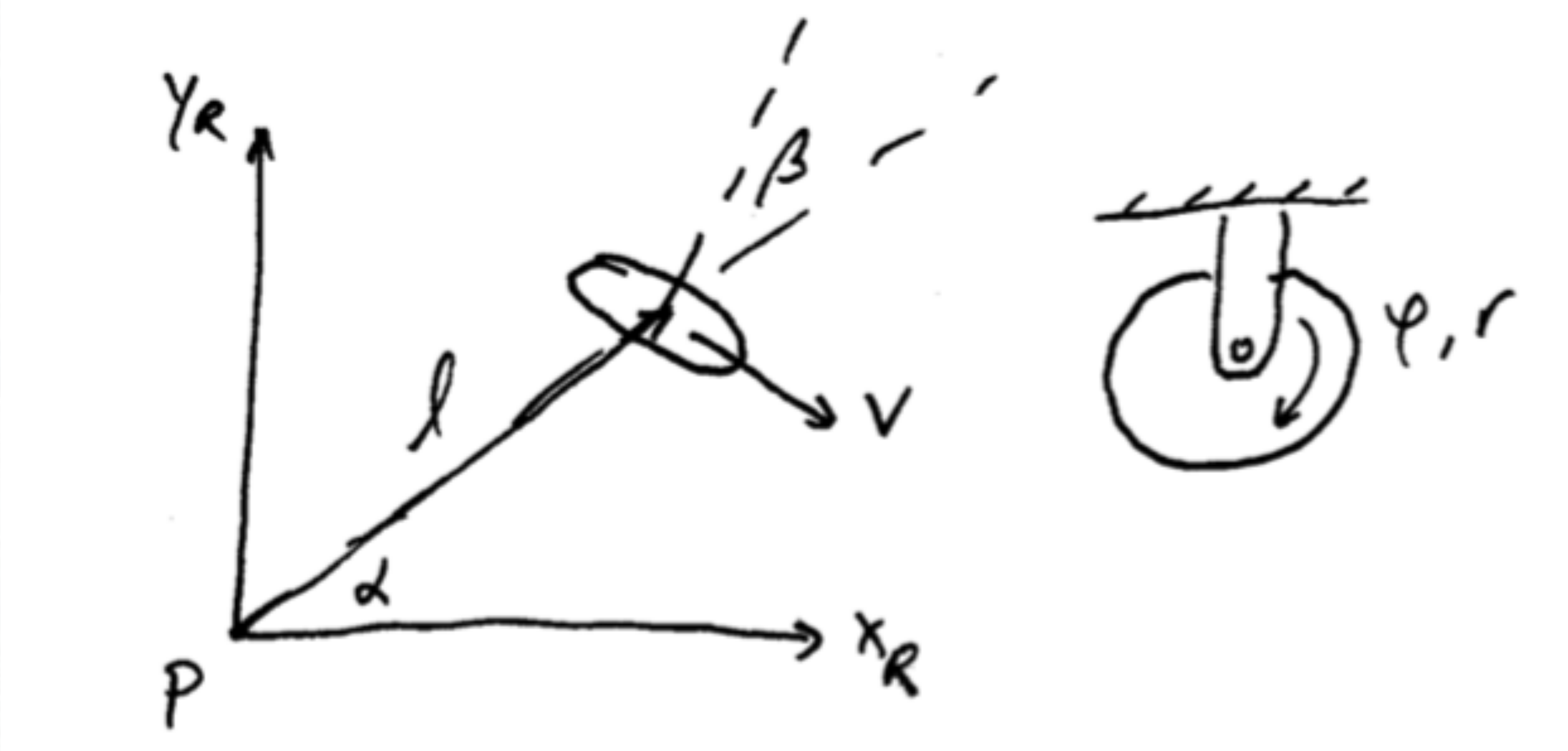
- Assumption: wheels roll but do not skid (laterally)
- Types of wheel mounts
  - Fixed
  - Castering
  - Steered
  - Spherical or Omni-wheel

Fixed/Steered Standard Wheel

In rolling direction, $[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad -l\cos\beta]R(\theta)\dot{\xi}_I = r\dot{\varphi}$

In transverse direction, $\left[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad l\sin\beta\right]R(\theta)\dot{\xi}_I = 0$

Steered standard wheel $\Longrightarrow$ same as fixed except $\beta = f(t) \Rightarrow \beta(t)$



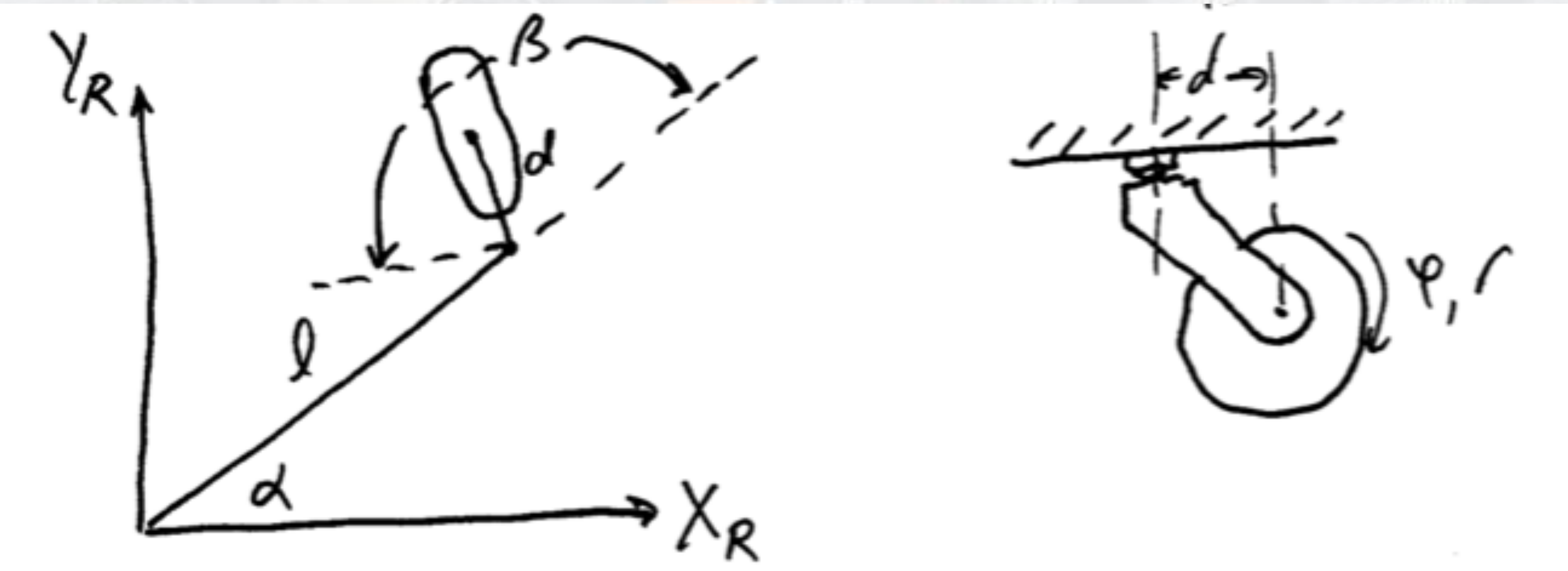
(Defining wheel positions in polar coordinates)

Castered Wheel

In rolling direction, $[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad -l\cos\beta]R(\theta)\dot{\xi}_I = r\dot{\varphi}$

(Same as before, but β is controlled by side loads)

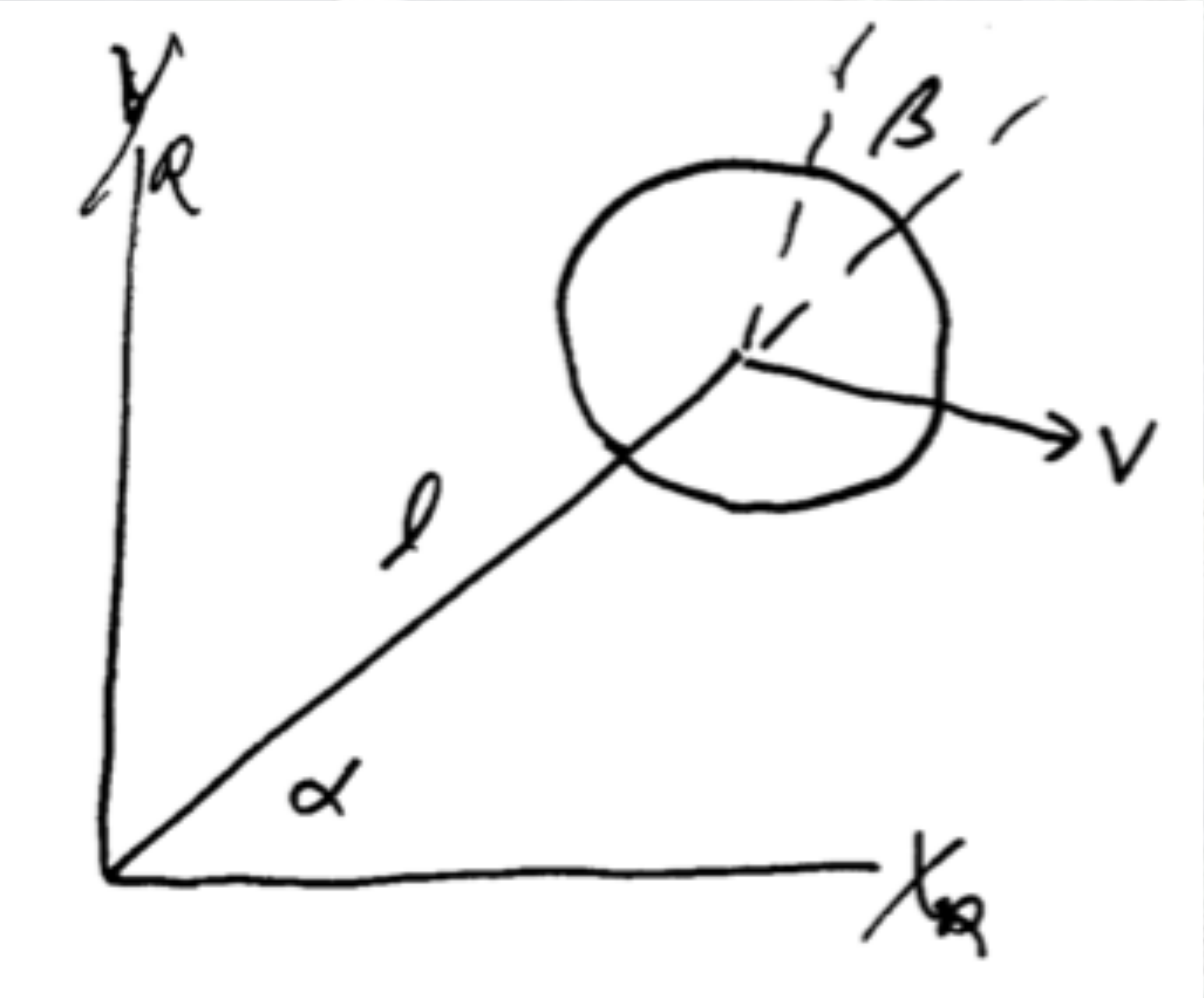
$$[\cos(\alpha+\beta) \quad \sin(\alpha+\beta)d \quad l\sin\beta]R(\theta)\dot{\xi}_I + d\dot{\beta} = 0$$



Spherical Wheels

$$\begin{bmatrix}\sin(\alpha+\beta) & -\cos(\alpha+\beta) & -l\cos\beta\end{bmatrix} R(\theta)\dot{\xi}_I = r\dot{\varphi}$$

$$\begin{bmatrix}\cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta\end{bmatrix} R(\theta)\dot{\xi}_I = 0$$



β is a free variable - example: if v is in the direction of y_R $\Rightarrow$ $\sin(\alpha+\beta)=0 \Rightarrow \alpha=-\beta$

Robot Kinematics – Constraints

- Spherical wheels and casters do not impose kinematic constraints
  - Only fixed and steered wheels control kinematics
- Assume N standard wheels
  - N_f fixed
  - N_s steered
- $\beta_s(t)$ are steering angles of N_s [drive angle of $\varphi_s(t)$]
- β_f are orientations of N_f [drive angle of $\varphi_f(t)$]

$$\varphi(t) = \begin{bmatrix} \varphi_f(t) \\ \varphi_s(t) \end{bmatrix}$$

Rolling Constraint for All Wheels

constraint for rolling wheels

$$J_1\left(\beta_s\right)R(\theta)\dot{\xi}_I - J_2\dot{\varphi} = 0$$

$$J_1\left(\beta_s\right) = \begin{bmatrix} J_{1f} \\ J_{1s}\left(\beta_s\right) \end{bmatrix} \Longleftarrow \text{not a function of } \beta_f$$

J_2 is $N \times N$ diagonal matrix of radii of all N wheels

J_{1F} is the constant matrix of projections for each wheel $(N_f \times 3)$

$$\left[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad -l\cos\beta\right] R(\theta)\ \dot{\xi}_I - r\dot{\varphi} = 0$$

$J_{1s}(\beta_s)$ is $N_s \times 3$ column matrix

$$\left[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad -l\cos\beta\right] R(\theta)\ \dot{\xi}_I - r\dot{\varphi} = 0$$

Constraint for sideslip (or lack thereof)

$$C_1(\beta_s)\,R(\theta)\dot{\xi}_I = 0 \qquad C_1(\beta_s) = \begin{bmatrix} C_1 f \\ C_{1s}(\beta_s) \end{bmatrix}$$

C_{1f} and C_{1s} are $\left(N_f \times 3\right)$ and $\left(N_s \times 3\right)$ matrices

$$[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad -l\cos\beta]\;R(\theta)\;\dot{\xi}_I - r\dot{\varphi} = 0$$

$$[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad l\sin\beta]\;R(\theta)\;\dot{\xi}_I = 0$$

Example: Differential Drive Robot

Rolling constraints J Sliding constraints C

$$\begin{bmatrix} J_1\left(\beta_s\right) \\ C_1\left(\beta_s\right) \end{bmatrix} R(\theta)\ \dot{\xi}_I = \begin{bmatrix} J_2\varphi \\ 0 \end{bmatrix}$$

Caster is unpowered and can move in any traction $\implies$ ignore

Two remaining wheels are fixed (i.e., unsteerable)

so $J_1\left(\beta_s\right) = J_{1f}$ and $C_1\left(\beta_s\right) = C_{1f}$

Travel in $+X_R$ direction

$$\left.\begin{array}{l} \text{right wheel } \alpha = -\frac{\pi}{2} \quad \beta = \pi \\ \text{left wheel } \alpha = \frac{\pi}{2} \quad \beta = 0 \end{array}\right\} \begin{array}{l} \text{produces forward } +X \text{ motion for } \dot{\varphi}_l \text{ and } \dot{\varphi}_r > 0, \\ \text{positive yaw for } \dot{\varphi}_l < 0 \text{ and } \dot{\varphi}_r > 0 \end{array}$$

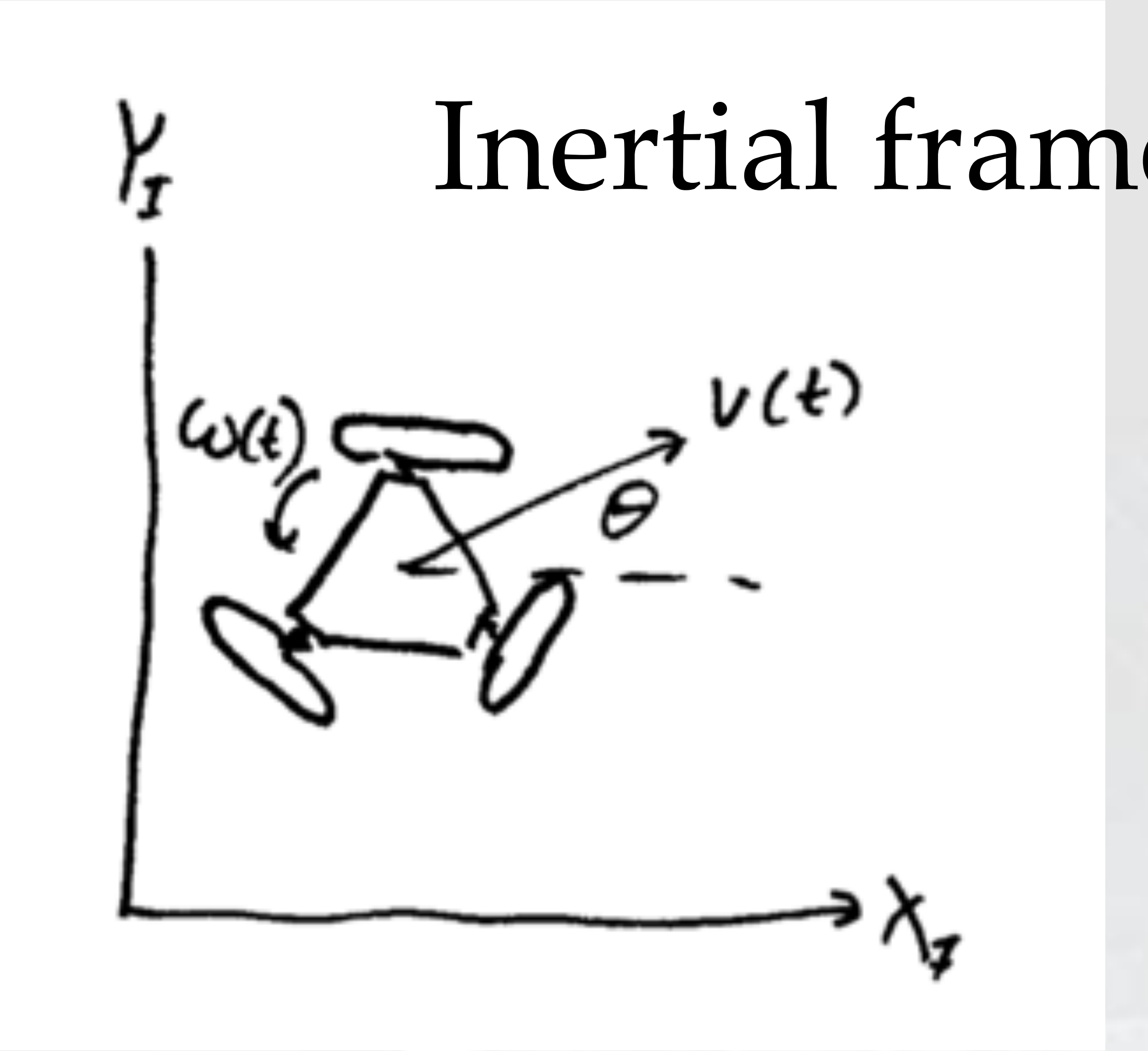
$$\begin{bmatrix} \begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{bmatrix} R(\theta)\dot{\xi}_I = \begin{bmatrix} J_2\dot{\phi} \\ 0 \end{bmatrix} \qquad J_2\dot{e} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix}$$

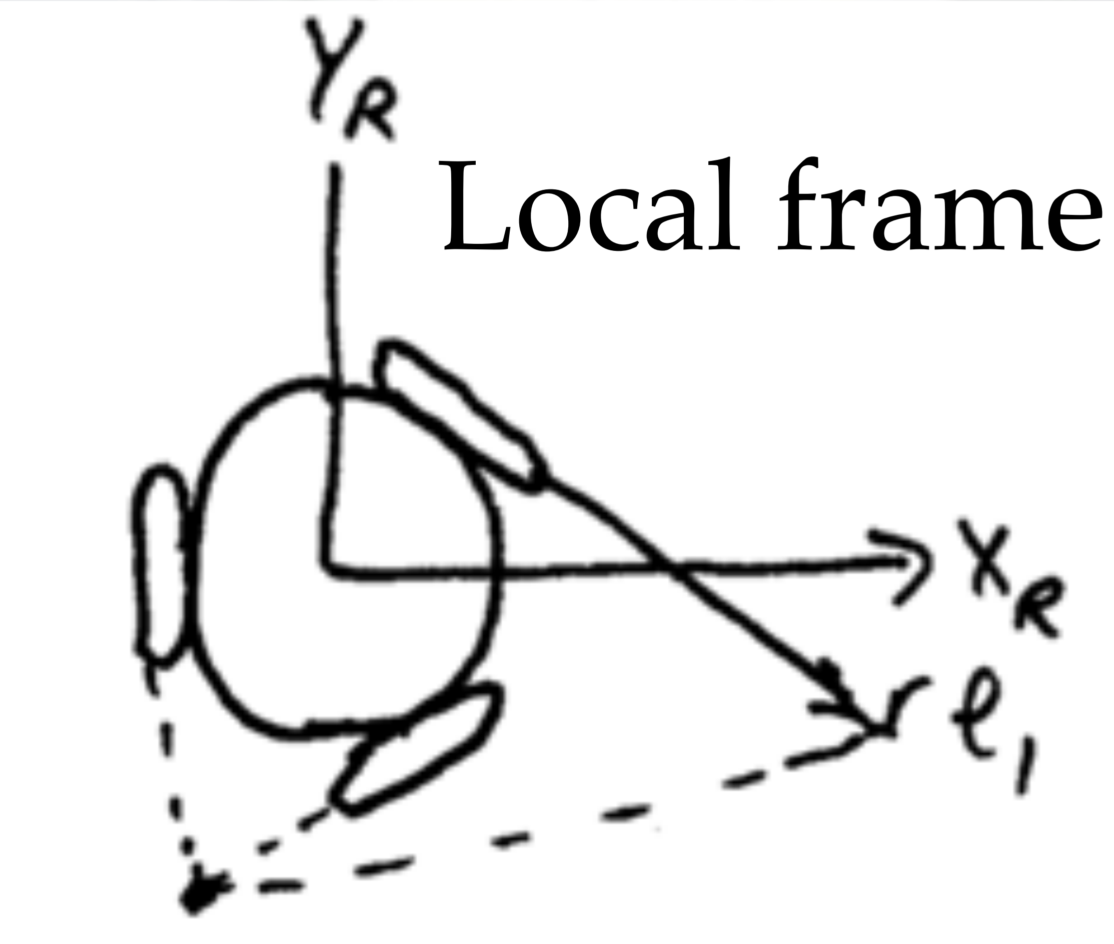
Invert to get kinematic control equation

$$\dot{\xi}_t = R(\theta)^{-1}\begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \\ 0 & 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} J_2\dot{\varphi} \\ 0 \end{bmatrix} = R(\theta)^{-1}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2l} & -\frac{1}{2l} & 0 \end{bmatrix}\begin{bmatrix} J_2\dot{\varphi} \\ 0 \end{bmatrix}$$

$$\text{reminder: } J_2\varphi = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{matrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{matrix} = \begin{bmatrix} r\dot{\varphi}_1 \\ r\dot{\varphi}_1 \end{bmatrix} \implies \begin{bmatrix} J_2\dot{\varphi} \\ 0 \end{bmatrix} = \begin{bmatrix} r\dot{\varphi}_1 \\ r\dot{\varphi}_1 \\ 0 \end{bmatrix}$$

Omniwheel Example





All wheels are the same distance *l* from center with radius *r*

All wheels are fixed

$$\dot{\xi}_I = R(\theta)^{-1} J_{1f}^{-1} J_2 \dot{\varphi}$$

$$\alpha_1 = \frac{\pi}{3} \quad \alpha_2 = \pi \quad \alpha_3 = -\pi/3$$

$\beta = 0$ for all (all wheels tangent to circular body)

$$J_{1f} = \begin{bmatrix} \sin\frac{\pi}{3} & -\cos\frac{\pi}{3} & -l \\ 0 & -\cos\pi & -l \\ \sin\left(-\frac{\pi}{3}\right) & -\cos\left(\frac{\pi}{3}\right) & -l \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -1 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & -1 \end{bmatrix}$$

$$\dot{\xi}_I = R(\theta)^{-1} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3l} & -\frac{1}{3l} & -\frac{1}{3l} \end{bmatrix} J_2\dot{\varphi}$$

J_{1f}^{-1}

$$\begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\varphi}_3 \end{bmatrix}$$

Assume $l = 1$ and $r = 1, \quad \theta = 0$

If $\dot{\varphi} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ what is the robot motion?

$$\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{3}} \\ -\frac{4}{3} \\ -\frac{7}{3} \end{bmatrix} \begin{matrix} \Leftarrow (\mathrm{m/sec}) \\ \Leftarrow (\mathrm{m/sec}) \\ \Leftarrow (\mathrm{rad/sec}) \end{matrix}$$

Maneuverability

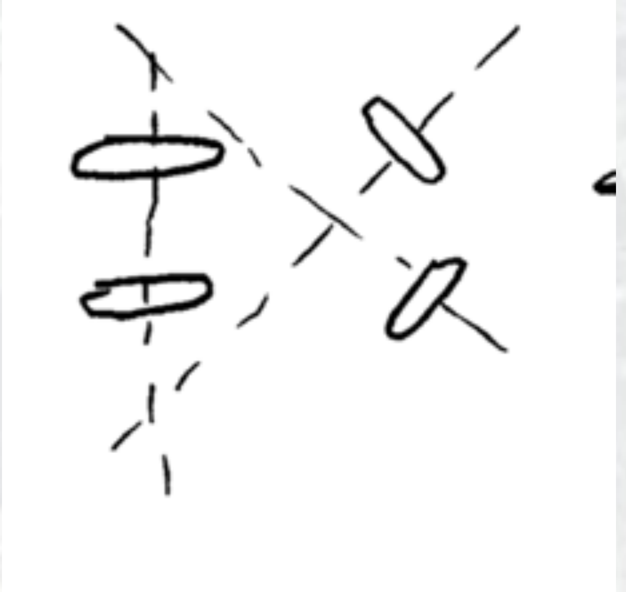
- Robot must move in the environment
- Each wheel must conform to its sliding constraint

$$\text{Start with } C_1\left(\beta_s\right) R(\theta)\dot{\xi}_I = 0$$

$$C_{1f} R(\theta)\dot{\xi}_I = 0$$

$$C_{1s}\left(\beta_s\right) R(\theta)\dot{\xi}_I = 0$$

To be satisfied, $R(\theta)\dot{\xi}_I$ must lie in null space of $C_1\left(\beta_s\right)$



(space N such that for any vector $\bar{n}$ in N, $C_1\left(\beta_s\right)\bar{n} = 0$)

Or, geometrically, all wheel drive axes must pass though the ICR

Consider a unicycle - one fixed wheel

$$C_1(\beta_s) = C_{1f} = \begin{bmatrix}\cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta\end{bmatrix} \qquad \Leftarrow \text{rank } [1]$$

Differential drive robot (two parallel fixed wheels)

Wheel 1: $\alpha_1,\ \beta_1\ l_1$ $\qquad l_1 = l_2 \qquad \alpha_1 + \pi = \alpha_2$

Wheel 2: $\alpha_2,\ \beta_2\ l_2$ $\qquad \beta_1 = \beta_2 = 0$

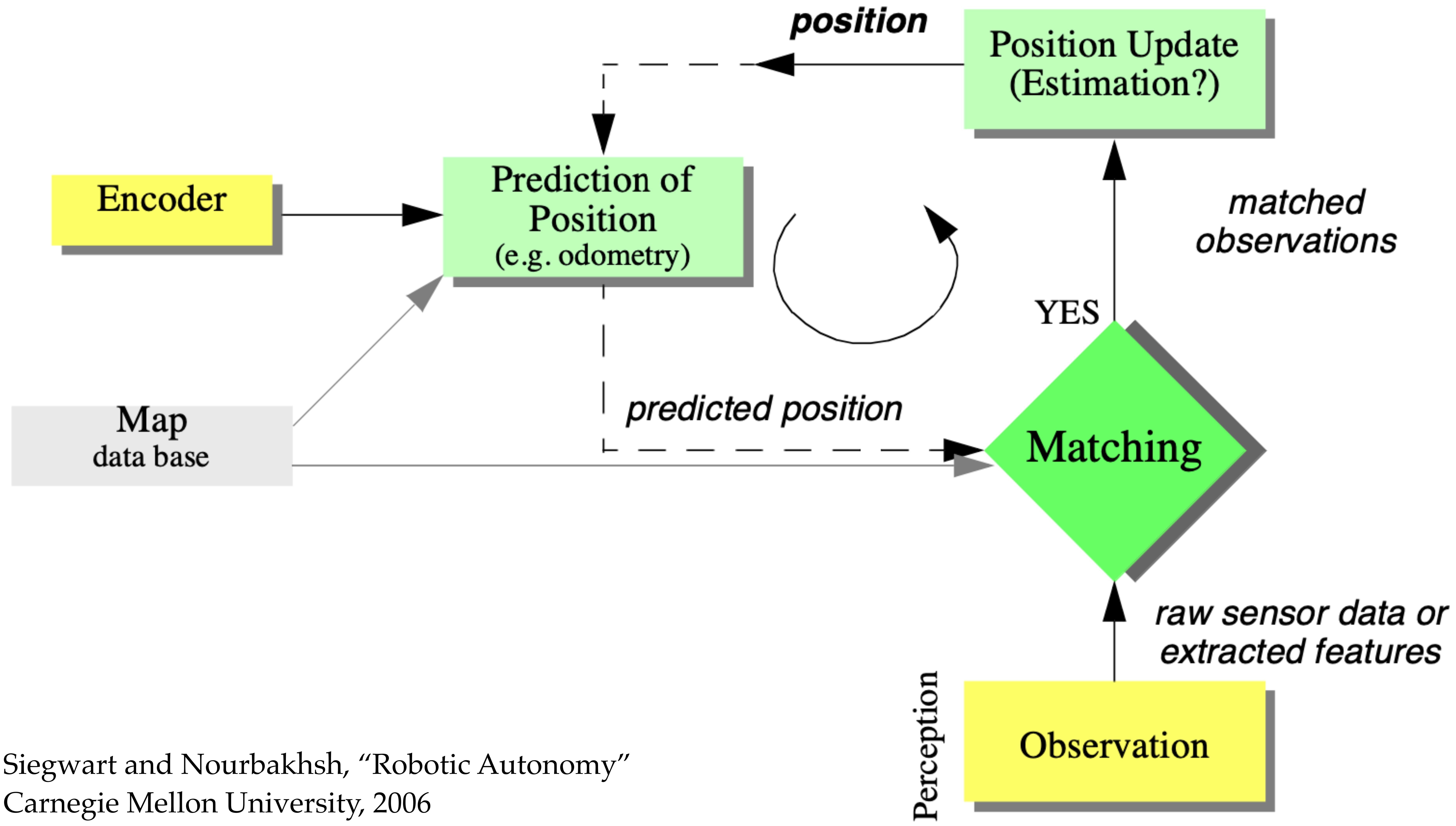
$$C_1(\beta_s) = C_{1f} = \begin{bmatrix}\cos\alpha_1 & \sin\alpha_1 & 0 \\ \cos(\alpha_1+\pi) & \sin(\alpha_1+\pi) & 0\end{bmatrix}$$

$\Leftarrow$ Two constraints, but still rank [1] because they are not independent

If rank [2] $\Rightarrow$ can only move in straight line or along circular arc

If rank [3] $\Rightarrow$ can't move

General Schematic for Localization



Siegwart and Nourbakhsh, “Robotic Autonomy”
Carnegie Mellon University, 2006

Wheel Odometry (Differential Drive)

$$\text{Current position} \implies \xi = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

$$\Delta x = \Delta s \cos\left(\theta + \frac{\Delta\theta}{2}\right) \qquad \Delta y = \Delta s \sin\left(\theta + \frac{\Delta\theta}{2}\right)$$

$$\Delta s = \frac{\Delta s_r + \Delta s_l}{2} \qquad \Delta\theta = \frac{\Delta s_r - \Delta s_l}{c}$$

c = distance between wheels

s_r, s_l = distance traveled by right and left wheels

Updated Position Estimate

$$\xi' = \begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \xi + \begin{bmatrix} \Delta s \cos\left(\theta + \frac{\Delta\theta}{2}\right) \\ \Delta s \sin\left(\theta + \frac{\Delta\theta}{2}\right) \\ \Delta\theta \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta s \cos\left(\theta + \frac{\Delta\theta}{2}\right) \\ \Delta s \sin\left(\theta + \frac{\Delta\theta}{2}\right) \\ \Delta\theta \end{bmatrix}$$

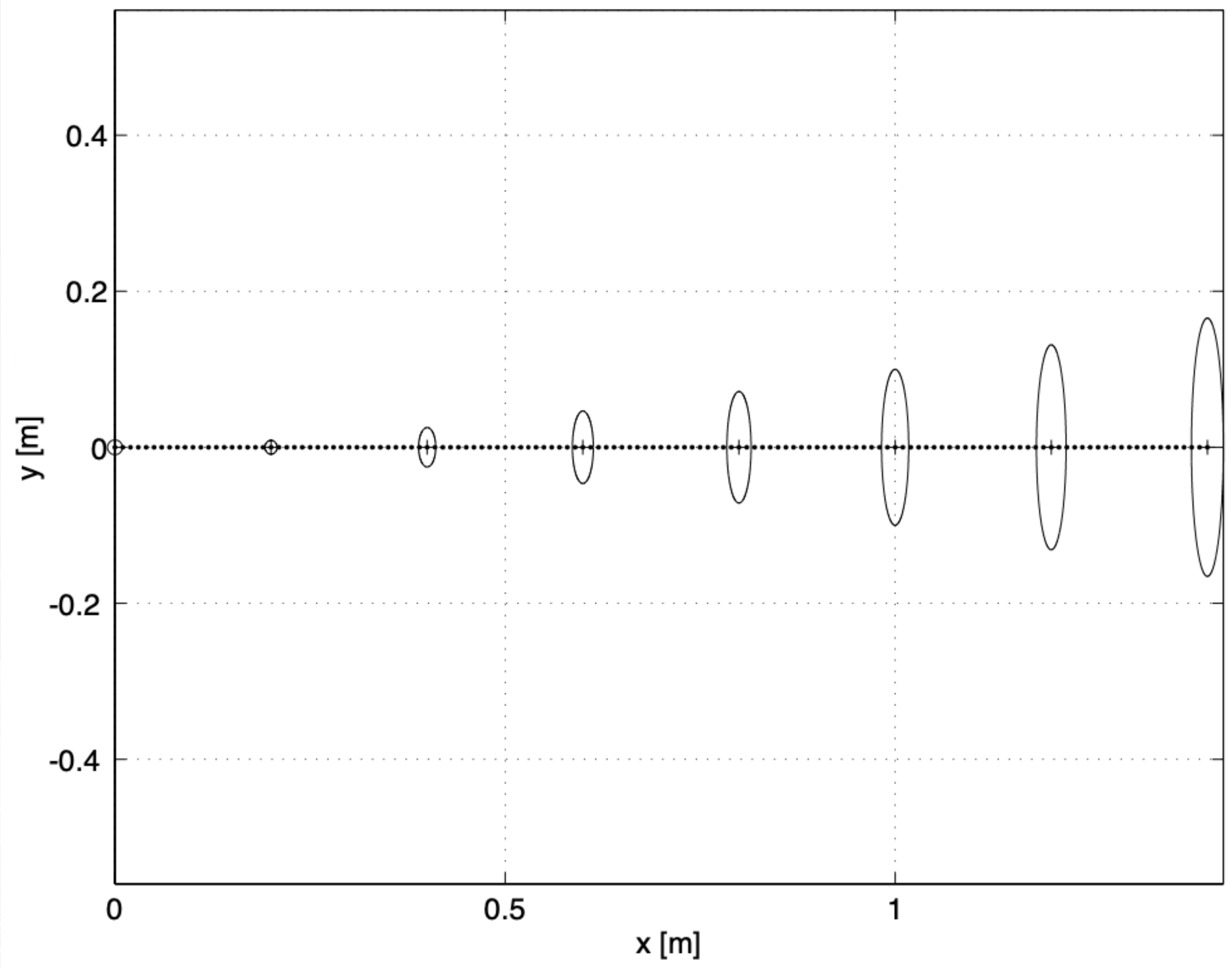
In terms of measurable parameters,

$$\xi' = f\left(x, y, \theta, \Delta s_r, \Delta s_l\right) = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} \frac{\Delta s_r + \Delta s_l}{2} \cos\left(\theta + \frac{\Delta s_r - \Delta s_l}{2c}\right) \\ \frac{\Delta s_r + \Delta s_l}{2} \sin\left(\theta + \frac{\Delta s_r - \Delta s_l}{2c}\right) \\ \frac{\Delta s_r - \Delta s_l}{c} \end{bmatrix}$$

Error Sources in Wheel Odometry

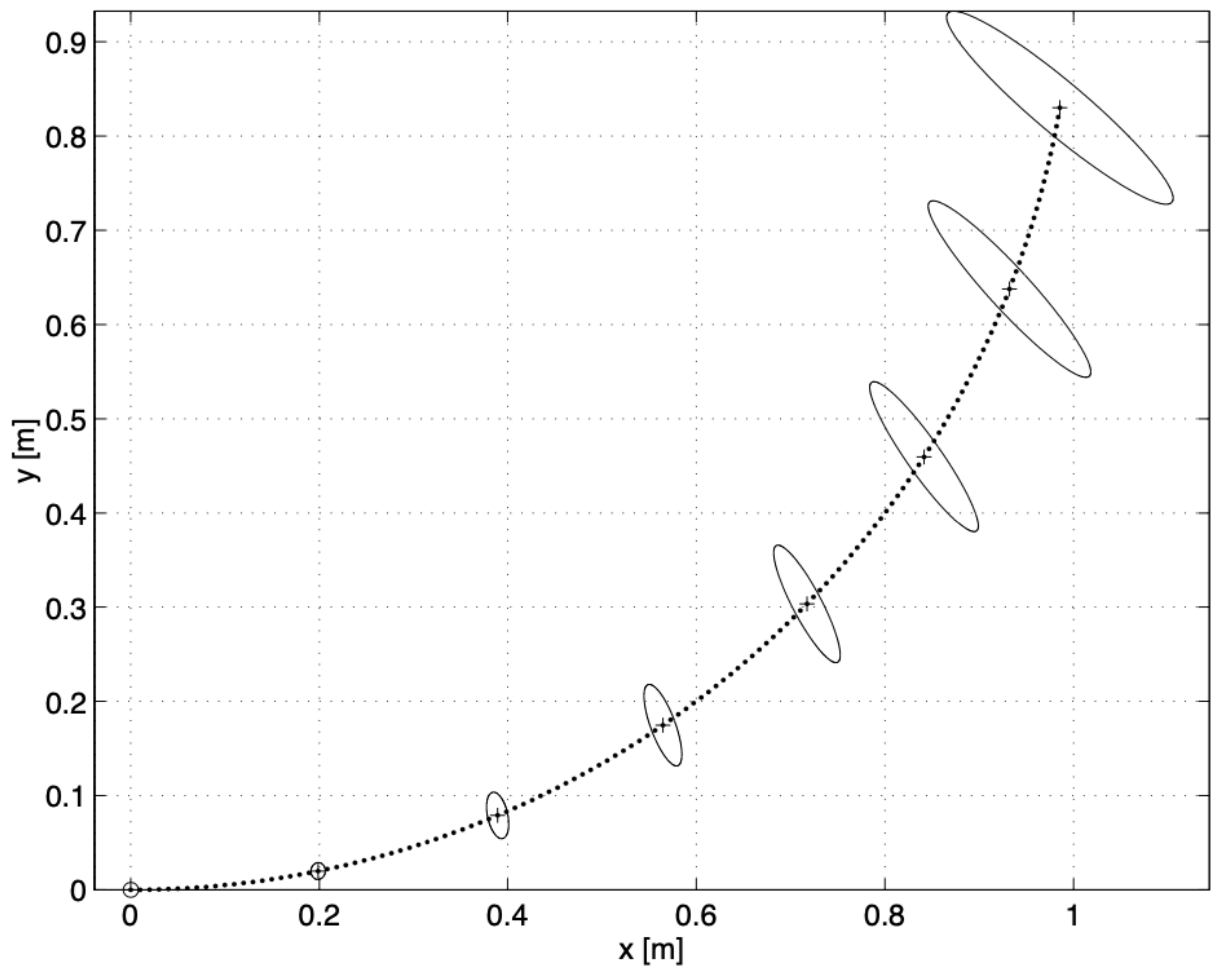
- Limited sampling resolution (e.g., time increments, measurement resolution)
- Misalignment of wheels
- Unequal wheel diameter
- Variation in contact point of wheels
- Unequal floor contact
- Slippage

Error Growth in Straight Line Motion



Siegwart and Nourbakhsh, "Robotic Autonomy" Carnegie Mellon University, 2006

Error Growth in Circular Motion



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