Terramechanics 1: Passive Rolling Resistance

- Origin and nature of lunar soil
- Soil mechanics
- Wheel-soil interactions
- Effects of tandem wheels

Lunar Regolith

- Broken down from larger pieces over time
- Major constituents
  – Rock fragments
  – Mineral fragments
  – Glassy particles
- Local environment
  – 10^{-12} torr (= 1.22×10^{-10} Pa = 1.93×10^{-14} psi)
  – Meteorites at velocities >10^5 m/sec
  – Galactic cosmic rays, solar particles
  – Temperature range +250°F – -250°F

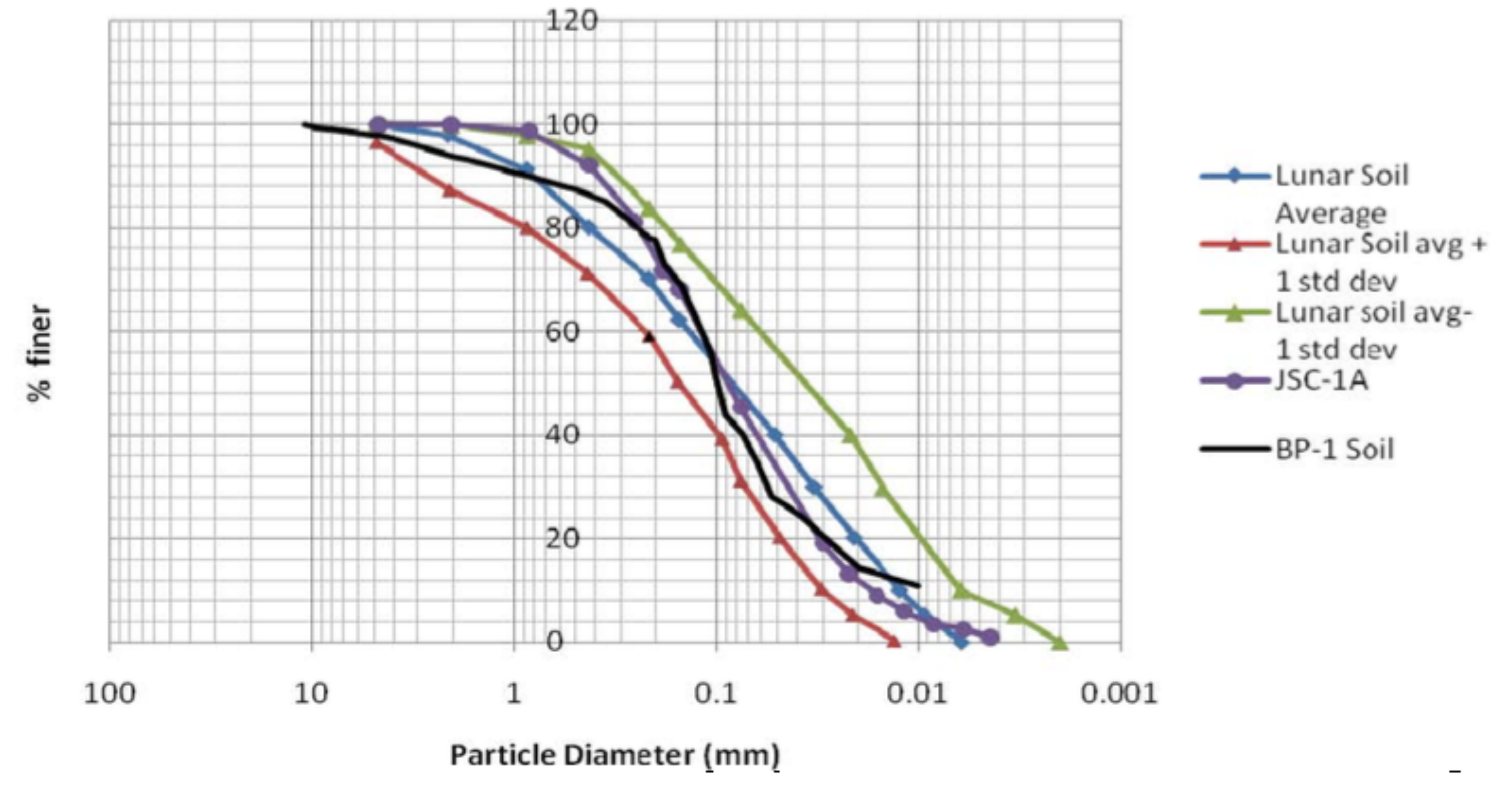
Regolith Creation Process

- Only “weathering” phenomenon on the moon is meteoritic impact
- Weathering processes
  - Comminution: breaking rocks and minerals into smaller particles
  - Agglutination: welding fragments together with molten glass formed by impact energy
  - Solar wind spallation and implantation (minuscule)
  - Fire fountaining (dormant)

JSC-1 Simulant

- Ash vented from Merriam Crater in San Francisco volcano field near Flagstaff, AZ
- K-Ar dated at 150,000 years old ± 30,000
- Major constituents SiO_2, TiO_2, Al_2O_3, Fe_2O_3, FeO, MgO, CaO, Na_2O, other <1%
- Represents low-Ti regolith from lunar mare
- MLS-1 simulant (U.Minn.) preferred for simulation of highland material
- BP-1 (Flagstaff, AZ) is ground basaltic lava - higher fidelity because of angular grain shapes

Particle Size Distribution in Regolith



Rahmatian and Metzger, "Soil Test Apparatus for Lunar Surfaces" *Earth and Space 2010*, ASCE

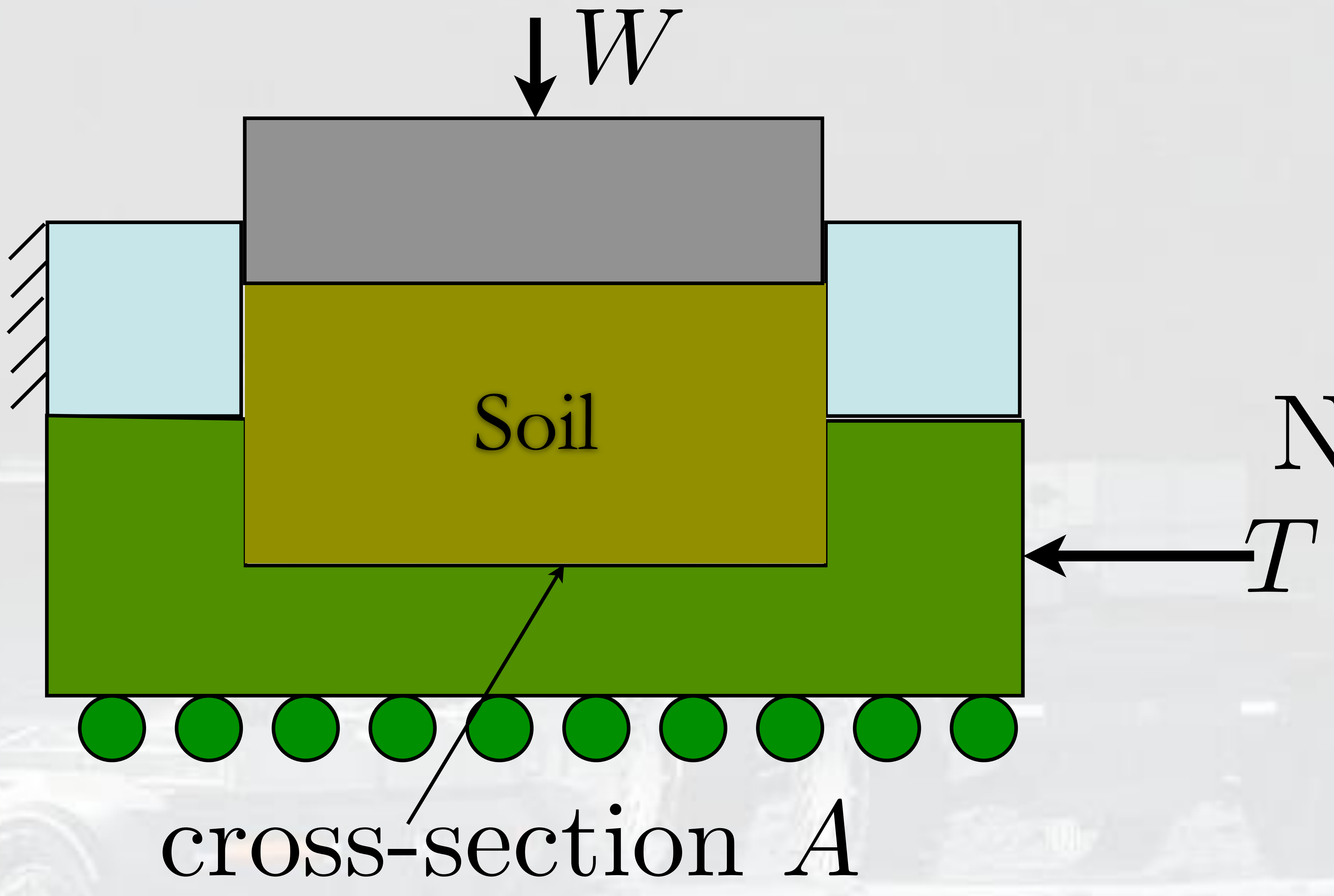
Soil Testing Apparatus

Bevameter (force vs. displacement)

Internal friction angle φ

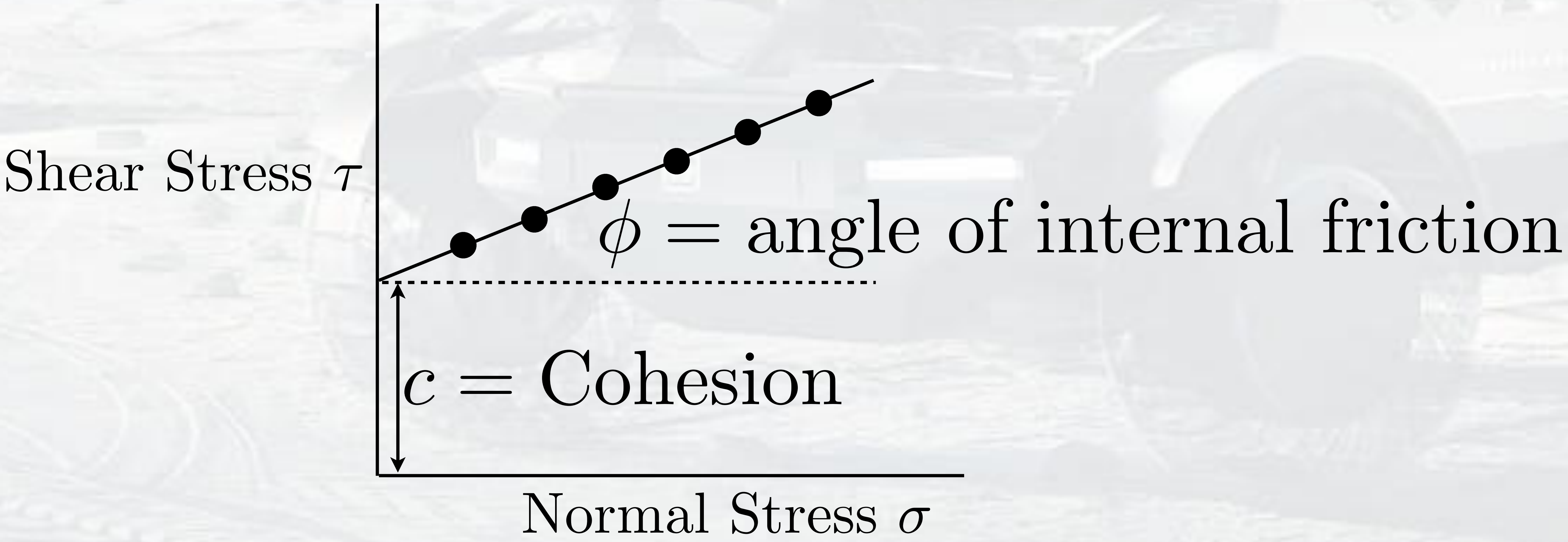
Shear deformation modulus K

Soil Characterization – Direct Shear



$$\text{Shear Stress } \tau = \frac{T}{A}$$

$$\text{Normal Stress } \sigma = \frac{W}{A}$$



Modeling Soil Reaction to a Wheel

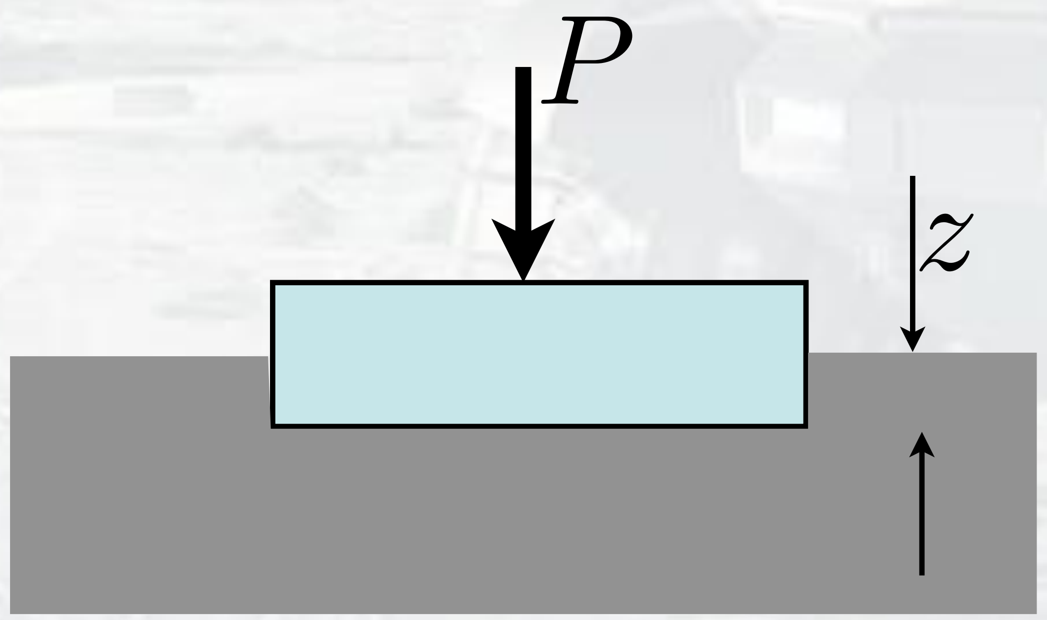
Assume soil reaction is like a (nonlinear) spring

$$P = kz^n$$

$$P = \text{applied pressure}$$

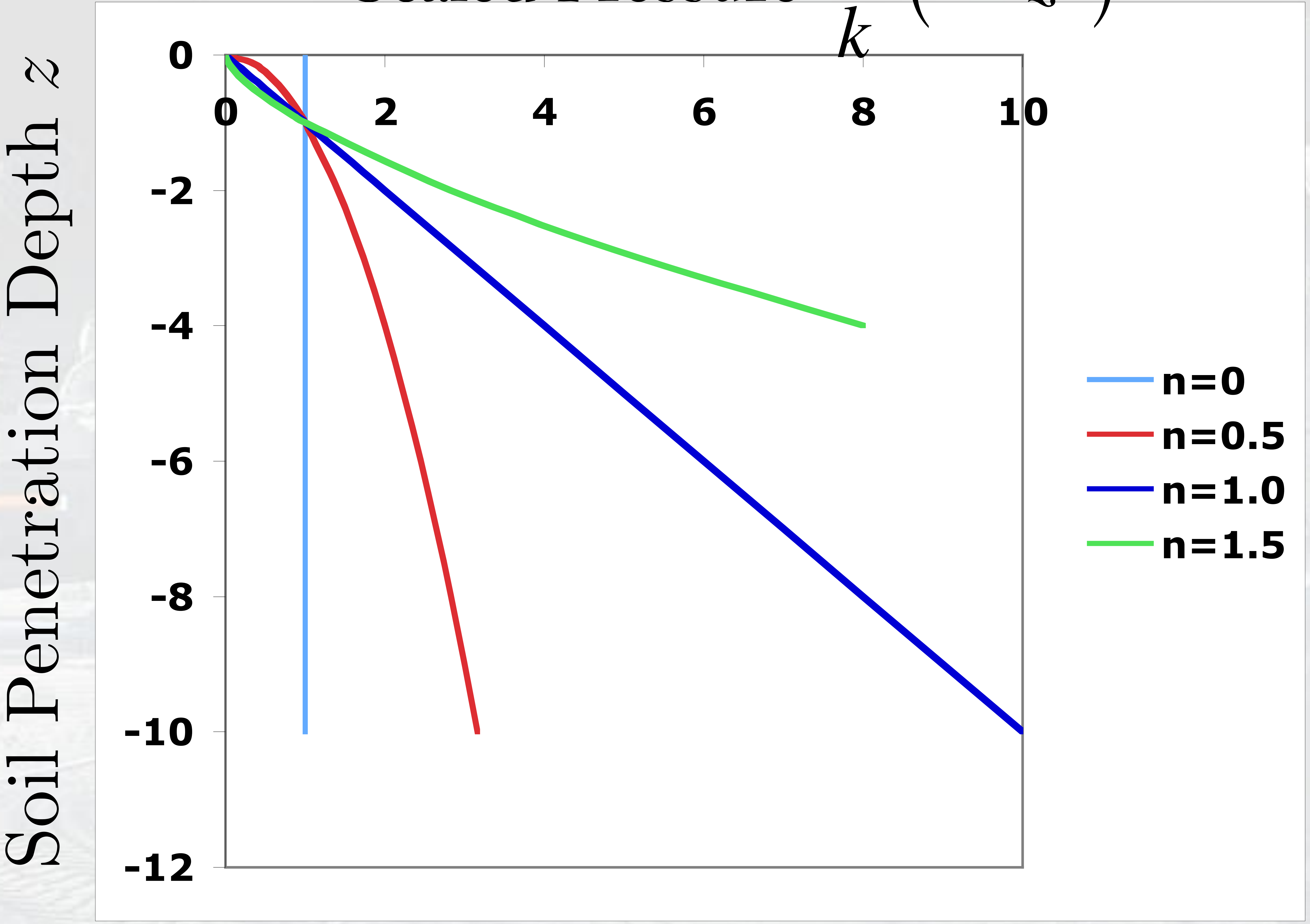
$$z = \text{compression depth}$$

$$k, n = \text{heuristic parameters}$$

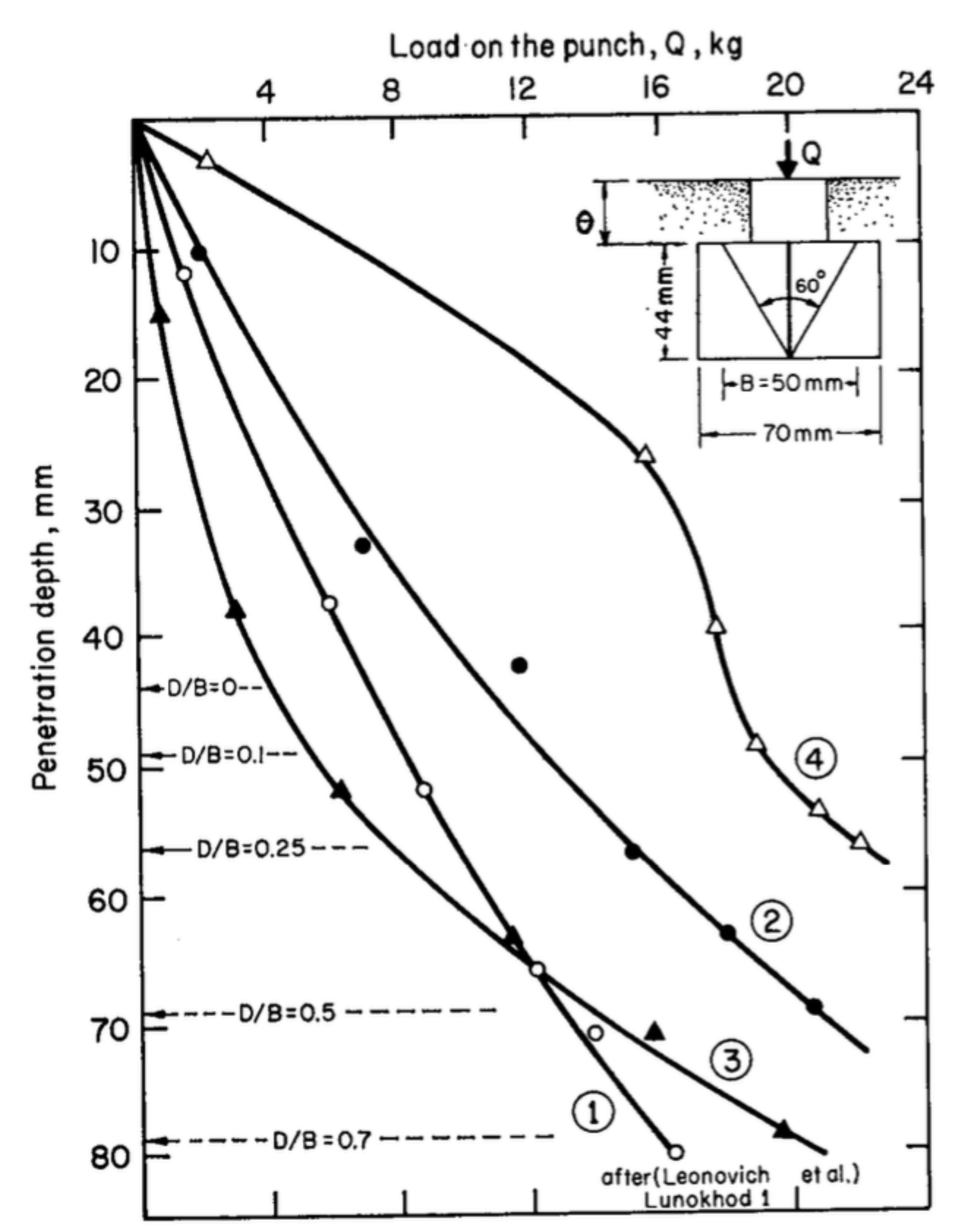


Effects of Soil Mechanics

Scaled Pressure $\frac{P}{k} \left(= z^n \right)$



Lunokhod 1 Penetrometer Data

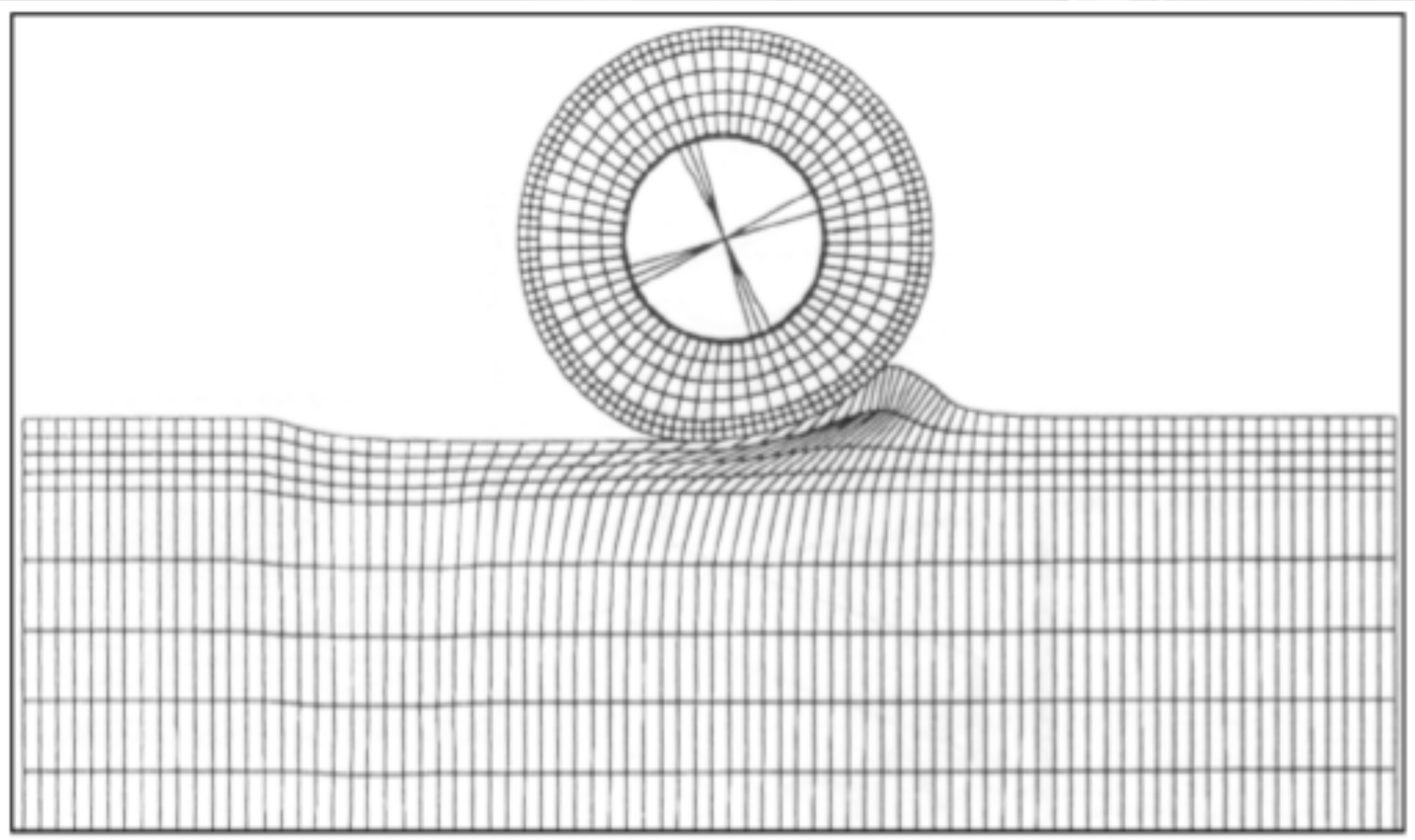


Mitchell et.al., Mechanical Properties of Lunar Soil: Density, Porosity, Cohesion, and Angle of Internal Friction"
Proceedings of the Third Lunar Science Conference, Vol. 3, MIT Press, 1972

Wheel-Soil Interaction

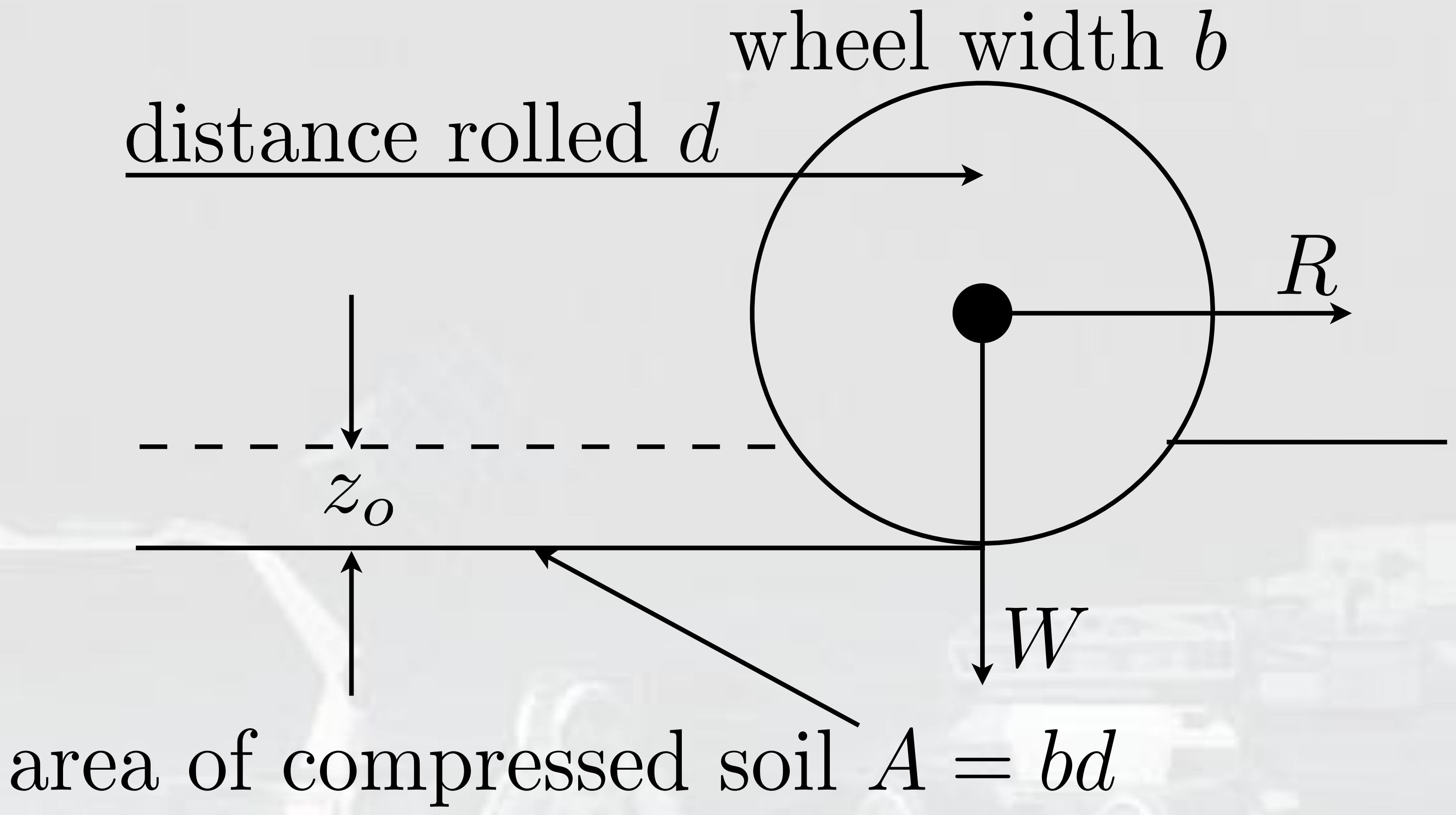
Wheel rolling over soil does physical work

- Compression
- "Bulldozing"



from Gibbesch and Schafer, "Advanced and Simulation Methods of Planetary Rover Mobility on Soft Terrain" *8th ESA Workshop on Advanced Space Technologies for Robotics and Automation*, Noordwijk, The Netherlands, November, 2004

Wheel-Soil Interactions



$$\text{Displacement Energy } \frac{E}{A} = \int \frac{F}{A} dz = \int P dz$$

$$\frac{E}{A} = \int_0^{z_o} P dz = \int_0^{z_o} k z^n dz = k \frac{z_o^{n+1}}{n+1}$$

Rolling Resistance

$$\text{Total Energy } \frac{E}{A}A = \frac{E}{A}bd = k\frac{z_o^{n+1}}{n+1}bd$$

Given a force resisting rolling $\equiv R$,

the energy required to roll a distance d is

$$E_{roll} = Rd$$

$$E_{roll} = E_{displacement} \Rightarrow Rd = \frac{E}{A}bd$$

Rolling Resistance

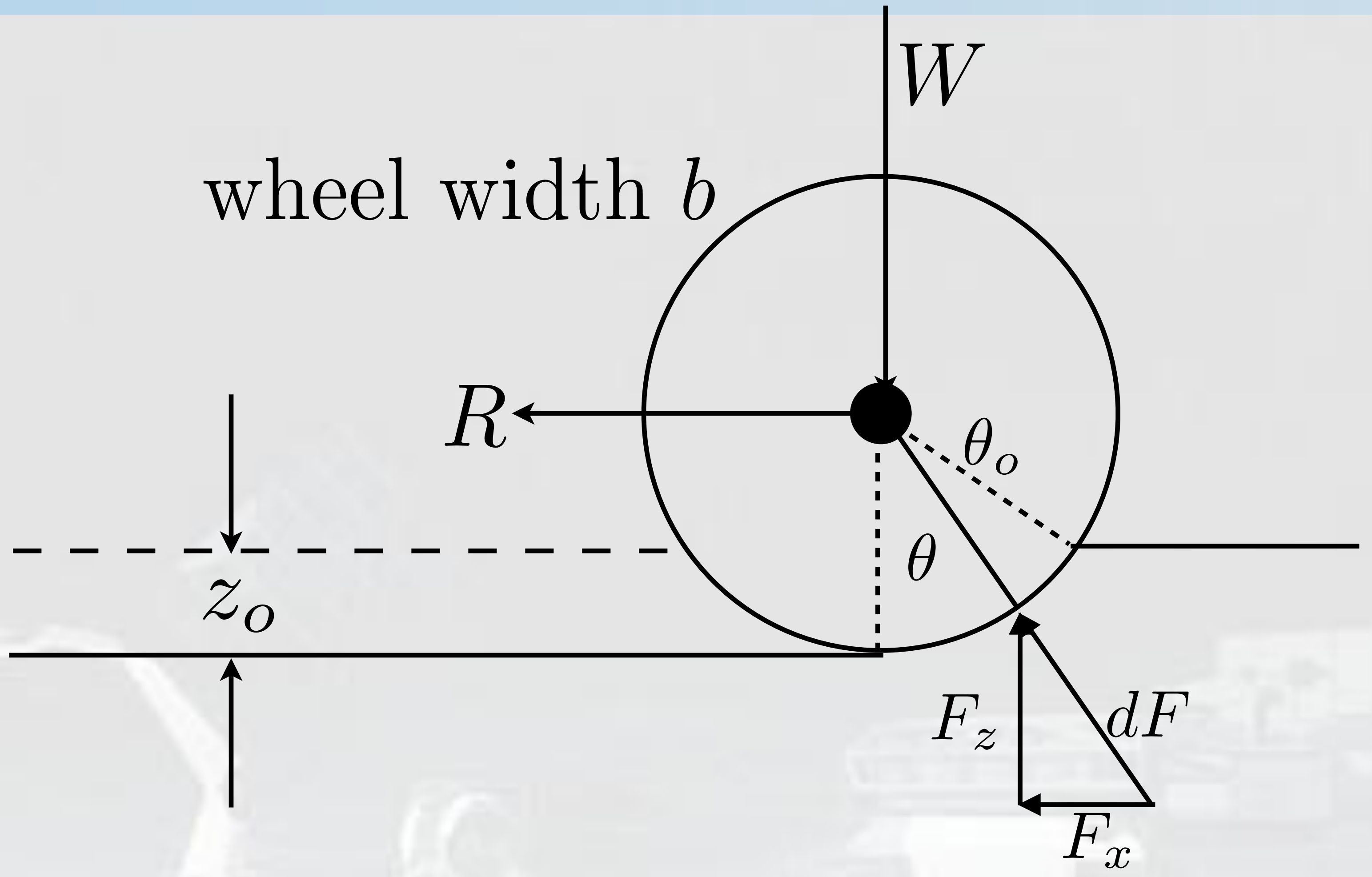
$$\text{For } n=1: P=kz;\ \frac{E}{A}=k\frac{z_o^2}{2};\ R=\frac{1}{2}kbz_o^2$$

$$\text{For } n=\frac{1}{2}: P^2=k^2z;\ \frac{E}{A}=\frac{2}{3}kz_o^{\frac{3}{2}};\ R=\frac{2}{3}kbz_o^{\frac{3}{2}}$$

$$\text{For } n=0: P=k;\ \frac{E}{A}=kz_o;\ R=kbz_o$$

$$\text{Generic case: } P=kz^n;\ \frac{E}{A}=k\frac{z_o^{n+1}}{n+1};\ R=kb\frac{z_o^{n+1}}{n+1}$$

Soil Displacement Calculations



$$R - \int_0^{\theta_o} dF \sin\theta = 0$$

$$-W + \int_0^{\theta_o} dF \cos\theta = 0$$

$$dF = Pbr\ d\theta$$

$$dF \cos\theta = -Pb\ dx$$

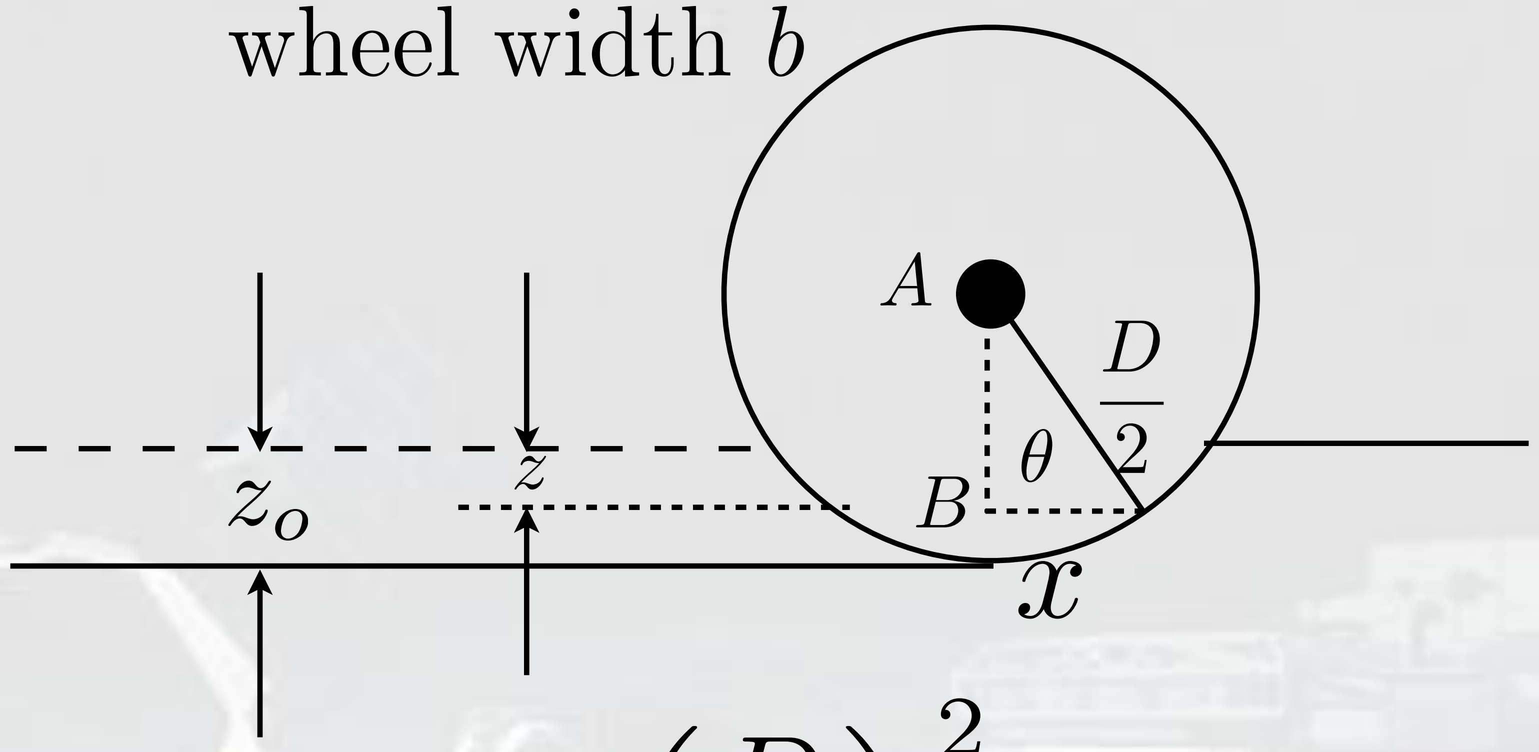
$$dF \sin\theta = Pb\ dz$$

$$R = \int_0^{\theta_o} Pb\ dz \qquad W = -\int_0^{\theta_o} Pb\ dx$$

In general, $P = kx^n$

$$W = -\int_0^{z_o} bkz^n dx$$

Soil Displacement Calculations



$$\bar{AB} = \frac{D}{2} - (z_o - z)$$

$$x^2 = \left(\frac{D}{2}\right)^2 - \bar{AB}^2 = \left(\frac{D}{2}\right)^2 - \left[\frac{D}{2} - (z_o - z)\right]^2$$

$$= \left(\frac{D}{2}\right)^2 - \left(\frac{D}{2}\right)^2 + 2\frac{D}{2}(z_o - z) - (z_o - z)^2$$

$$x^2 = [D - (z_o - z)](z_o - z)$$

Soil Compression Calculations

But $D \gg z_o - z$

$$x^2 \approx D(z_o - z) \Rightarrow 2x\ dx = -D\ dz$$

$$\text{so from } W = -\int_0^{z_o} bkz^n dx \text{ we get } W = -\int_0^{z_o} bkz^n \frac{-D}{2x} dz$$

$$W = -bk \int_0^{z_o} z^n \left(\frac{-D}{2\sqrt{D}\sqrt{z_o - z}} \right) dz$$

$$W = bk \int_0^{z_0} \left(\frac{\sqrt{D}}{2\sqrt{z_0 - z}} \right) z^n dz$$

Soil Displacement Calculations

$$\text{Define } z_o - z \equiv t^2 \Rightarrow dz = -2t\,dt$$

$$W = bk\sqrt{D}\int_0^{\sqrt{z_o}} (z_o - t^2)^n\,dt$$

$$\text{Taylor Series expansion } (z_o - t^2)^n \cong z_o^n - n z_o^{n-1} t^2 + \cdots$$

$$W \approx \frac{bk\sqrt{Dz_o}}{3} z_o^n (3 - n)$$

$$\text{for } n = 1 \Rightarrow W = \frac{2}{3} bk z_o \sqrt{Dz_o}$$

$$\text{for } n = \frac{1}{2} \Rightarrow W = \frac{5}{6} bk z_o \sqrt{D}$$

$$\text{for } n = 0 \Rightarrow W = bk\sqrt{Dz_o}$$

Rolling Resistance as f(W)

$$\text{for } n = 0 \Rightarrow W = bk\sqrt{Dz_o} \Rightarrow z_o = \left(\frac{W}{bk}\right)^2 \frac{1}{D}$$

$$R = kbz_o \Rightarrow R = \frac{kb}{(kb)^2}\frac{W^2}{D} \Rightarrow R = \frac{W^2}{kbD}$$

$$\text{for } n = \frac{1}{2} \Rightarrow W = \frac{5}{6}bkz_o\sqrt{D} \Rightarrow z_o = \frac{6}{5}\frac{W}{bk\sqrt{D}}$$

$$R = \frac{2}{3}kbz_o^{\frac{3}{2}} \Rightarrow R = \frac{2}{3}kb\left(\frac{6}{5}\frac{W}{kb\sqrt{D}}\right)^{\frac{3}{2}} = \frac{2}{3}\left(\frac{6}{5}\right)^{\frac{3}{2}}\frac{W^{\frac{3}{2}}}{\sqrt{kb}D^{\frac{3}{4}}}$$

$$R = 0.876\frac{W^{\frac{3}{2}}}{\sqrt{kb}D^{\frac{3}{4}}}$$

Rolling Resistance as f(W)

$$\text{for } n = 1 \Rightarrow W = \frac{2}{3} b k z_o^{\frac{3}{2}} \sqrt{D} \Rightarrow z_o^2 = \left(\frac{3W}{2kb\sqrt{D}} \right)^{\frac{4}{3}}$$

$$R = \frac{1}{2} k b z_o^2 \Rightarrow R = \frac{1}{2} k b \left(\frac{3W}{2kb\sqrt{D}} \right)^{\frac{4}{3}} = \frac{1}{2} \left(\frac{3}{2} \right)^{\frac{4}{3}} \left(\frac{W^4}{kbD^2} \right)^{\frac{1}{3}}$$

$$R = 0.859 \left(\frac{W^4}{kbD^2} \right)^{\frac{1}{3}}$$

Rolling Resistance as f(W) (Generic)

$$W = \frac{bk\sqrt{Dz_o}}{3} z_o^n (3-n) = \frac{bk\sqrt{D}}{3} z_o^{n+\frac{1}{2}} (3-n)$$

$$z_o^{n+\frac{1}{2}} = \frac{3}{(3-n)} \frac{W}{bk\sqrt{D}}$$

$$z_o^{n+1} = \left(\frac{3}{3-n} \frac{W}{bk\sqrt{D}}\right)^{\frac{n+1}{n+\frac{1}{2}}} = \left(\frac{3}{3-n} \frac{W}{bk\sqrt{D}}\right)^{\frac{2(n+1)}{2n+1}}$$

$$R = \frac{bk}{n+1} z_o^{n+1} = \frac{bk}{n+1} \left(\frac{3}{3-n} \frac{W}{bk\sqrt{D}}\right)^{\frac{2(n+1)}{2n+1}}$$

$$R = \frac{1}{n+1} \left(\frac{3}{3-n} \frac{W}{\sqrt{D}}\right)^{\frac{2(n+1)}{2n+1}} \left(\frac{1}{bk}\right)^{\frac{1}{2n+1}}$$

Soil Characteristics

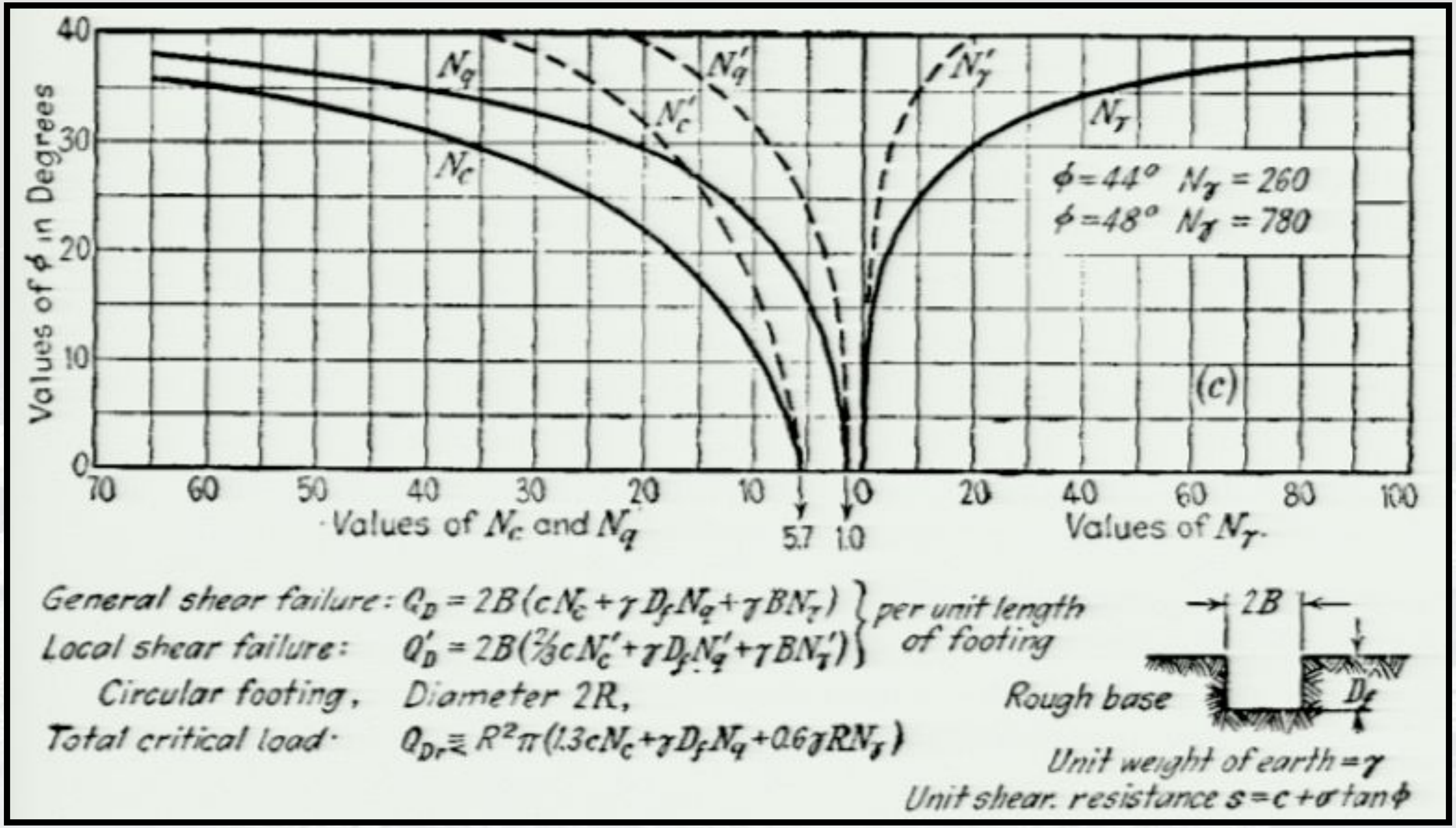
soil type	n	$k_c \ \langle \frac{N}{m^{n+1}} \rangle$	$k_\phi \ \langle \frac{N}{m^{n+2}} \rangle$
Dry Sand	1.1	990	1,528,000
Lunar Regolith	1	1400	820,000
Sandy Loam	0.7	5270	1,515,000
Sandy Loam (MER-B)	1	28,000	7,600,000
Slope Soil (MER-B)	0.8	6800	210,000
Clay (Earth)	0.5	13,190	692,200

More Earth Soil Characteristics

Terrain	n	k_c (kN/m^{n+1})	k_φ (kN/m^{n+2})	c (kPa)	φ (deg)
Dry Sand	1.1	0.99	1528.43	1.04	28
Sandy Loam	0.7	5.27	1515.04	1.72	29
Sandy Loam	0.2	2.56	43.12	1.38	38
Sandy Loam	0.9	52.53	1127.97	4.83	20
Sandy Loam	0.4	11.42	808.96	9.65	35
Sandy Loam	0.3	2.79	141.11	13.79	22
Sandy Loam	0.5	0.77	51.91	5.17	11
Clayey Soil	0.5	13.19	692.15	4.14	13
Clayey Soil	0.7	16.03	1262.53	2.07	10
Heavy Clay	0.13	12.7	1555.95	68.95	34
Heavy Clay	0.11	1.84	103.27	20.69	6
Lean Clay	0.2	16.43	1724.69	68.95	20
Lean Clay	0.15	1.52	119.61	13.79	11
LETE Sand	0.79	102	5301	1.3	31.1
Upland Sandy Loam	1.1	74.6	2080	3.3	33.7
Rubicon Sandy Loam	0.66	6.9	752	3.7	29.8
North Gower Clayey Loam	0.73	41.6	2471	6.1	26.6
Grenville Loam	1.01	0.06	5880	3.1	29.8
Snow (USA)	1.6	4.37	196.72	1.03	19.7
Snow (USA)	1.6	2.49	245.9	0.62	23.2
Snow (Sweden)	1.44	10.55	66.08	6	20.7

D. Michel and K. McIsaac, “New Rocker-Bogie and Terramechanics-Based Wheel/Soil Interaction Models for Planetary Rovers” 2012 International Conference On Mechatronics and Automation

Terzaghi Analysis of Soil Deformation



Terzaghi Soil Bearing Capacity Factors

$$N_q = \frac{\exp\left[\left(\frac{3\pi}{2} - \phi\right)\tan\phi\right]}{2\cos^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right)}$$

$$N_c = \cot\phi\left\{\frac{\exp\left[\left(\frac{3\pi}{2} - \phi\right)\tan\phi\right]}{2\cos^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right)} - 1\right\} = \frac{N_q - 1}{\tan\phi}$$

$$N_\gamma = \frac{2(N_q + 1)\tan(\phi)}{1 + 0.4\sin(4\phi)}$$

ϕ = Angle of internal resistance of soil

Parameter Definition

$$K_c \equiv \text{Cohesive modulus of soil deformation}$$

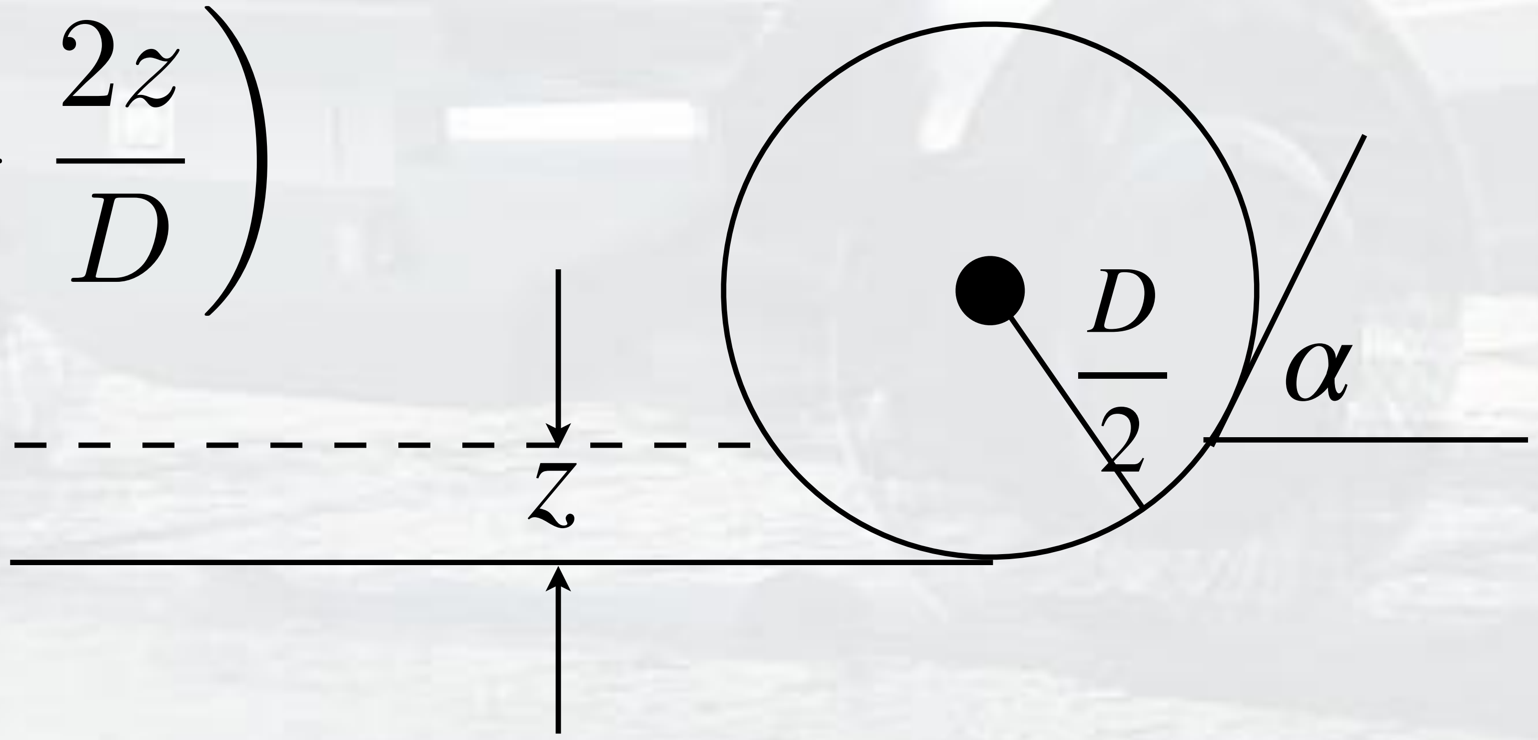
$$K_c = (N_c - \tan\phi)\cos^2\phi$$

$$K_\phi \equiv \text{Frictional modulus of soil deformation}$$

$$K_\gamma = \left[\frac{2N_\gamma}{\tan\phi} + 1\right]\cos^2\phi$$

$$\alpha \equiv \text{Angle of approach of wheel to soil}$$

$$\alpha = \cos^{-1}\left(1 - \frac{2z}{D}\right)$$



$$\gamma \equiv \text{Weight density of soil } \left(\frac{N}{m^3}\right)$$

Lunar Soil Canonical Values

$$\text{Soil density: } \rho = 1600 \ \frac{kg}{m^3} \Longrightarrow \gamma = 2470 \ \frac{N}{m^3}$$

$$n = 1$$

$$k_c = 1400 \ N/m^2$$

$$k_\phi = 830{,}000 \ N/m^3$$

$$\phi = 33^{O} = 0.576 \ rad$$

$$\text{Cohesion } c = 170 \ N/m^2$$

$$N_q = 32.23$$

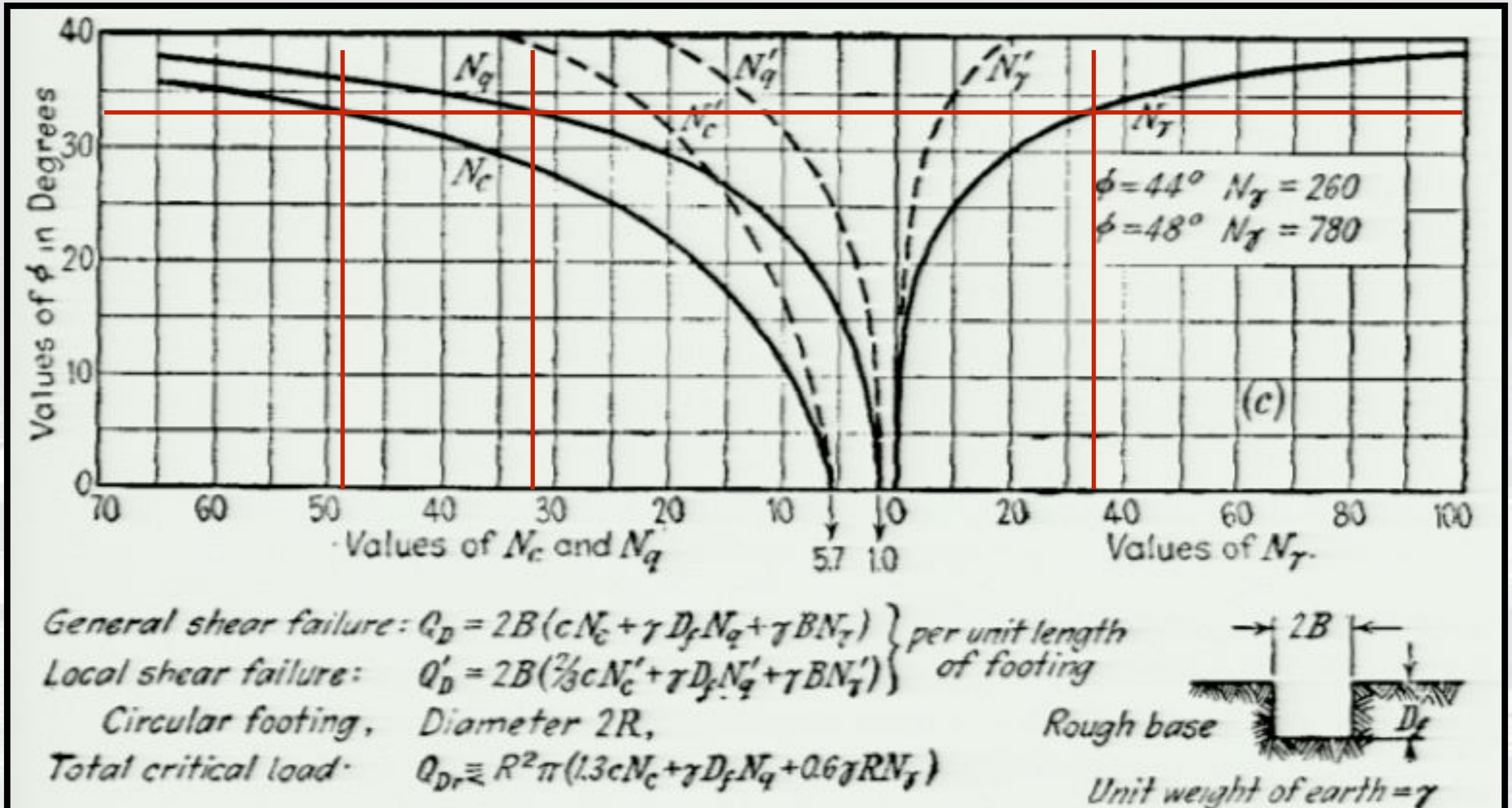
$$N_c = 48.09$$

$$N_\gamma = 33.27$$

$$K_c = 33.37$$

$$K_\phi = 72.77$$

LRV Terzhagi Parameters



Red lines represent values calculated for this example

More Detailed Soil Compression Equation

$$k = \frac{k_c}{b} + k_\phi$$

$$k_c = \text{modulus of cohesion of soil deformation}$$

$$k_c \text{ units} \Rightarrow < N/m^{(n+1)} >$$

$$k_\phi = \text{modulus of friction of soil deformation}$$

$$k_\phi \text{ units} \Rightarrow < N/m^{(n+2)} >$$

$$b = \text{wheel width}$$

$$P = \left(\frac{k_c}{b} + k_\phi\right) z^n$$

Equations for Compression Resistance

From page 21,

$$z = \left(\frac{3}{3-n} \frac{W_w}{bk\sqrt{D}} \right)^{\frac{2}{2n+1}}$$

$$W_w = \text{weight on wheel}$$

$$D = \text{wheel diameter}$$

$$R_c = \left(\frac{bk}{n+1} \right) z^{n+1}$$

$$R_c = \text{compression resistance (per wheel)}$$

$$k = \frac{k_c}{b} + k_\phi \Longrightarrow bk = k_c + bk_\phi$$

Equations for Compression Resistance

$$z = \left(\frac{3}{3-n} \frac{W_w}{(k_c + b k_\phi)\sqrt{D}} \right)^{\frac{2}{2n+1}}$$

$$R_c = \left(\frac{k_c + b k_\phi}{n+1} \right) z^{n+1}$$

$$R_c = \left(\frac{k_c + b k_\phi}{n+1} \right) \left(\frac{3}{3-n} \frac{W_w}{(k_c + b k_\phi)\sqrt{D}} \right)^{\frac{2(n+1)}{2n+1}}$$

Compression Resistance (Lunar Soil)

$$R_c = \frac{1}{n+1}(k_c + bk_\phi)^{\frac{-1}{2n+1}}\left(\frac{3W_w}{(3-n)\sqrt{D}}\right)^{\frac{2(n+1)}{2n+1}}$$

$$n = 1$$

$$z = \left(\frac{3}{2}\frac{W_w}{(k_c + bk_\phi)\sqrt{D}}\right)^{\frac{2}{3}}$$

$$R_c = \frac{1}{2}(k_c + bk_\phi)^{\frac{-1}{3}}\left(\frac{3W_w}{2\sqrt{D}}\right)^{\frac{4}{3}}$$

Relevant LRV Parameters

$$W_w = 370\ lbs = 168\ kg = 259\ N \text{ (on Moon)}$$

$$D = 32\ in = 0.813\ m$$

$$b = 9\ in = 0.229\ m$$

$$z = \left(\frac{3}{2} \frac{259\ N}{\left(1400\ \frac{N}{m^2} + (0.229\ m)830000\ \frac{N}{m^3}\right)\sqrt{0.813\ m}} \right)^{\frac{2}{3}} = 0.0151\ m$$

$$\text{Units check: } \left(\frac{N}{(\frac{N}{m^2} + \frac{Nm}{m^3})\sqrt{m}} \right)^{\frac{2}{3}} = \left(m^{\frac{3}{2}} \right)^{\frac{2}{3}} = m$$

Soil Bearing Limit

Safe weight on the soil

$$W_s = A\left(cN_c + \gamma z N_q + \frac{1}{2}\gamma b N_\gamma\right)$$

Safe soil pressure $P_s = \dfrac{W_s}{A}$

$c \equiv$ Soil cohesion (Pa)

$b \equiv$ Wheel width (m)

For our LRV example case, $P_s = 18{,}790\ Pa$

For $W_v = 1004\ N$, the minimum contact area is $534\ cm^2$ or $\ell > 5.8\ cm$

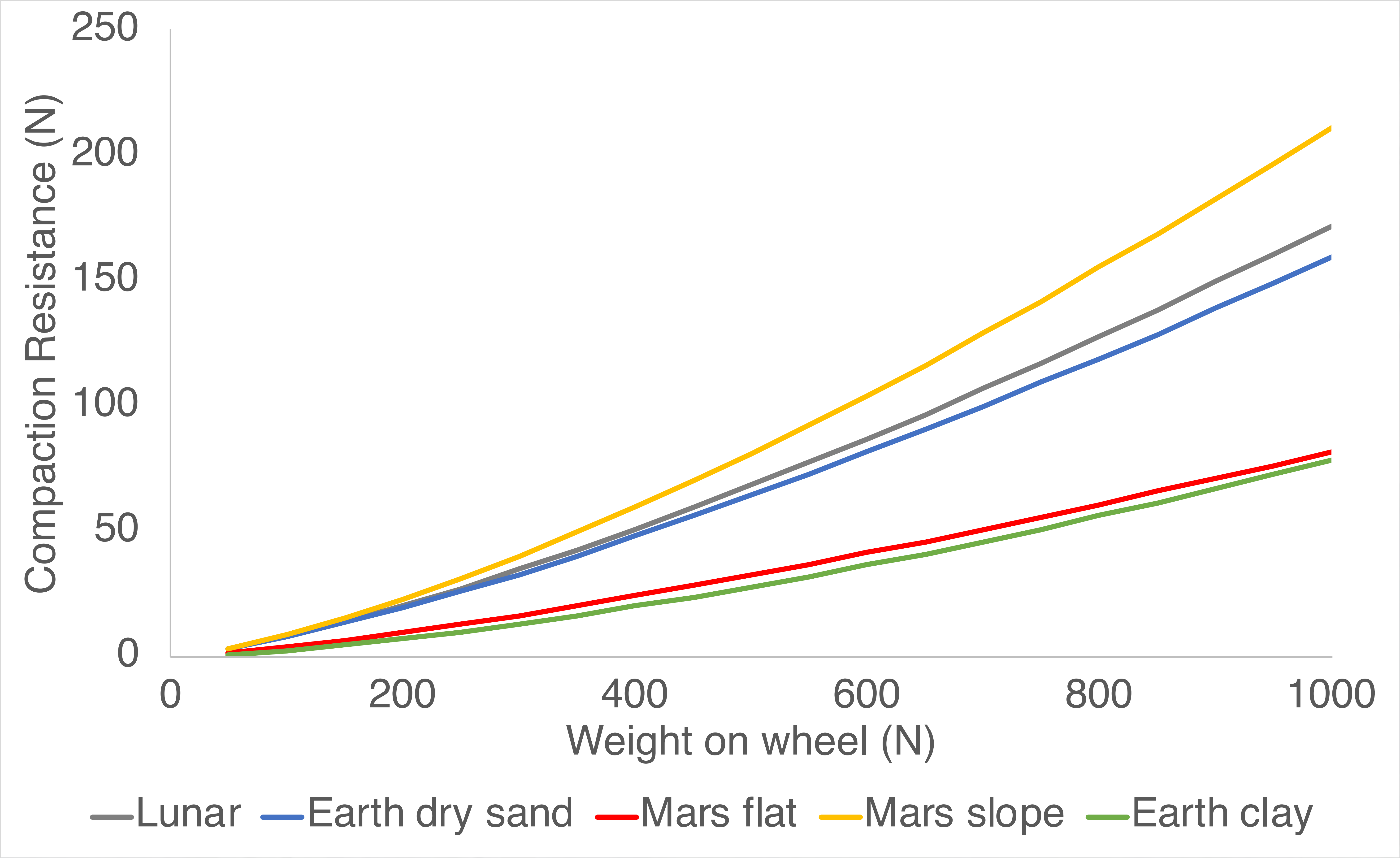
LRV Compression Resistance

$$R_c = \frac{1}{2}\left(1400\ \frac{N}{m^2} + (0.229\ m)820000\ \frac{N}{m^3}\right)^{\frac{-1}{3}}\left(\frac{3(259\ N)}{2\sqrt{0.813\ m}}\right)^{\frac{4}{3}} = 28.3\ N$$

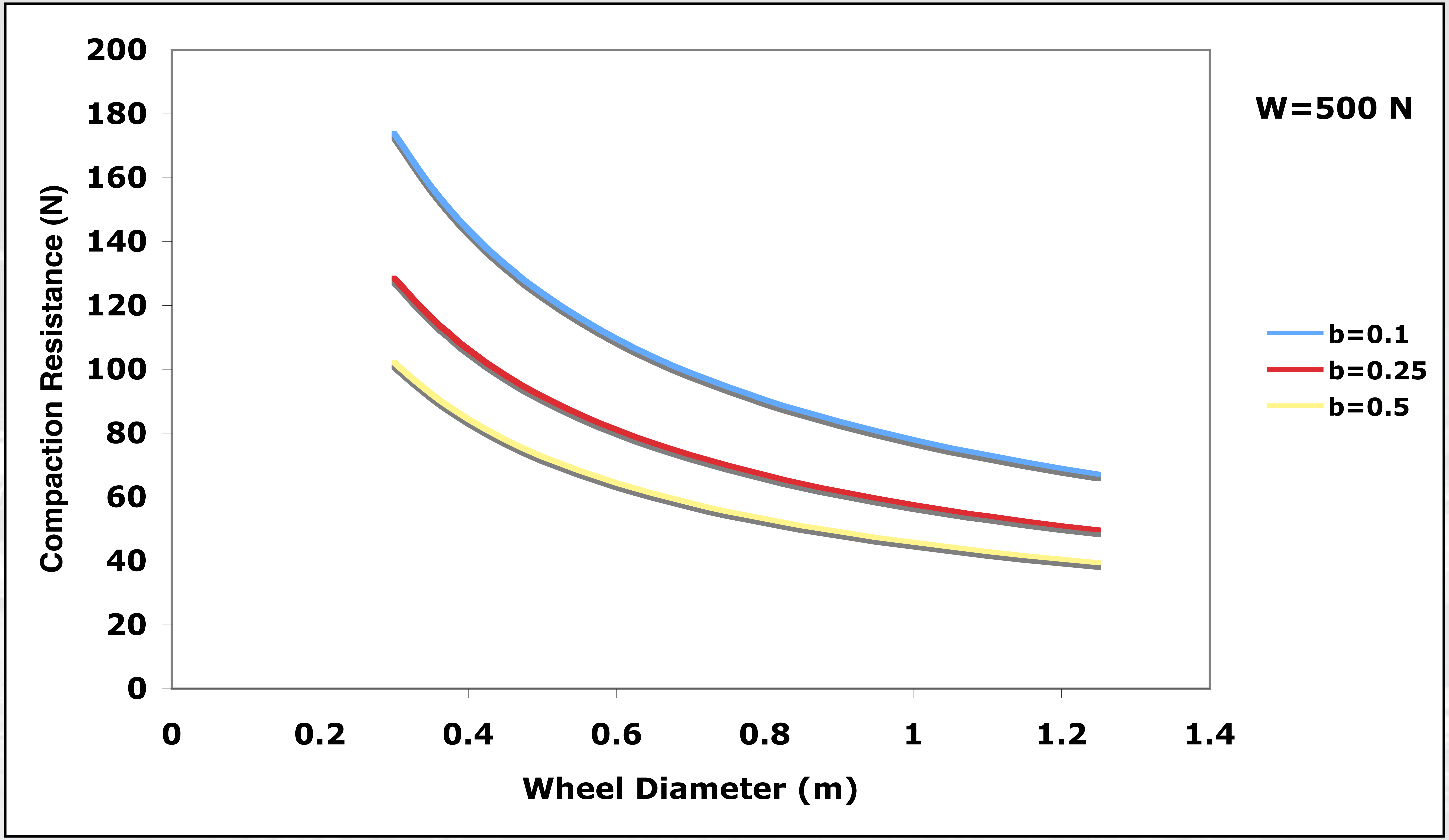
$$\text{Check units: } \left(\frac{N}{m^2} + \frac{Nm}{m^3}\right)^{\frac{-1}{3}}\left(\frac{N}{\sqrt{m}}\right)^{\frac{4}{3}} = \left(\frac{m^{2/3}}{N^{1/3}}\right)\left(\frac{N^{4/3}}{m^{2/3}}\right) = N$$

LRV Wheel - Different Weights and Soils

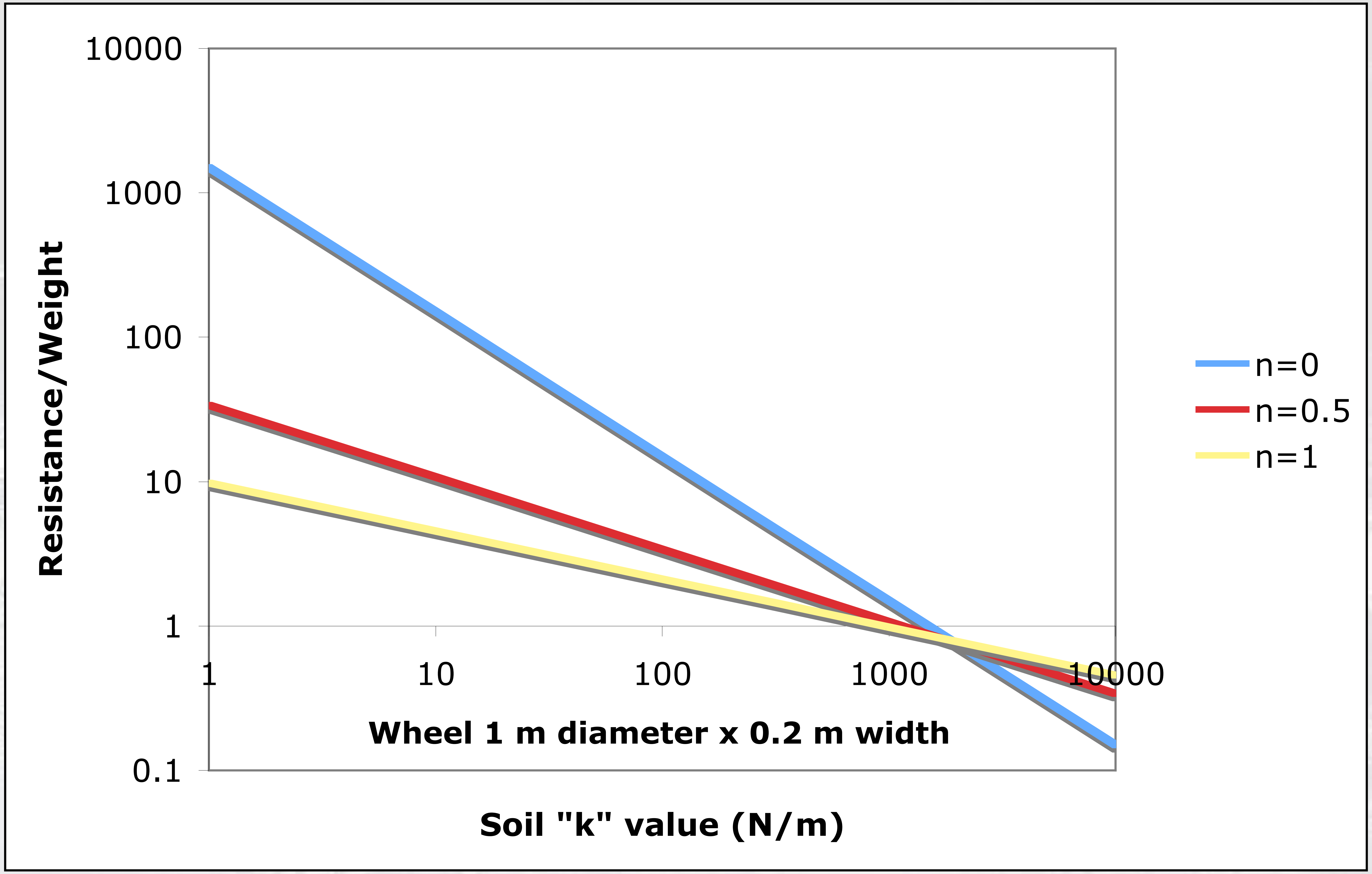
$$R = (k_c + k_\phi b)^{\frac{-1}{2n+1}} W^{\frac{2(n+1)}{2n+1}} \frac{1}{n+1} \left(\frac{3}{3-n}\right)^{\frac{2(n+1)}{2n+1}} D^{\frac{-(n+1)}{2n+1}}$$



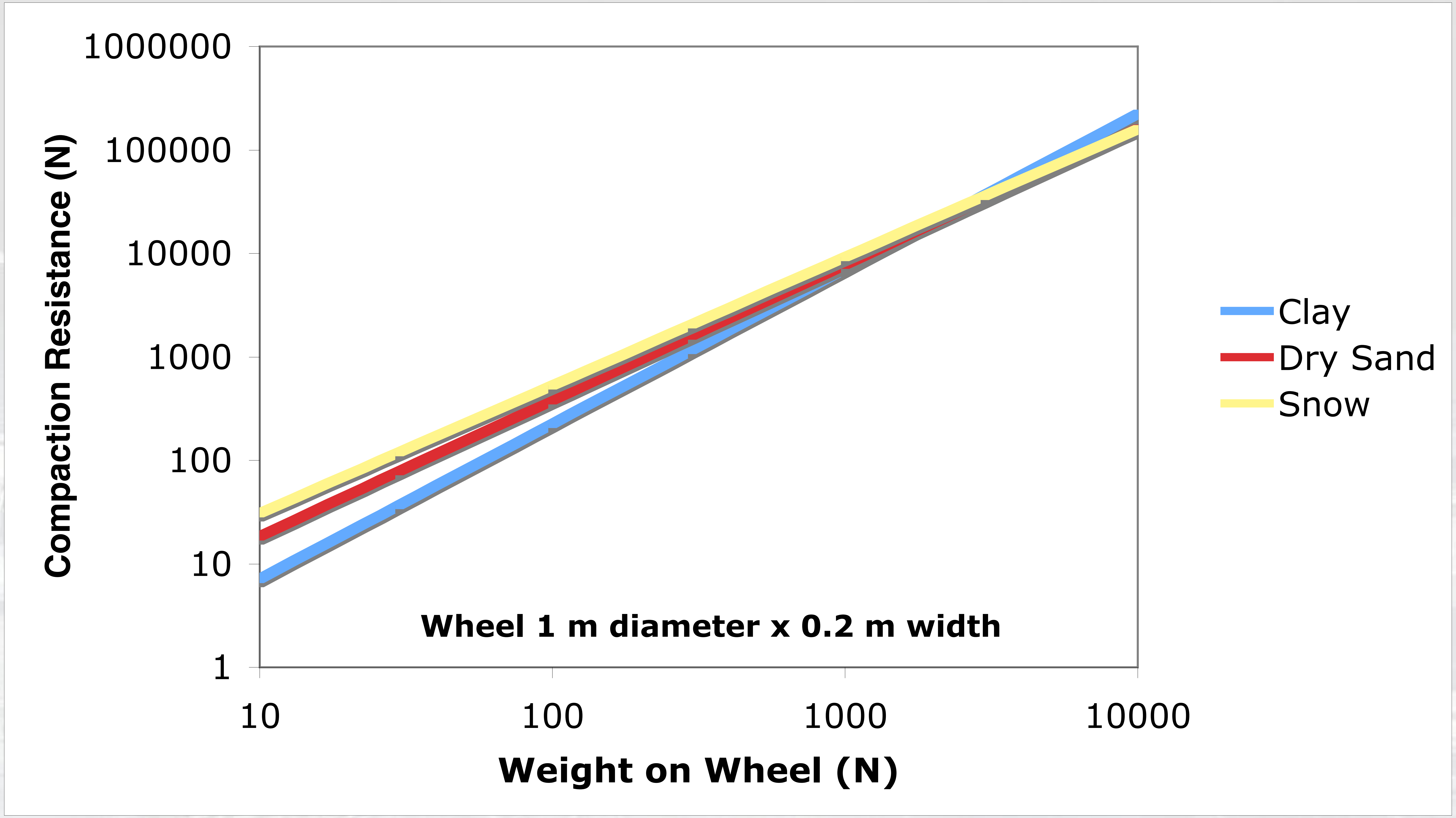
Effects of Wheel Parameters



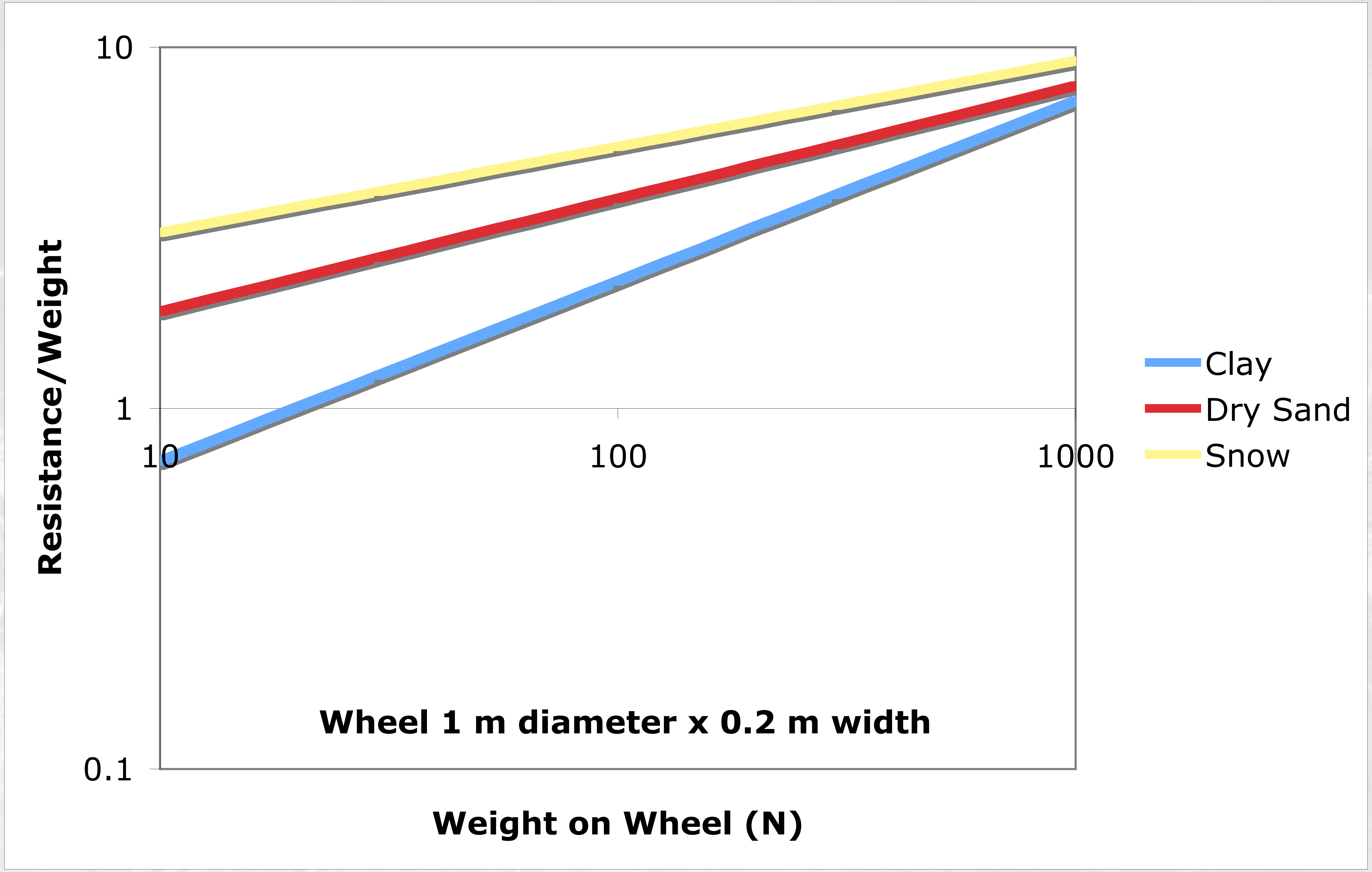
Effect of Soil “Spring Constant” on R/W



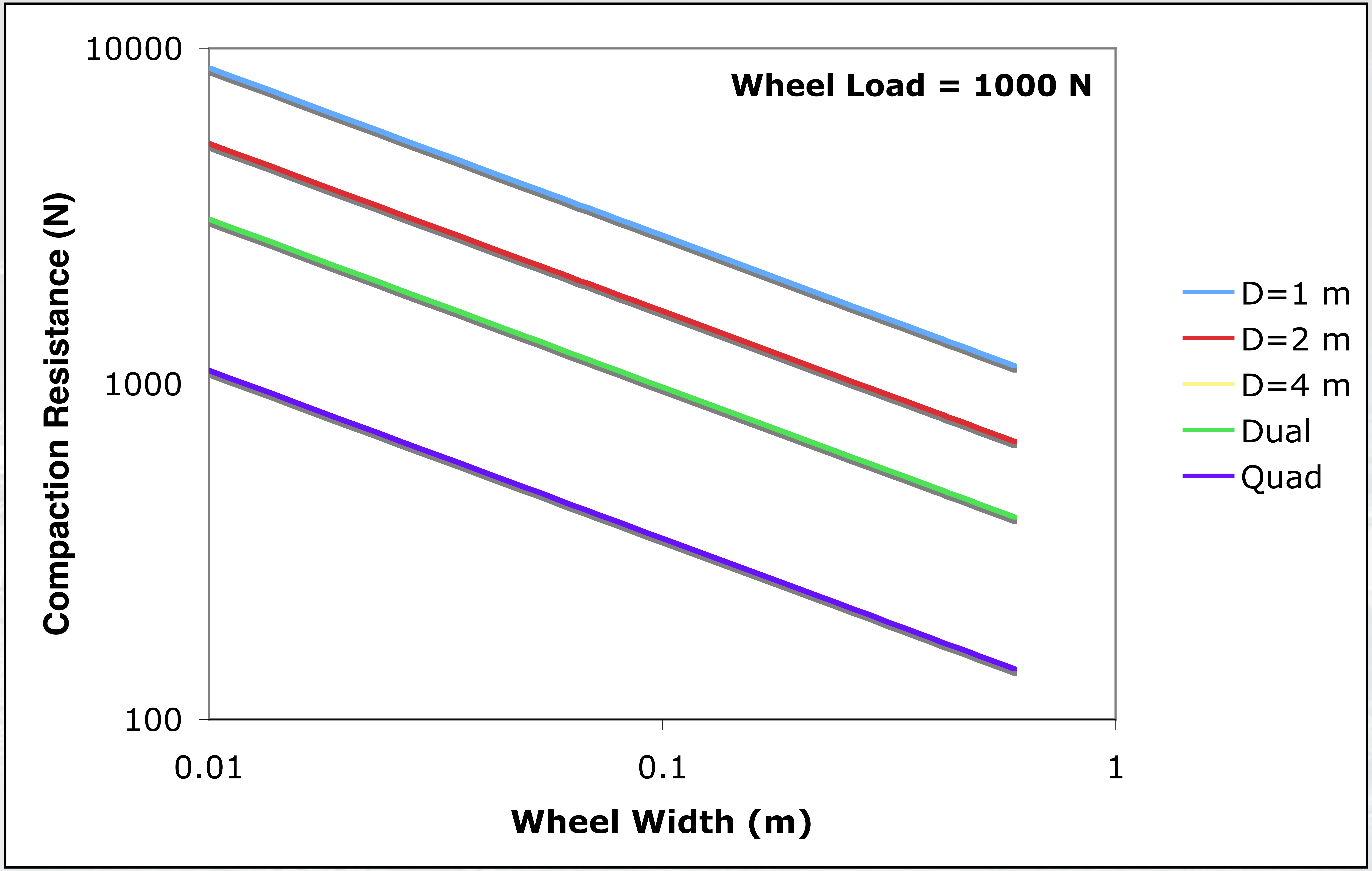
Soil Type and Wheel Load



Soil Type and Specific Resistance



Effect of Wheel Diameter and Width



Rolling and Gravitation Resistance

- Rolling resistance (tires, bearings, etc.)

$$R_r = W_v c_f$$

$$W_v = \text{weight of vehicle}$$

$$c_f = \text{coefficient of friction (typ. 0.05)}$$

- Gravitational resistance

$$R_g = W_v \sin \theta_{slope}$$

- LRV examples (15° slope, W_v=1004 N)

$$R_r = 50\ N \qquad R_g = 260\ N$$

Bulldozing Resistance

General case:

All angles in radians!

$$R_b = \frac{b \sin(\alpha + \phi)}{2 \sin\alpha \cos\phi} \left(2zcK_c + \gamma z^2 K_\gamma\right) + \frac{\ell_o^3 \gamma}{3}\left(\frac{\pi}{2} - \phi\right) + c\ell_o^2 \left[1 + \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right]$$

$$\alpha = \text{angle of attack of wheel in soil} \equiv \cos^{-1}\left(1 - \frac{2z}{D}\right)$$

$$\gamma = \text{density of soil} \left\langle \frac{N}{m^3} \right\rangle \qquad \ell_o = z \tan^2\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$$

For tracked vehicles, only the first term applies:

$$R_b = \frac{b \sin(\alpha + \phi)}{2 \sin\alpha \cos\phi} \left(2zcK_c + \gamma z^2 K_\gamma\right)$$

LRV Bulldozing Example (1)

$$\phi = 33^o = 0.576\ rad$$

$$\alpha = \cos^{-1}\left(1 - \frac{2z}{D}\right) = \cos^{-1}\left(1 - \frac{2(1.812)}{81.2}\right) = 17.18^\circ = 0.2999\ \text{rad}$$

$$\ell_o = z\tan^2\left(\frac{\pi}{4} - \frac{\phi}{2}\right) = 0.0051\ m$$

ℓ_o is length of soil ruptured by compression

Soil parameter values calculated on page 26

$$\ell = \frac{D}{2}\cos^{-1}\left(1 - \frac{2z}{D}\right) = 12.18\ \text{cm}$$

ℓ is the length of the wheel's circumference in contact with the soil

LRV Bulldozing Example (2)

$$R_b = \frac{b \sin(\alpha + \phi)}{2 \sin \alpha \cos \phi} \left(2zcK_c + \gamma z^2 K_\gamma \right)$$

$$+ \frac{\ell_o^3 \gamma}{3} \left(\frac{\pi}{2} - \phi \right) + c\ell_o^2 \left[1 + \tan \left(\frac{\pi}{4} + \frac{\phi}{2} \right) \right]$$

$$\langle R_b \rangle = cm \left(cm \, \frac{N}{cm^2} + \frac{N}{cm^3} cm^2 \right) + cm^3 \frac{N}{cm^3} + \frac{N}{cm^2} cm^2$$

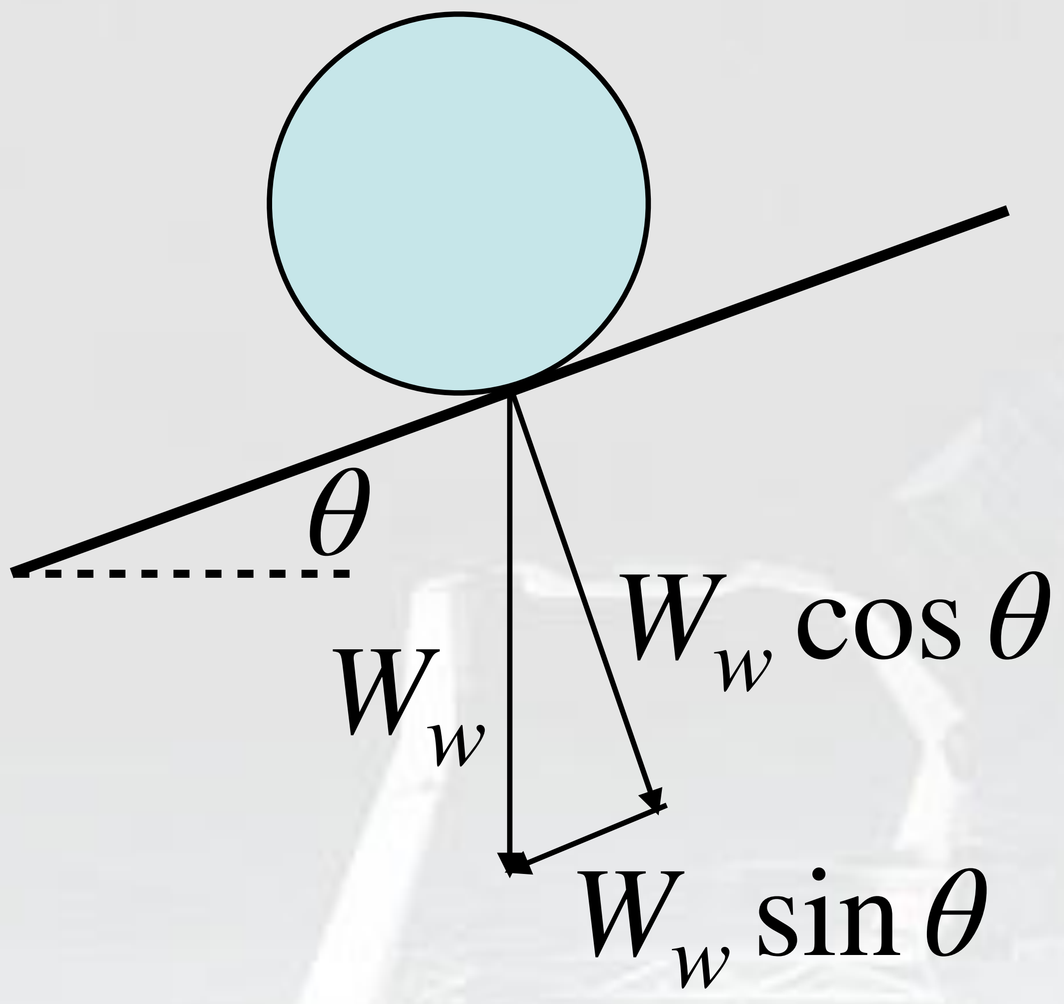
$$R_b = 94.98 + 0.000131 + 0.014 = 95.00 \; N \text{ per leading wheel}$$

$$R_{b,total} = 190.0 \; N$$

Total LRV Rolling Resistance Estimate

• Compression resistance R_c (4 wheels)	113 N
• Rolling resistance R_r	50 N
• Bulldozing resistance R_b (2 front wheels)	190 N
• *Total resistance, flat ground*	*352 N*
• Gravitational resistance, 15° slope	260 N
• *Total resistance, 15° slope*	*613 N*

Revisiting Rolling Resistance and Slopes



LRV example - $n = 1$

$$R_c = \frac{1}{2}(k_c + bk_\phi)^{\frac{-1}{3}}\left(\frac{3W_w \cos\theta}{2\sqrt{D}}\right)^{\frac{4}{3}}$$

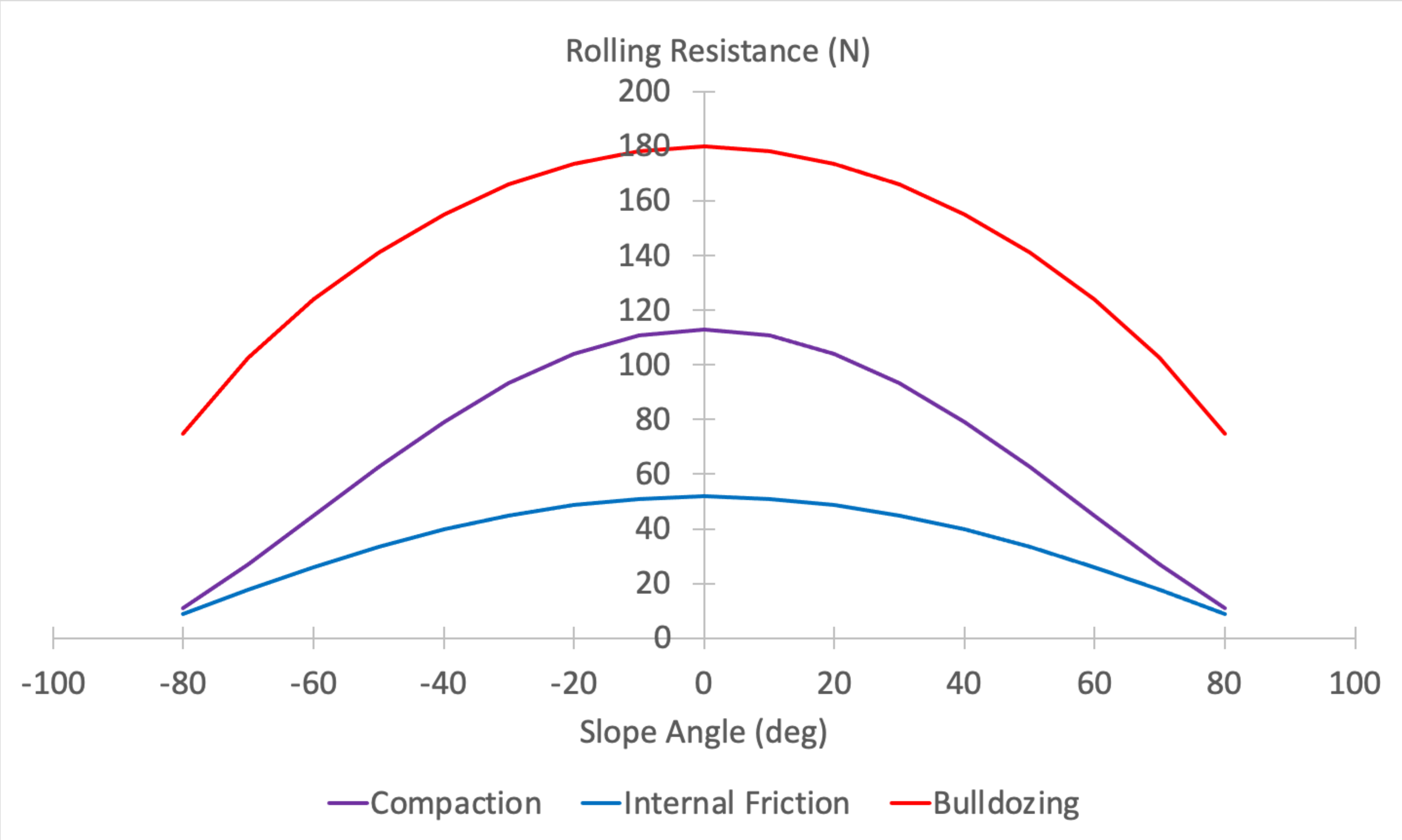
$$R_g = W_v \sin\theta$$

$$R_r = W_v \cos\theta \; c_f$$

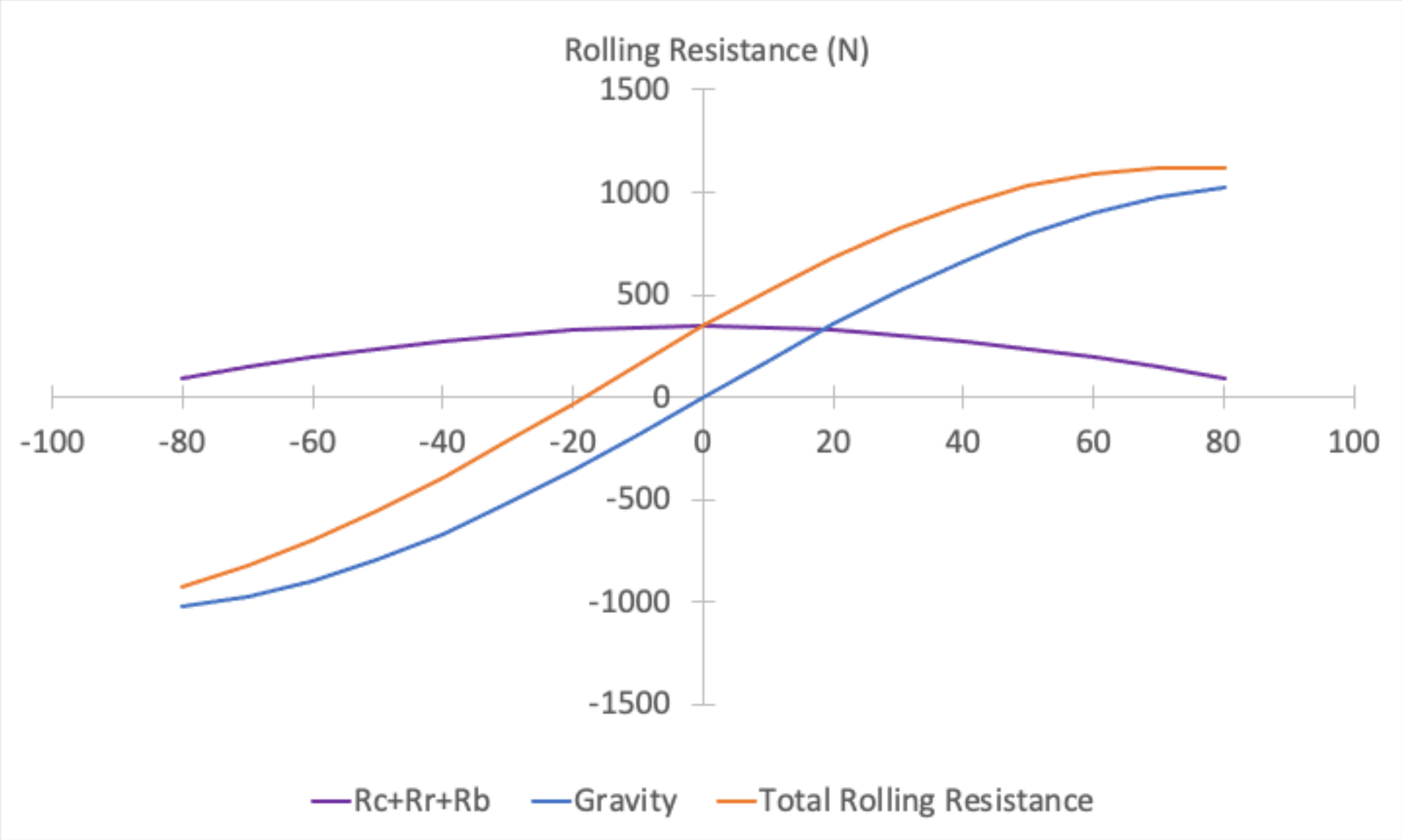
$$z = \left(\frac{3}{2}\frac{W_w \cos\theta}{(k_c + bk_\phi)\sqrt{D}}\right)^{\frac{2}{3}}$$

Bulldozing equation on pg. 43 is unchanged, but solution is altered by changing value of z

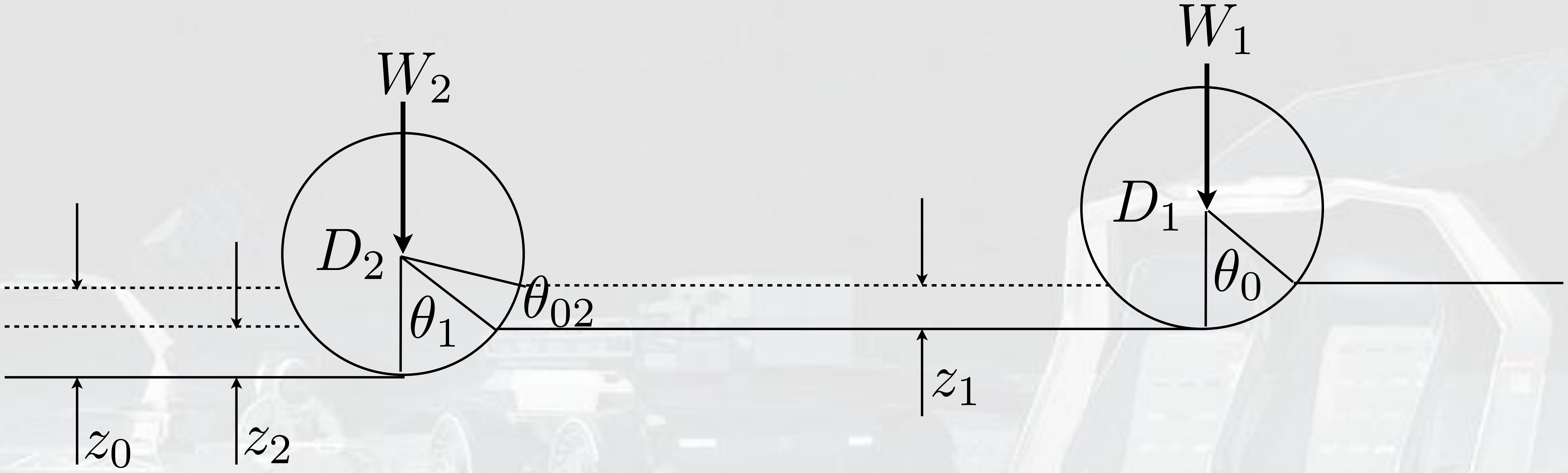
Slope Effects on LRV Rolling Resistance



Slope Effects on Rolling Resistance



Tandem Wheels



$$z_0 = z_1 + z_2$$

$$\text{Assume } n = \frac{1}{2} \Longrightarrow P = k\sqrt{z}$$

$$P_1 = k\sqrt{z_1} \qquad P_2 = k\sqrt{z_1 + z_2}$$

Soil Weight Bearing Analysis

In general,

$$W = \int_0^{\theta_0} dF \cos\theta = \int_0^{\theta_0} Pb\, ds \cos\theta$$

$$W = \int_0^{\theta_0} bk\sqrt{z}\cos\theta\ ds$$

$$ds = r\ d\theta \qquad z = r(\cos\theta - \cos\theta_0)$$

$$W = \int_0^{\theta_0} bkr\sqrt{r(\cos\theta - \cos\theta_0)}\cos\theta\ d\theta$$

Generic Wheel Soil Suspension

Assuming small sinkage,

$$z \to \text{ small}, \ \theta \to \text{ small},$$

$$\cos\theta \approx 1 \qquad \cos\theta \; d\theta \approx d\theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2} + (\text{higher order terms})$$

$$W = \frac{bkr^{3/2}}{\sqrt{2}} \int_0^{\theta_0} \sqrt{\theta_0^2 - \theta^2} \; d\theta$$

$$W = \frac{bkr^{3/2}}{\sqrt{2}} \frac{1}{2} \left[\theta_0^2 \sin^{-1}\left(\frac{\theta}{\theta_0}\right) + \theta\sqrt{\theta_0^2 - \theta^2} \right]_0^{\theta_0}$$

Weight on the Front Wheel

$$W = \frac{\pi b k r^{3/2}}{4\sqrt{2}} \theta_0^2$$

$$\text{Front wheel: } z_1 = r_1 - r_1 \cos \theta_0$$

$$z_1 = r_1 - r_1 \left(1 - \frac{\theta_0^2}{2} + \cdots \right) \implies \theta_0^2 \cong 2 \frac{z_1}{r_1}$$

$$\boxed{W_1 \cong \frac{\pi b k z_1 \sqrt{r_1}}{2\sqrt{2}}}$$

Weight on Back Wheel

Change to limits of integration:

$$0 \longrightarrow \theta_0, \quad r \longrightarrow r_2$$

$$W_2 = \frac{bkr_2^{3/2}}{\sqrt{2}} \int_0^{\theta_0} \sqrt{\theta_{02}^2 - \theta^2}\; d\theta$$

$$\sqrt{\theta_{02}^2 - \theta^2} \cong \theta_{02}\left(1 - \frac{1}{2}\frac{\theta^2}{\theta_{02}^2} + \cdots\right)$$

$$\int_0^{\theta_0} \sqrt{\theta_{02}^2 - \theta^2}\; d\theta \cong \theta_{02}\theta_1 - \frac{1}{6}\frac{\theta_1^3}{\theta_{02}} = \theta_{02}\theta_1\left(1 - \frac{1}{6}\frac{\theta_1^2}{\theta_{02}^2}\right) \cong \theta_{02}\theta_1$$

Weight on Back Wheel

$$W_2 = \frac{bkr^{3/2}}{\sqrt{2}}\theta_{02}\theta_1$$

$$z_0 \cong r_2 \frac{\theta_{02}^2}{2} \qquad z_2 \cong r_2 \frac{\theta_1^2}{2}$$

$$\theta_{02} \cong \sqrt{2\frac{z_0}{r_2}} \qquad \theta_1 \cong \sqrt{2\frac{z_2}{r_2}}$$

$$\boxed{W_2 = bk\sqrt{2r_2}\sqrt{z_0 z_2}}$$

Track Depth of Tandem Wheels

$$\text{Front: } z_1 = \frac{2\sqrt{2}W_1}{\pi b k \sqrt{r_1}}$$

$$\text{Back: } z_2 = \left[\frac{W_2}{bk\sqrt{2r_2}}\frac{1}{\sqrt{z_0}}\right]^2$$

$$z_0 = z_1 + z_2 = \frac{2\sqrt{2}W_1}{\pi b k \sqrt{r_1}} + \frac{W_2^2}{(bk)^2 2r_2}\frac{1}{z_0}$$

Much algebra then ensues...

$$z_0^2 - \frac{2\sqrt{2}W_1}{\pi b k \sqrt{r_1}} z_0 + \frac{W_2^2}{(bk)^2 2r_2} = 0$$

Rolling Resistance of Tandem Wheels

Solve the quadratic equation to get

$$z_0 = \frac{1}{bk}\left(\frac{\sqrt{2}W_1}{\pi\sqrt{r_1}} + \sqrt{\frac{2W_1^2}{\pi^2 r_1} + \frac{W_2^2}{2r_2}}\right)$$

This was all done for $n = \frac{1}{2} \implies R = \frac{2}{3}bkz_0^{3/2}$

$$R = \frac{2}{3}\frac{1}{\sqrt{bk}}\left(\frac{\sqrt{2}W_1}{\pi\sqrt{r_1}} + \sqrt{\frac{2W_1^2}{\pi^2 r_1} + \frac{W_2^2}{2r_2}}\right)^{3/2}$$

$$R = \frac{2}{3}\frac{1}{\sqrt{bk}}\left(\frac{2W_1}{\pi\sqrt{D_1}} + \sqrt{\frac{4W_1^2}{\pi^2 D_1} + \frac{W_2^2}{D_2}}\right)^{3/2}$$

Nondimensional Forms

Total wheel load $W = W_1 + W_2$ Wheel weight ratio $a \equiv \frac{W_1}{W_2}$

$$\text{For } W_1 = W_2 = \frac{W}{2} \implies a = 1$$

$$W_1 = \frac{a}{1+a}W \qquad W_2 = \frac{1}{1+a}W$$

$$R = \frac{2}{3}\frac{1}{(a+1)^{3/2}}\frac{W^{3/2}}{\sqrt{bk}}\left(\frac{2a}{\pi\sqrt{D_1}} + \sqrt{\frac{4a^2}{\pi^2 D_1} + \frac{1}{D_2}}\right)^{3/2}$$

Define wheel diameter ratio $\rho \equiv \frac{D_1}{D_2}$

Nondimensional Forms

$$R = \frac{2}{3}\frac{1}{(a+1)^{3/2}}\frac{W^{3/2}}{D_2^{3/4}\sqrt{bk}}\left(\frac{2a}{\pi\sqrt{\rho}} + \sqrt{1+\frac{4a^2}{\pi^2\rho}}\right)^{3/2}$$

$$\text{Let } \xi \equiv \frac{2}{3}\frac{1}{(a+1)^{3/2}}\left(\frac{2a}{\pi\sqrt{\rho}} + \sqrt{1+\frac{4a^2}{\pi^2\rho}}\right)^{3/2}$$

$$\boxed{R = \frac{\xi}{\sqrt{bk}}\frac{W^{3/2}}{D_2^{3/4}}}$$

Simple Example Case

$$\text{Consider } \rho = 1 \quad (D_1 = D_2 = D)$$

$$a = 1 \quad \left(W_1 = W_2 = \frac{W}{2} \right)$$

$$\text{For tandem wheels, } R = \frac{0.580}{\sqrt{bk}} \frac{W^{3/2}}{D_2^{3/4}}$$

$$\text{For single wheel (n=1/2), } R = \frac{0.876}{\sqrt{bk}} \frac{W^{3/2}}{D_2^{3/4}}$$

Tandem wheels reduce rolling resistance by 34%

Dual Wheels

Equivalent to single wheel case twice as wide $b \Longrightarrow 2b$

$$R = \frac{0.876}{\sqrt{2}} \frac{1}{\sqrt{bk}} \frac{W^{3/2}}{D_2^{3/4}}$$

$$R_{dual} = \frac{0.619}{\sqrt{bk}} \frac{W^{3/2}}{D_2^{3/4}}$$

Dual wheel rolling resistance 29% less than single,

7% higher than tandem

But all of this assumes $n = 1/2$...

Equations for Compression Resistance

$$z = \left[\frac{3W_w}{(3-n)bk\sqrt{D}} \right]^{\frac{2}{2n+1}}$$

$$W_w = \text{weight on wheel}$$

$$D \equiv \text{wheel diameter}$$

$$R_c = \left(\frac{bk}{n+1} \right) z^{n+1}$$

$$R_c = \text{compression resistance (per wheel)}$$

LRV Sinkage in Lunar Regolith

$$z = \left[\frac{3W_w}{2\left(k_c + bk_\phi\right)\sqrt{D}} \right]^{\frac{2}{3}}$$

$$z_1 = 1.51\ cm$$

Back wheel is a tandem wheel

$$W_w = \frac{2}{3}\left(k_c + bk_\phi\right)\sqrt{D}z^{\frac{3}{2}}$$

Tandem Wheels, n=1

$$W_i = \frac{2}{3}\left(k_c + bk_\phi\right)\sqrt{D}\left(z_i^{\frac{3}{2}} - z_{i-1}^{\frac{3}{2}}\right)$$

We already calculated $z_1 = 1.51\ cm$

$$W_2 = \frac{2}{3}\left(k_c + bk_\phi\right)\sqrt{D}\left(z_2^{\frac{3}{2}} - z_1^{\frac{3}{2}}\right)$$

$$z_2 = \left[\frac{3}{2}\frac{W_2}{\left(k_c + bk_\phi\right)\sqrt{D}} + z_1^{\frac{3}{2}}\right]^{\frac{2}{3}} = 2.443\ cm$$

This is the total depth (from undisturbed soil) of wheel 2

Compaction Resistance, n=1

$$R_c = \frac{1}{2}(k_c + bk_\phi)z^2$$

$$R_c = 56.92\ N\ \text{(per side)} \implies$$

$$R_c = 113.8\ N$$

Or, You Could Cheat…

Approximation formulas for n=1

Two wheels in tandem

$$R_c = \frac{1.7}{2}\left(k_c + bk_\phi\right) z_1^2$$

Three wheels in tandem

$$R_c = \frac{2.3}{2}\left(k_c + bk_\phi\right) z_1^2$$

z_1 is the sinkage depth of the front wheel in both cases

$R_g,\ R_f$ are straightforward