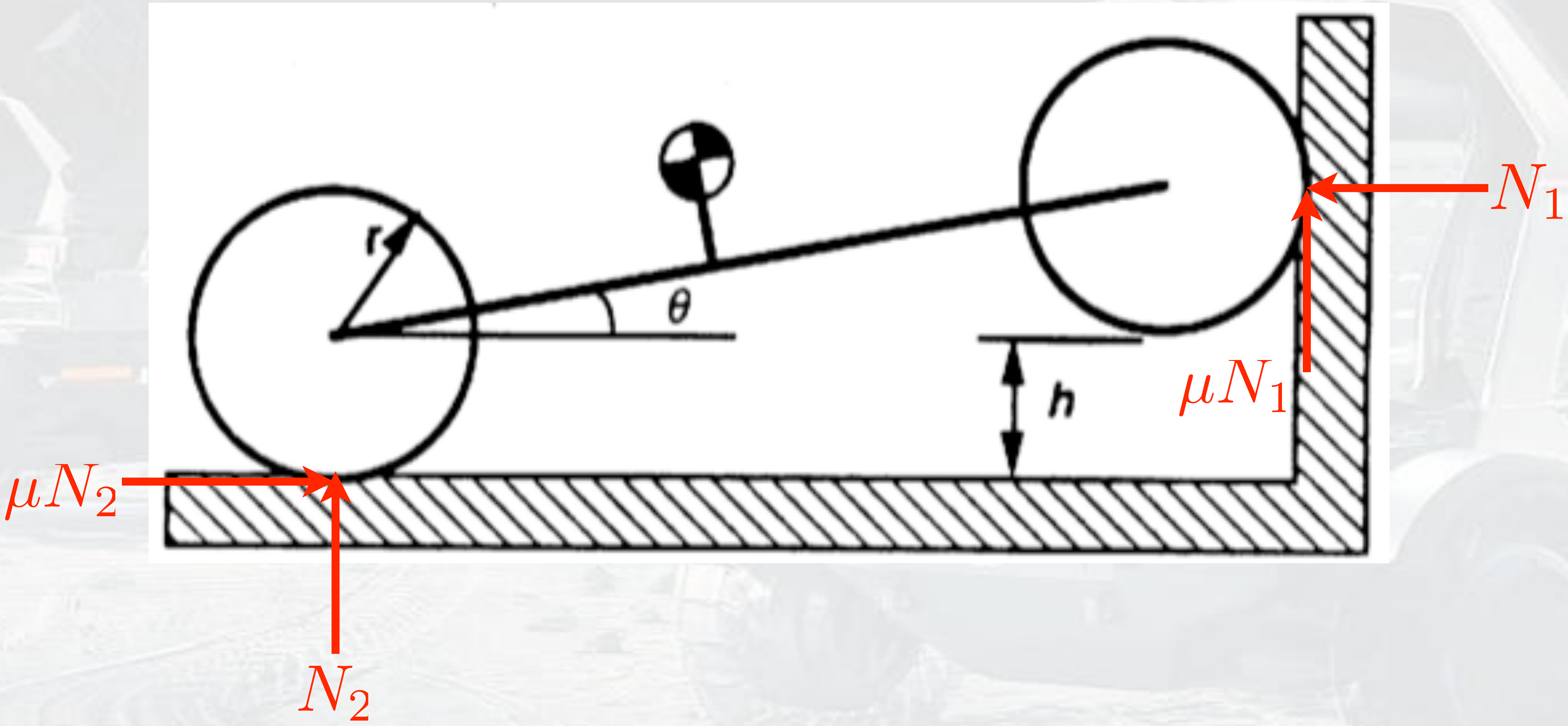
Suspension Systems

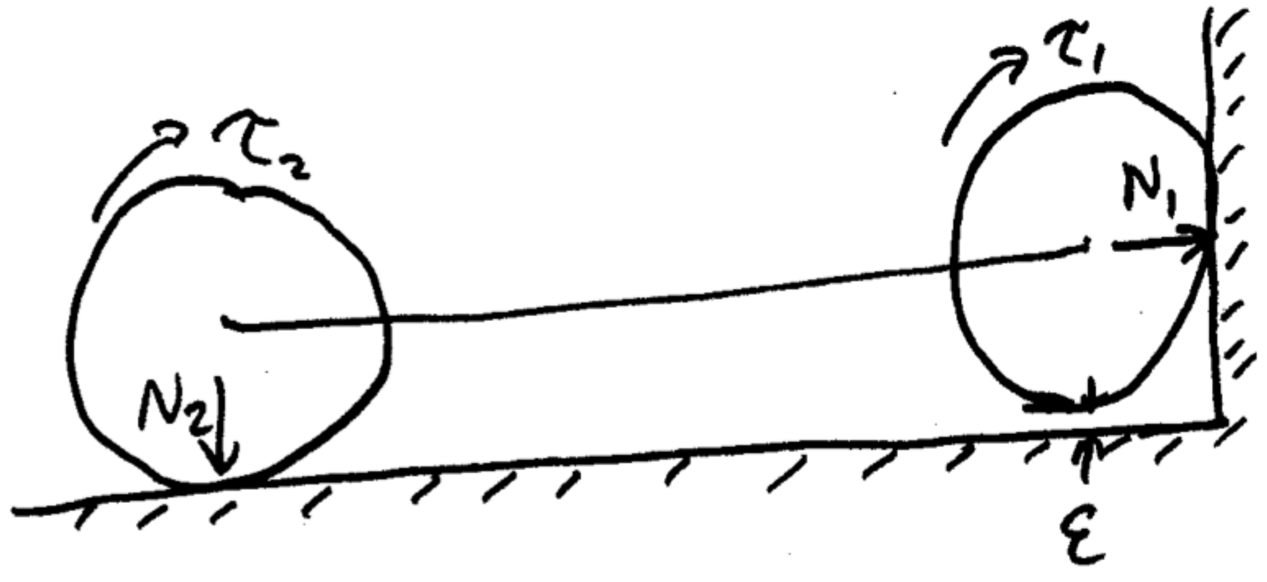
- Obstacle climbing with multiwheel systems
- Planar rocker analysis
- Planar rocker-bogey analysis
- Suspension dynamics
- 3D vehicle wheel loading
- Spring-damper suspension dynamics

Four-Wheeled Vehicle Climbing a Wall



from Howard Eisen, "Scale and Computer Modeling of Wheeled Vehicles for Planetary Exploration" S.M. Thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, May, 1990

Wall Climbing



$$\sum F_{vertical} \Rightarrow N_2 + \mu N_1 = W$$

$$\sum F_{horizontal} \Rightarrow \mu N_2 = N_1$$

$$\sum M_{rear\ axle} \Rightarrow \mu N_2 r + \mu N_1 (l + r) = W(l - a)$$

$$N_2 + \mu^2 N_2 = W \Rightarrow N_2 = \frac{W}{1+\mu^2} \Rightarrow N_1 = \frac{\mu}{1+\mu^2} W$$

$$\frac{\mu}{1+\mu^2} W r + \frac{\mu^2}{1+\mu^2} W (l + r) = W(l - a)$$

$$(r+a)\mu^2 + r\mu - (l - a) = 0$$

$$\mu = \frac{-r \pm \sqrt{r^2 + 4(r+a)(\ell - a)}}{2(r+a)}$$

Let $\alpha \equiv \frac{a}{r}$ $\lambda \equiv \frac{\ell}{r}$

$$\mu = \frac{-1 \pm \sqrt{1 + 4(1+\alpha)(\lambda - \alpha)}}{2(1+\alpha)}$$

Assume $\alpha = \frac{\lambda}{2}$

$$\mu = -\frac{1}{2+\lambda} \pm \frac{\sqrt{1 + 4\left(1 + \frac{\lambda}{2}\right)\left(\frac{\lambda}{2}\right)}}{2+\lambda}$$

$\rightarrow$ $1 + 4\frac{\lambda}{2} + 4\frac{\lambda^2}{4}$

$= 1 + 2\lambda + \lambda^2$

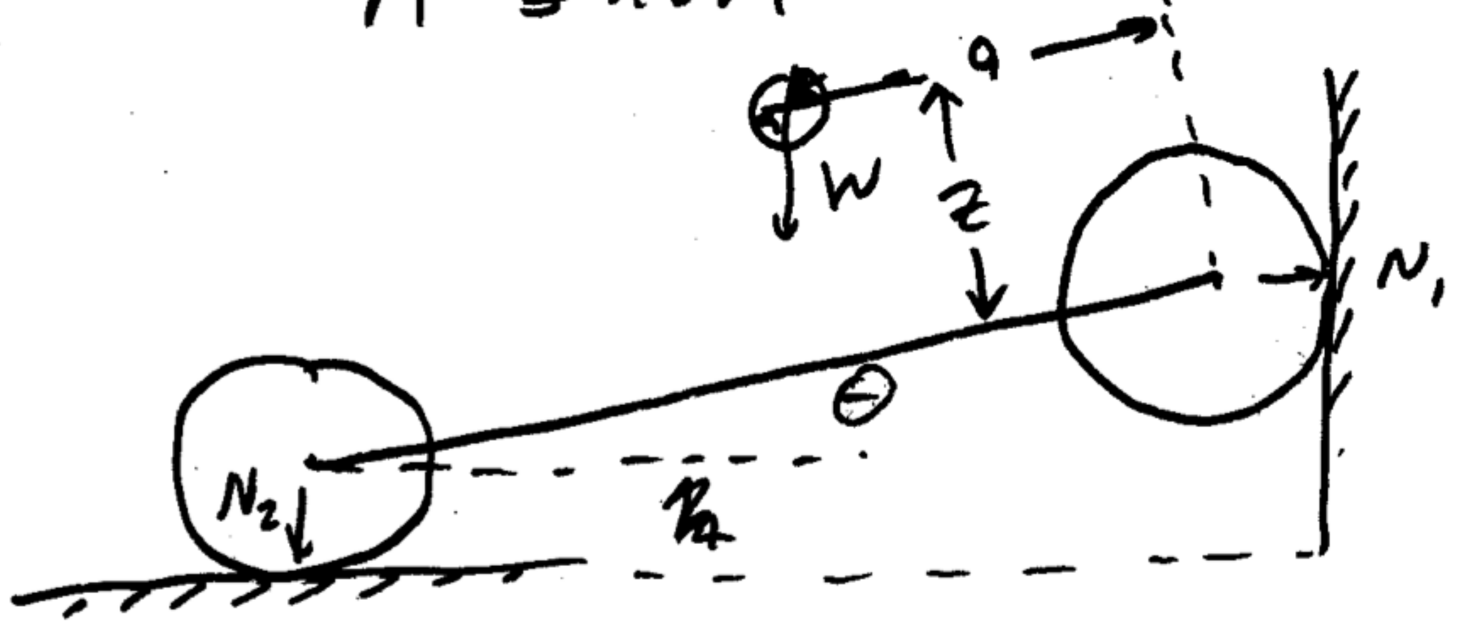
$= (\lambda + 1)^2$

$$= -\frac{1 \pm (\lambda+1)}{2+\lambda} = \frac{\lambda}{2+\lambda}, \; -1$$

$\lambda \to 0$ $\mu_{limit} \to 0$ $\lambda \to \infty$ $\mu_{limit} \to 1$

Shorter is better!

A Short Time Later...



$$h = l \sin\theta$$

$$\Sigma F_{vert} \Rightarrow \mu N_1 + N_2 = W$$

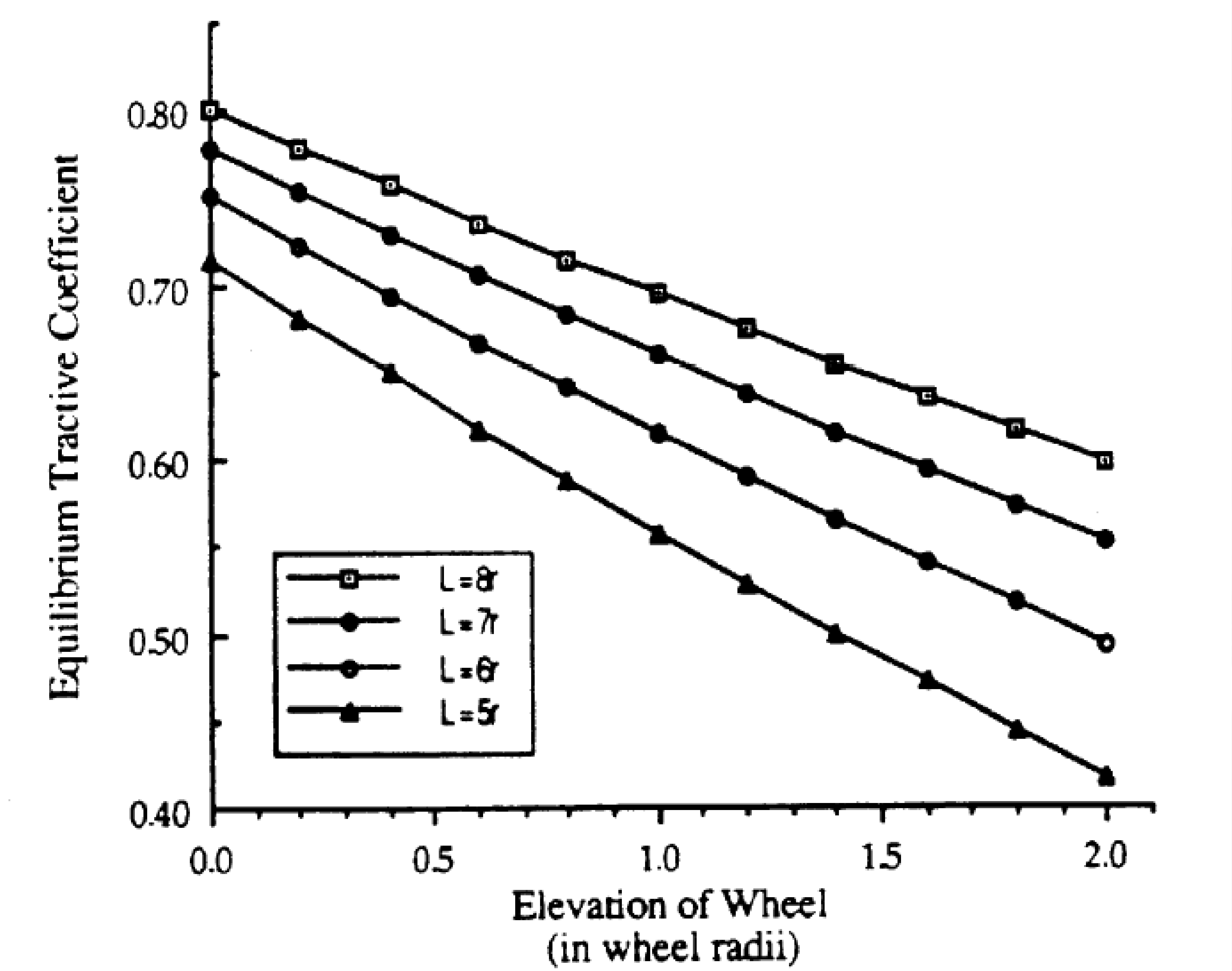
$$\Sigma F_{horiz} \Rightarrow \mu N_2 = N_1$$

$$N_1 = \frac{\mu}{1+\mu^2} W \qquad N_2 = \frac{W}{1+\mu^2}$$

$$\Sigma M_{rear} \Rightarrow \mu N_2 r + N_1 l \sin\theta + \mu N_1 (r + l\cos\theta) = W\left[(l-a)\cos\theta - z\sin\theta\right]$$

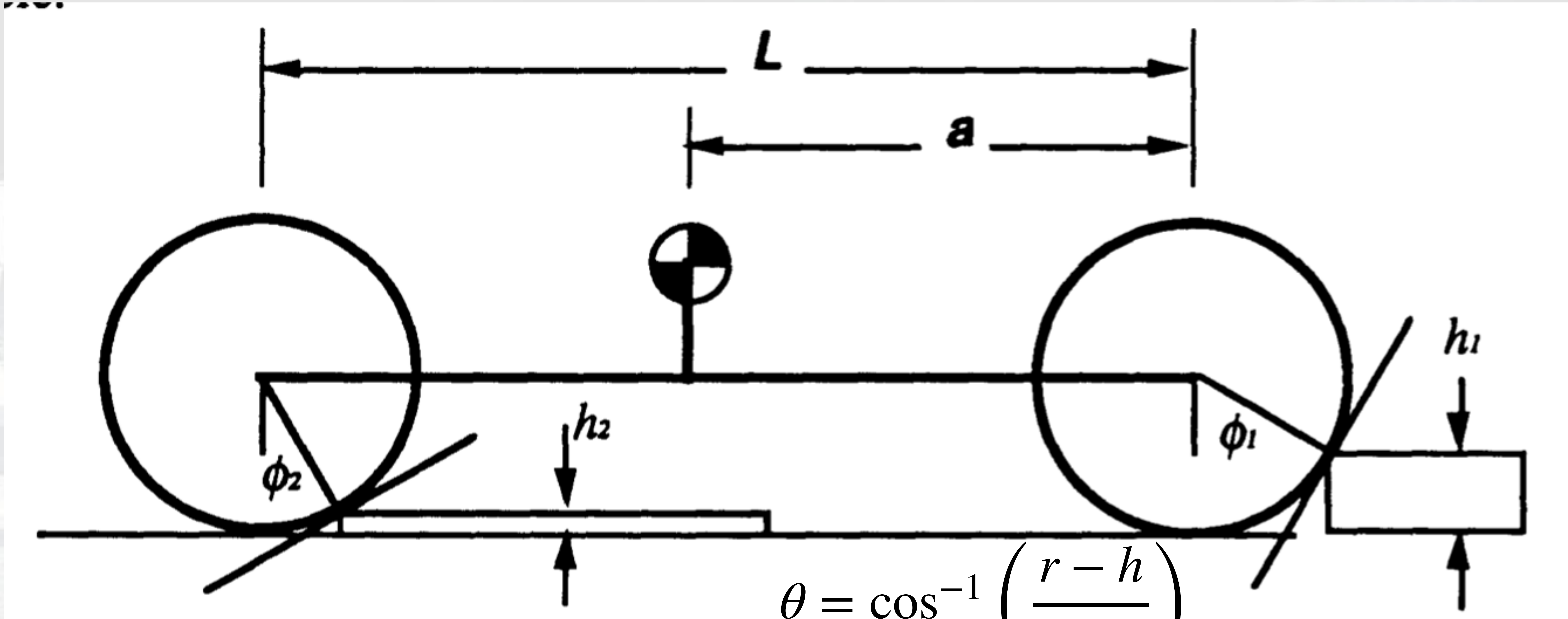
As θ increases, effect of W decreases and effect of N_1 increases ⇒ hardest point of the climb is at the start!

Required Traction for Wall Climbing



from Howard Eisen, "Scale and Computer Modeling of Wheeled Vehicles for Planetary Exploration" S.M. Thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, May, 1990

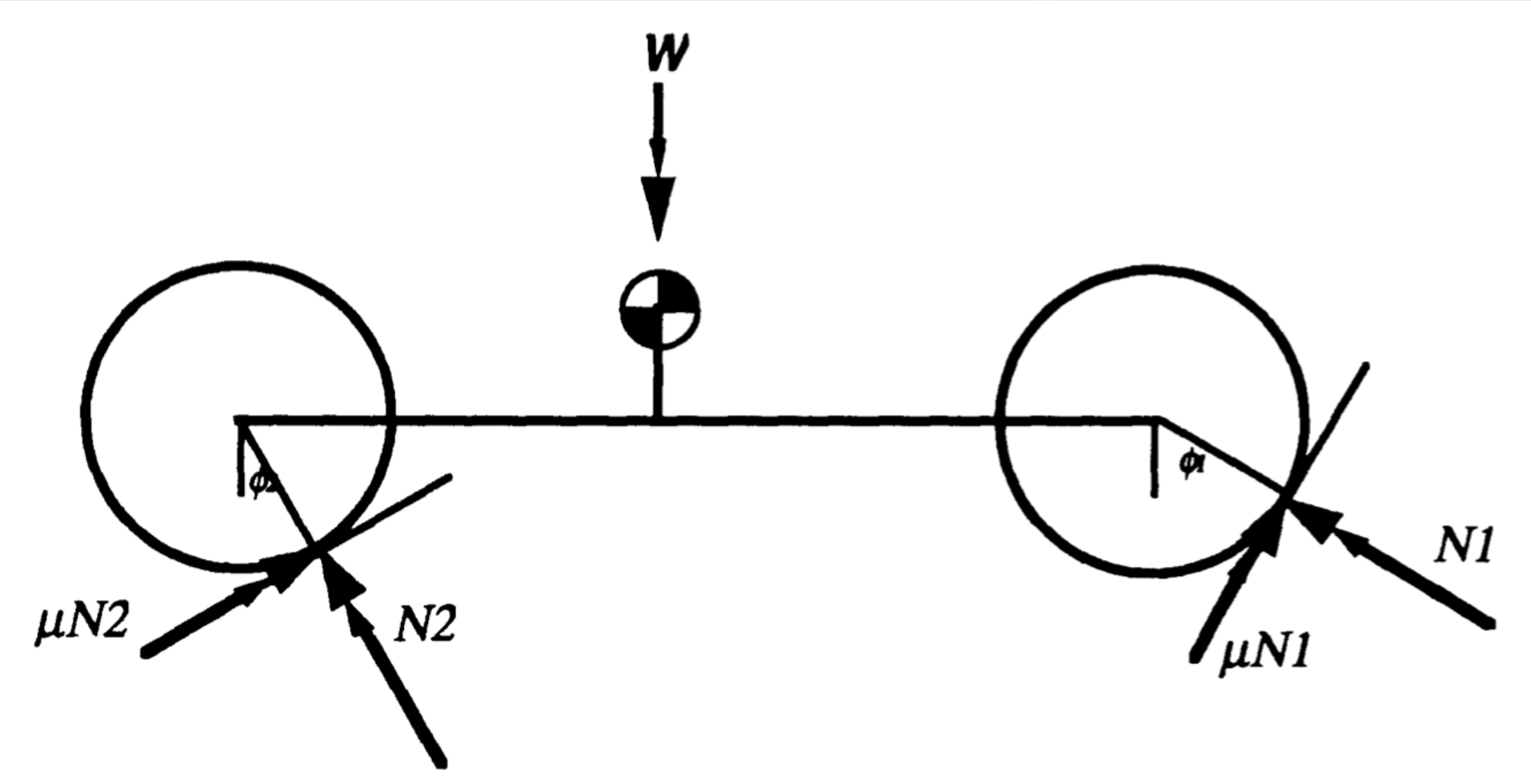
Wheel Interaction with Slope



$$\theta = \cos^{-1}\left(\frac{r-h}{r}\right)$$

from Howard Eisen, "Scale and Computer Modeling of Wheeled Vehicles for Planetary Exploration" S.M. Thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, May, 1990

Equations for Slopes under Wheels



Sum of Horizontal forces:

$$\mu N_2 \sin\phi_2 + N_2 \cos\phi_2 + \mu N_1 \sin\phi_1 + N_1 \cos\phi_1 - W = 0$$

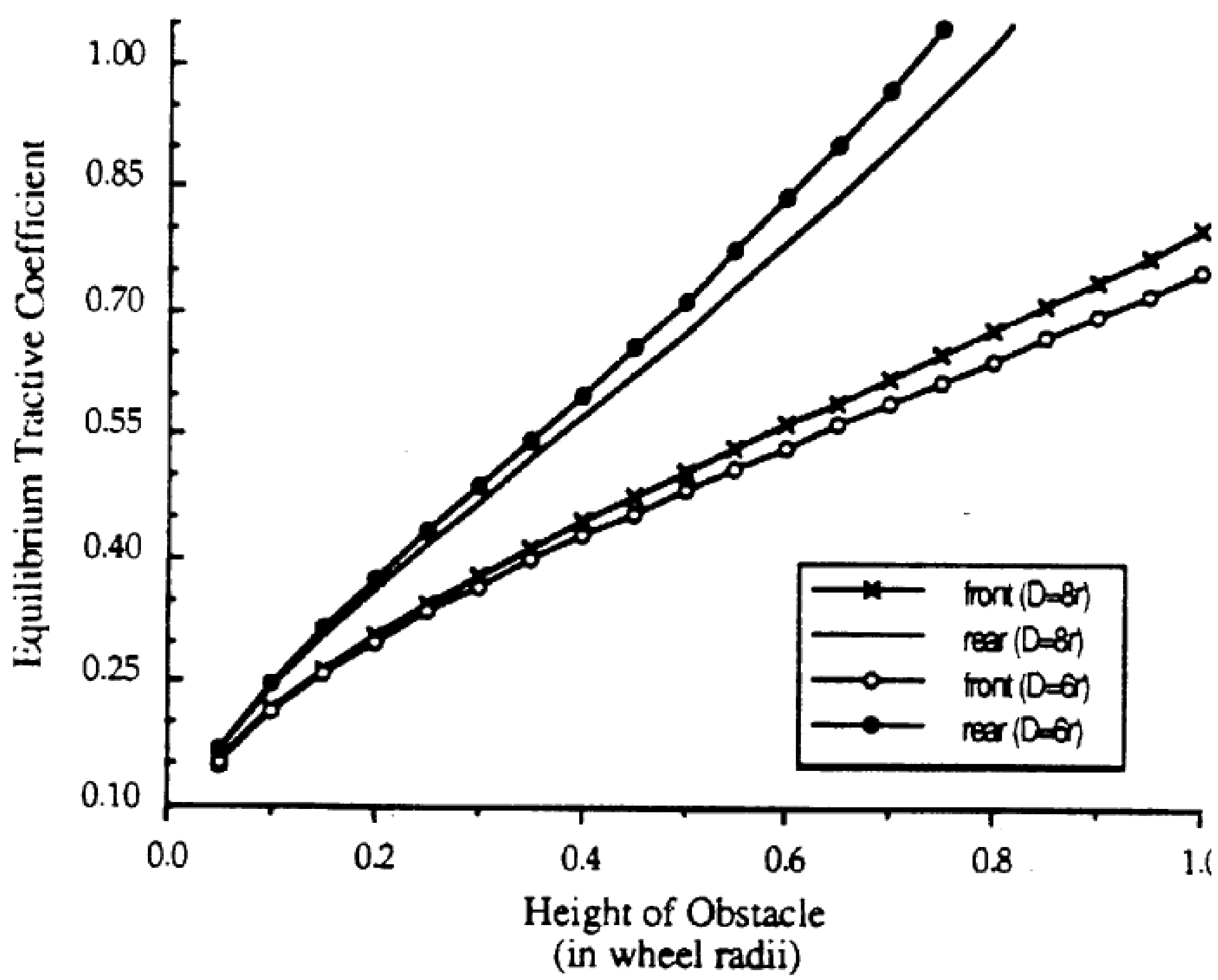
Sum of vertical forces:

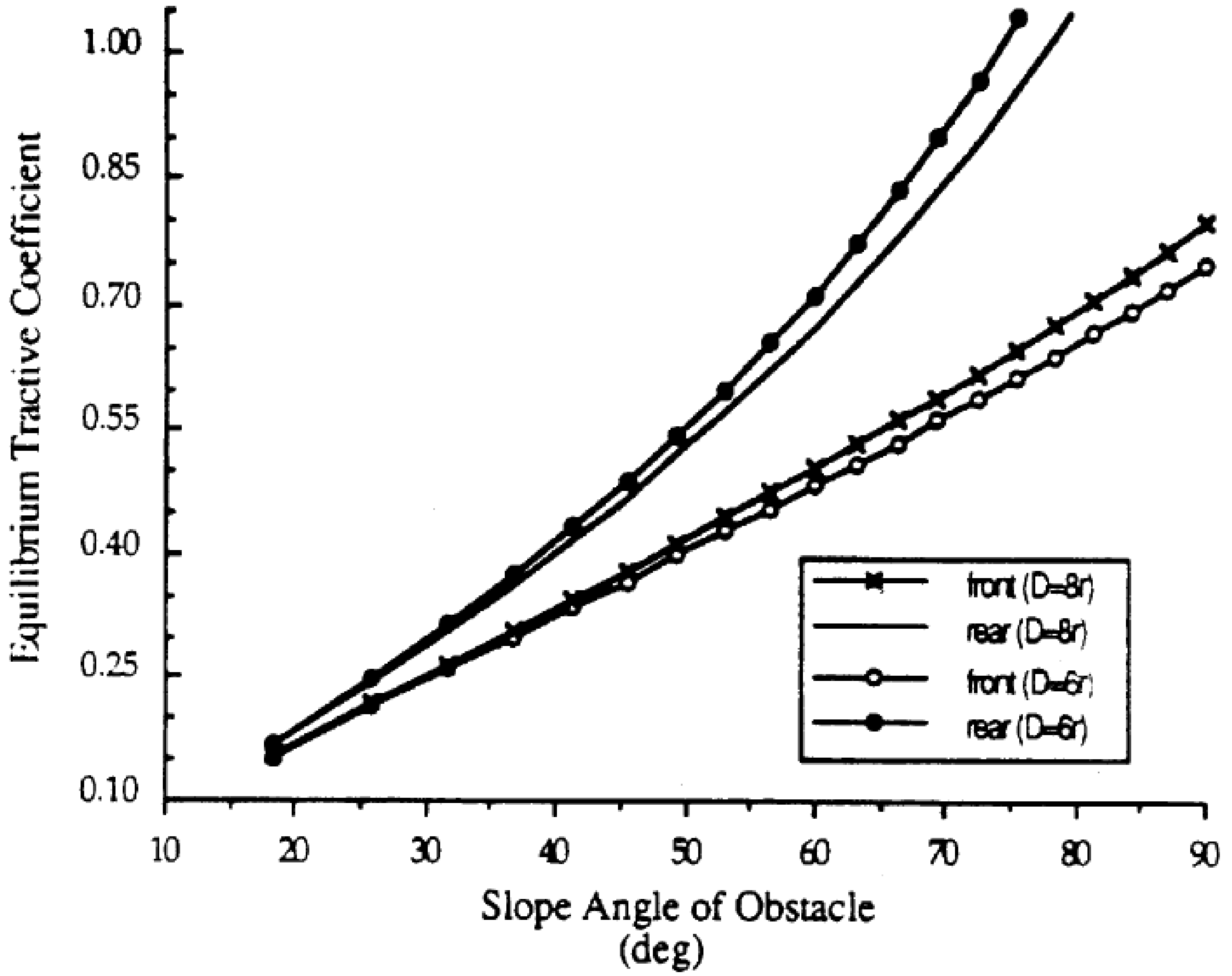
$$\mu N_2 \cos\phi_2 - N_2 \sin\phi_2 + \mu N_1 \cos\phi_1 - N_1 \sin\phi_1 = 0$$

Sum of forces around the rear axle:

$$\Big(\mu N_2 r - W(L-a) + N_1 L \cos\phi_1 + \mu N_1 \left(r + L\sin\phi_1\right) = 0$$

Bump/Slope Traction Requirements





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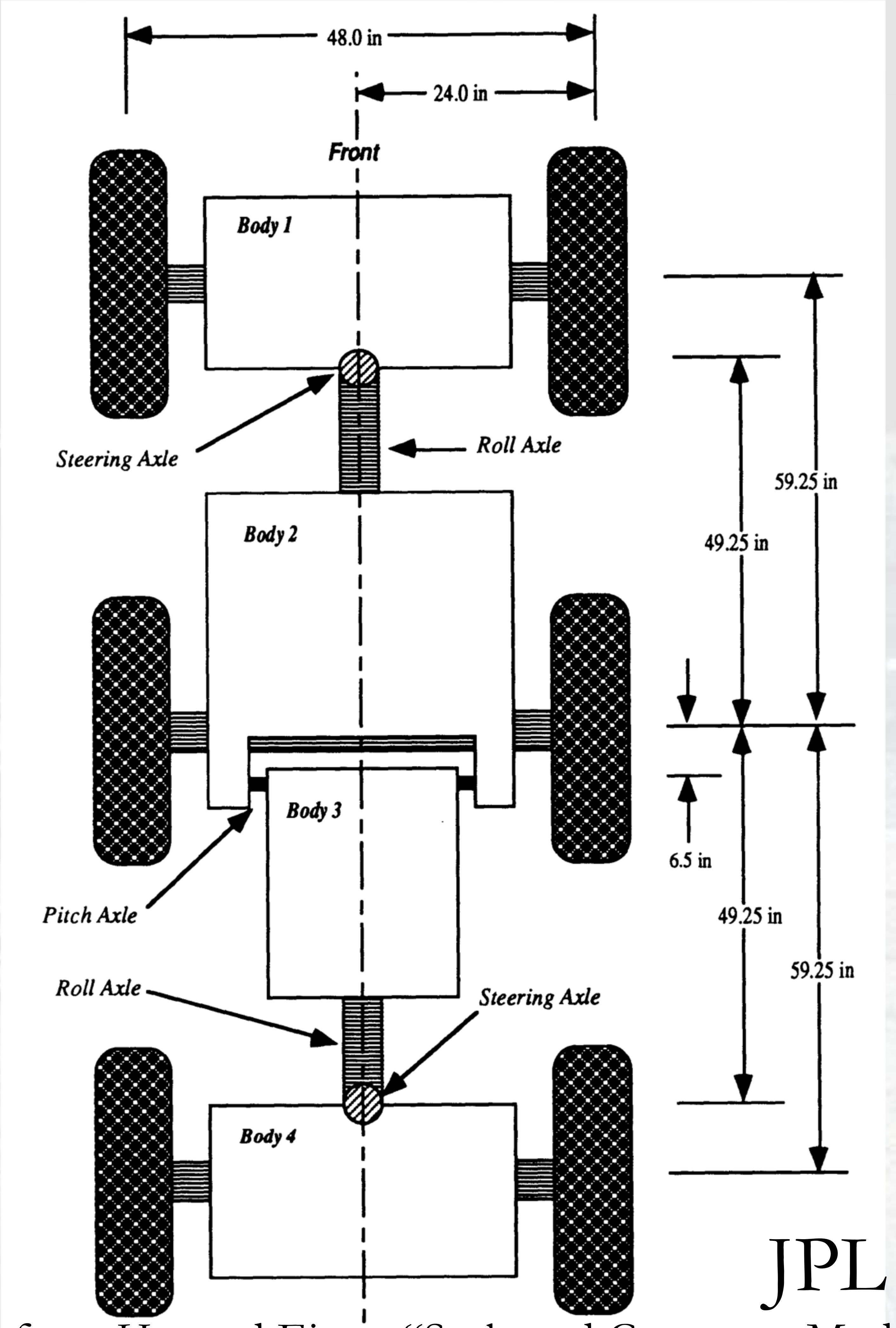
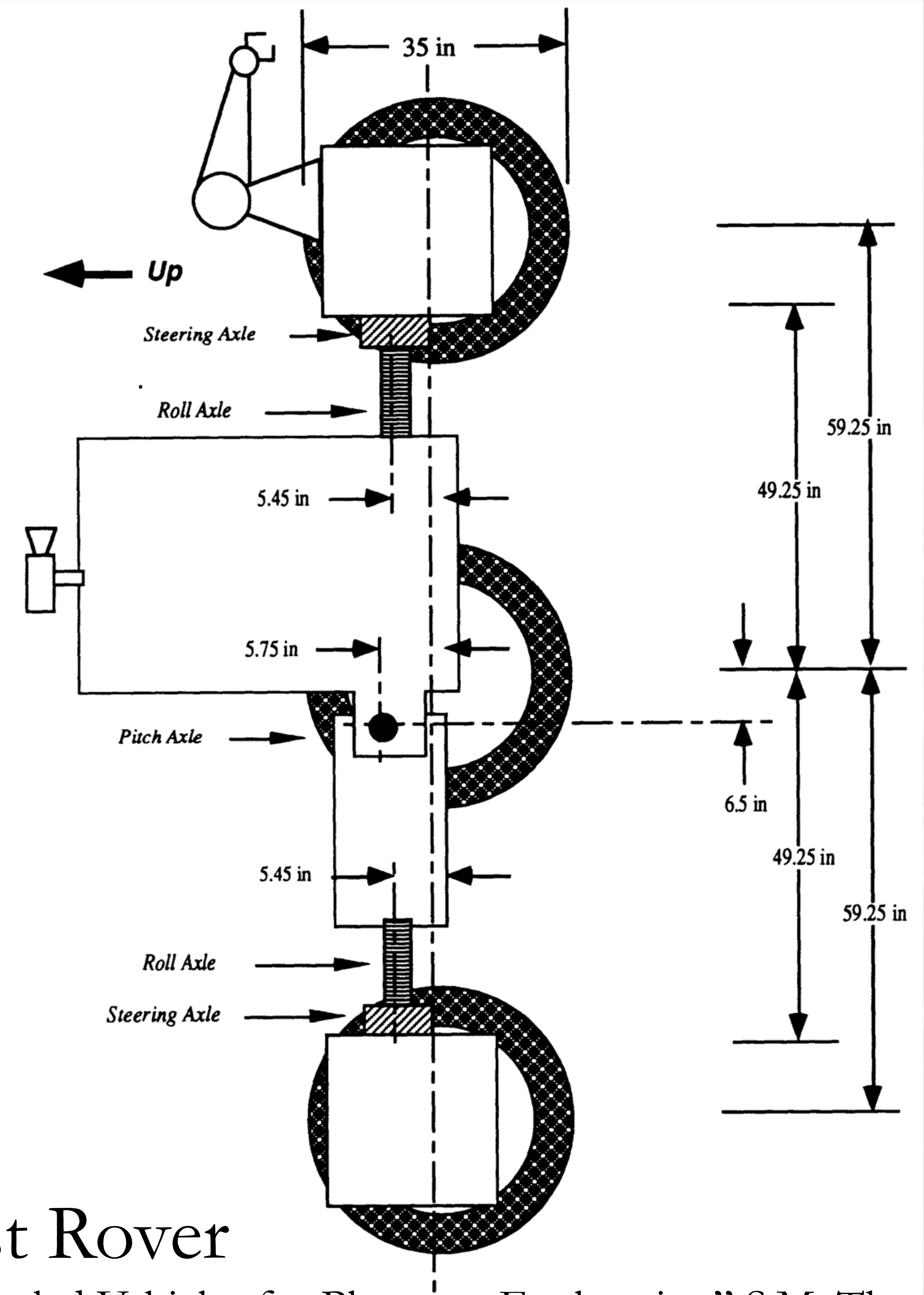
Six-Wheel Articulated Body Rover

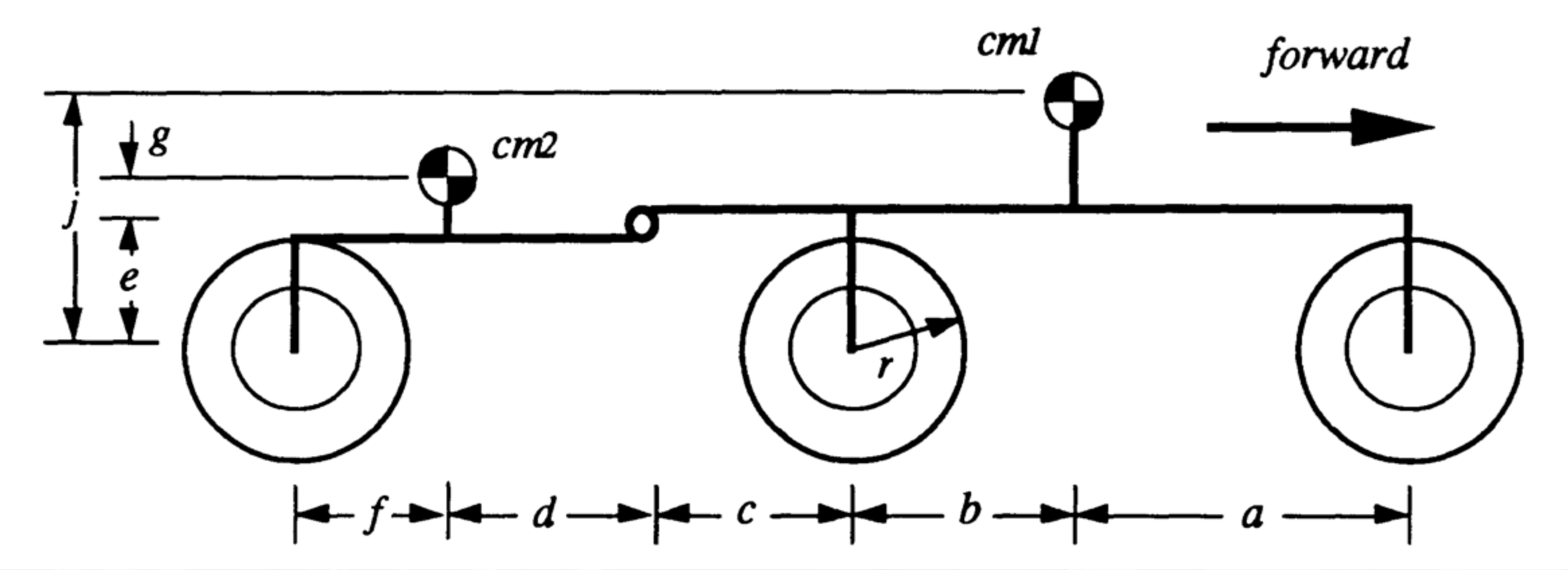




JPL Navtest Rover

from Howard Eisen, "Scale and Computer Modeling of Wheeled Vehicles for Planetary Exploration" S.M. Thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, May, 1990

Model of Six-Wheel Vehicle



Sum of vertical forces:

$$N_3 + \mu N_2 + N_1 - Wf - Wb = 0$$

Sum of horizontal forces:

$$\mu N_3 - N_2 + \mu N_1 = 0$$

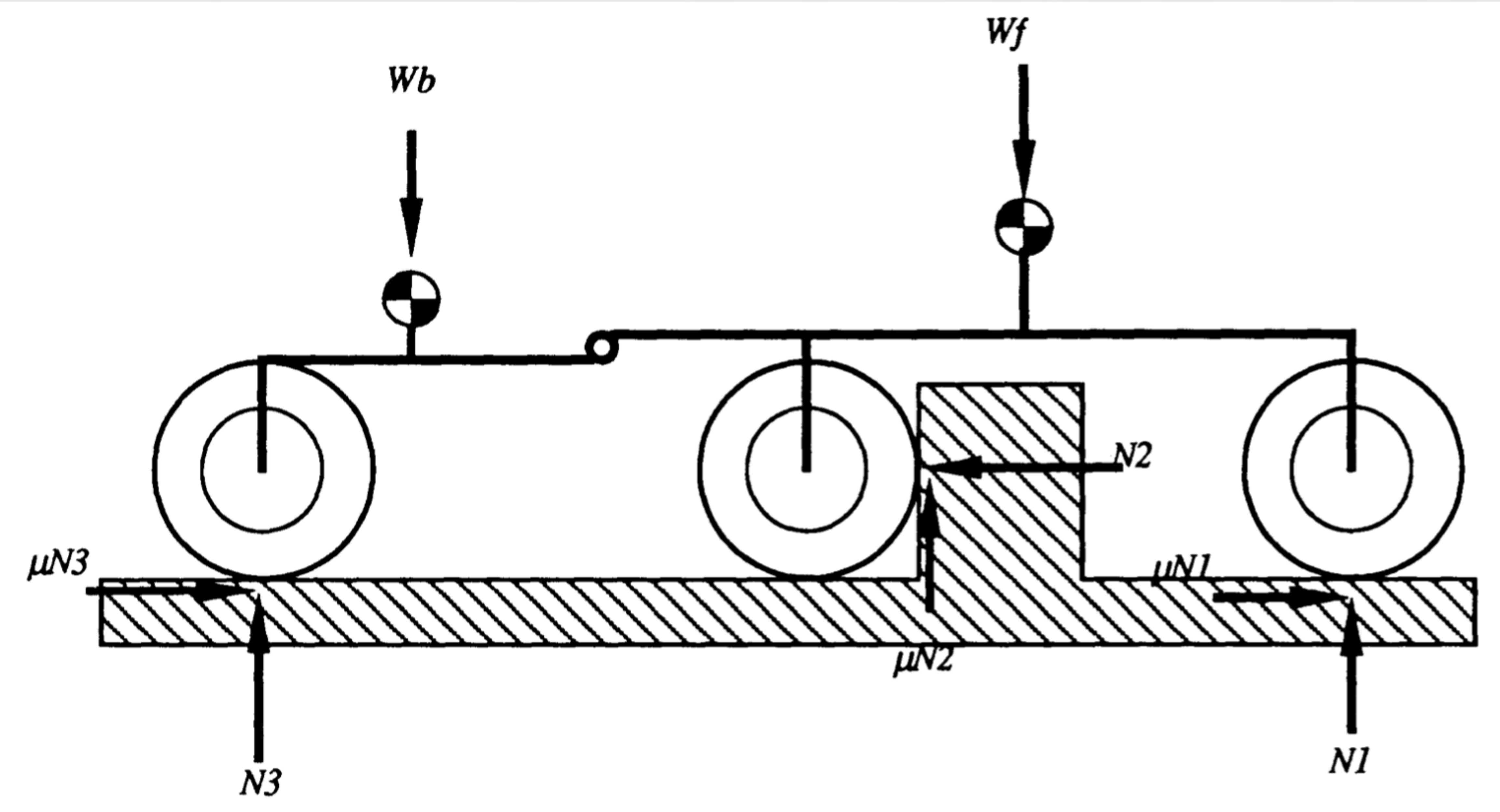
Sum of moments for front body around pitch axis

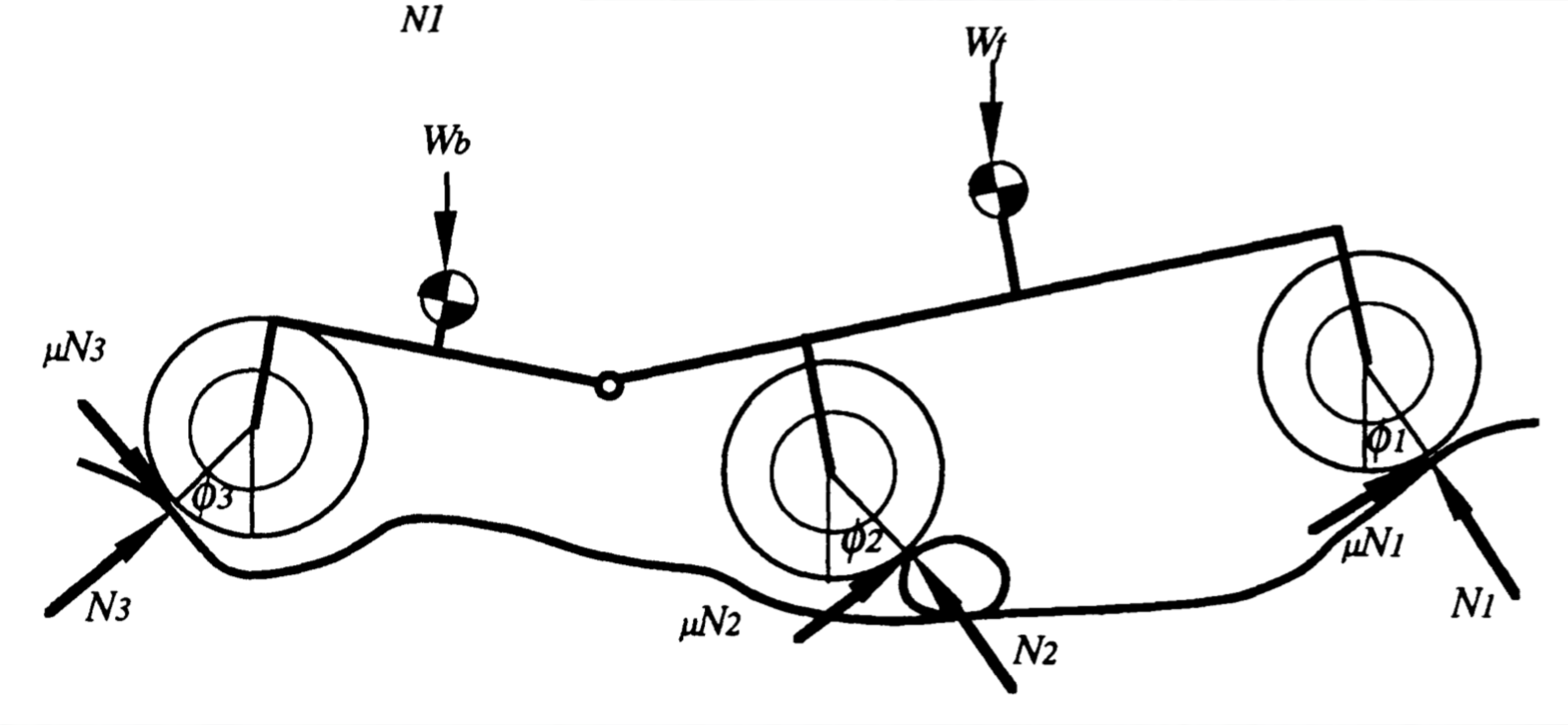
$$\mu N_1(r + e) + N_1(a + b + c) + \mu N_2(r + c) + -N_2 e - Wf(b + c) = 0$$

Sum of moments for rear body around pitch axis

$$Wbd + \mu N_3(r + e) - N_3(d + f) = 0$$

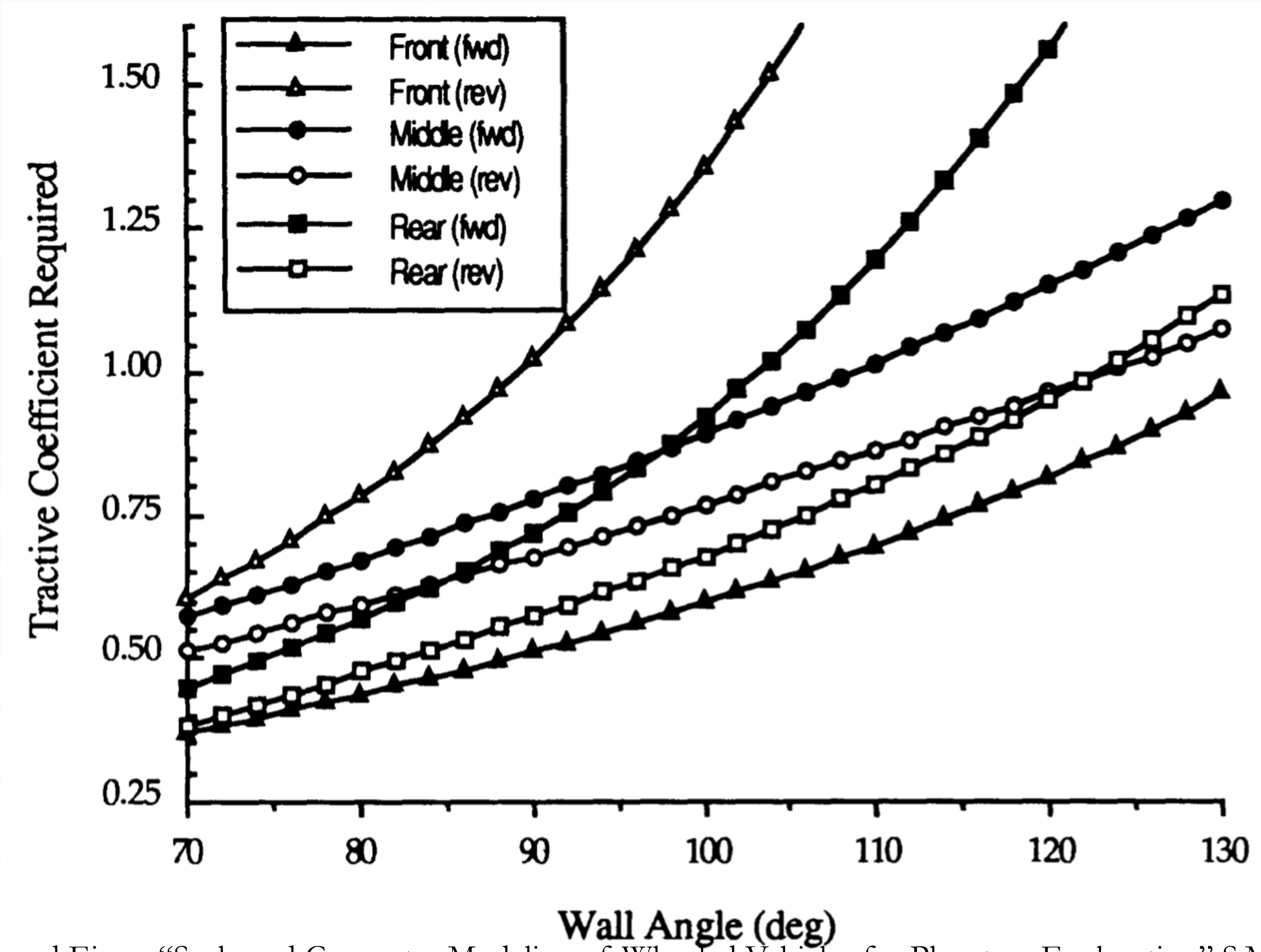
Navtest Rover with Walls and Slopes





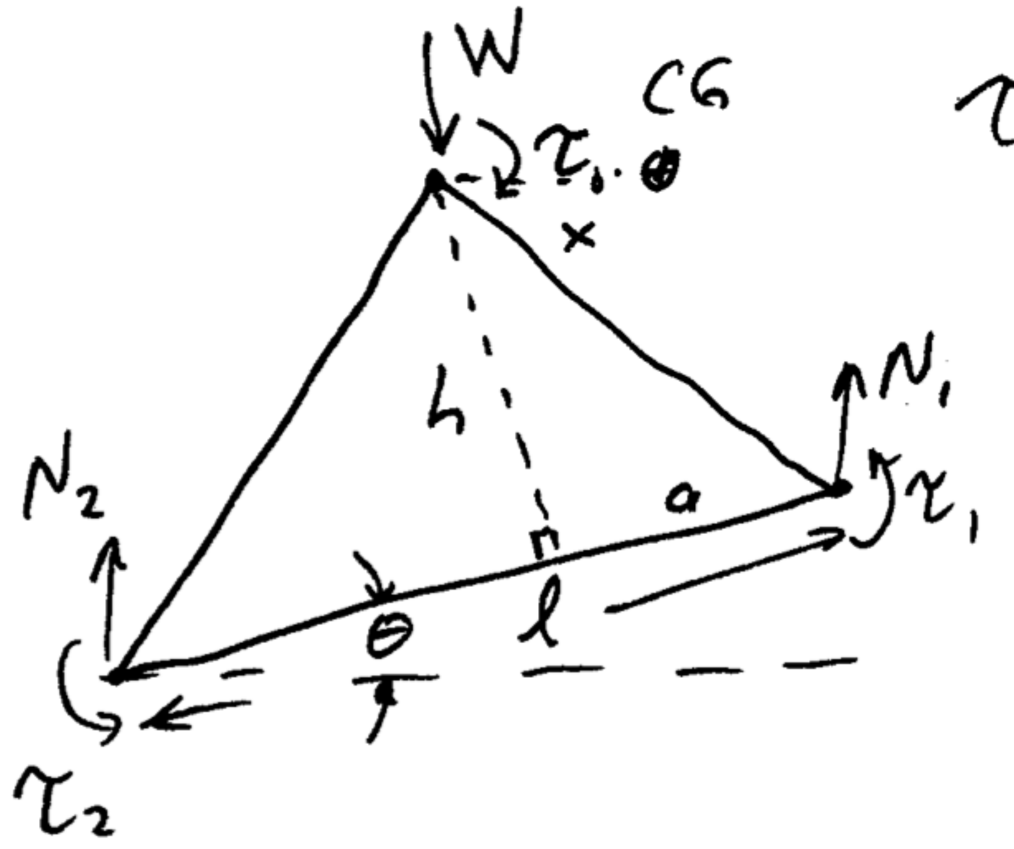
from Howard Eisen, “Scale and Computer Modeling of Wheeled Vehicles for Planetary Exploration” S.M. Thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, May, 1990

Six-Wheel Rover, Slope Climbing



from Howard Eisen, "Scale and Computer Modeling of Wheeled Vehicles for Planetary Exploration" S.M. Thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, May, 1990

Four-Wheel Rocker Suspension



$\tau_0 = xW$

Planar Rocker Analysis

Σ Forces: $N_1 + N_2 = W$

Σ Moment (rear axle)

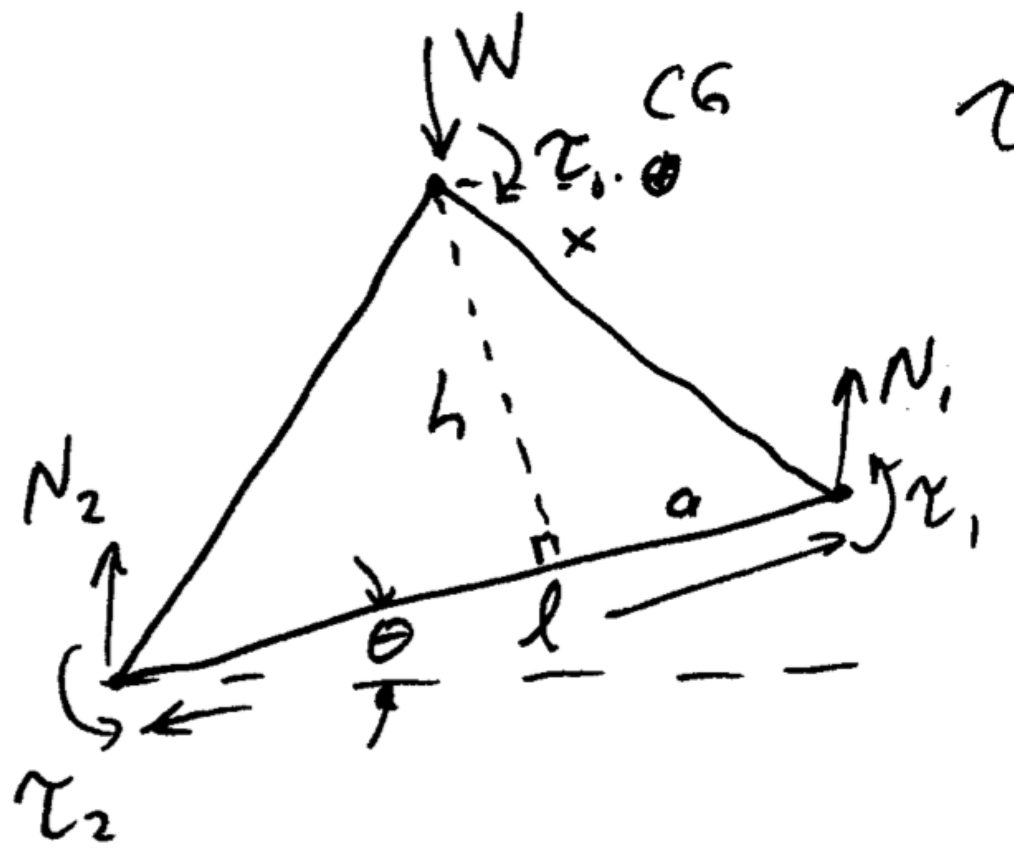
$$\tau_1 + \tau_2 + N_1 \ell \cos\theta = \tau_0 + W[(\ell - a)\cos\theta - h\sin\theta]$$

$$N_1 \ell \cos\theta = \tau_0 - \tau_1 - \tau_2 + W[(\ell - a)\cos\theta - h\sin\theta]$$

$$\boxed{N_1 = \frac{\tau_0 - \tau_1 - \tau_2}{\ell\cos\theta} + W\left(\frac{\ell - a}{\ell} - \frac{h}{\ell}\tan\theta\right)}$$

$$N_2 = W - N_1 = W\left(1 - \frac{\ell - a}{\ell} + \frac{h}{\ell}\tan\theta\right) - \frac{\tau_0 - \tau_1 - \tau_2}{\ell\cos\theta}$$

$$\boxed{N_2 = W\left(\frac{a}{\ell} + \frac{h}{\ell}\tan\theta\right) + \frac{\tau_1 + \tau_2 - \tau_0}{\ell\cos\theta}}$$



$\tau_0 = xW$

Nondimensionalize

$$\tau_0 = Wx \Rightarrow \frac{\tau_0}{Wl} = \frac{x}{l}$$

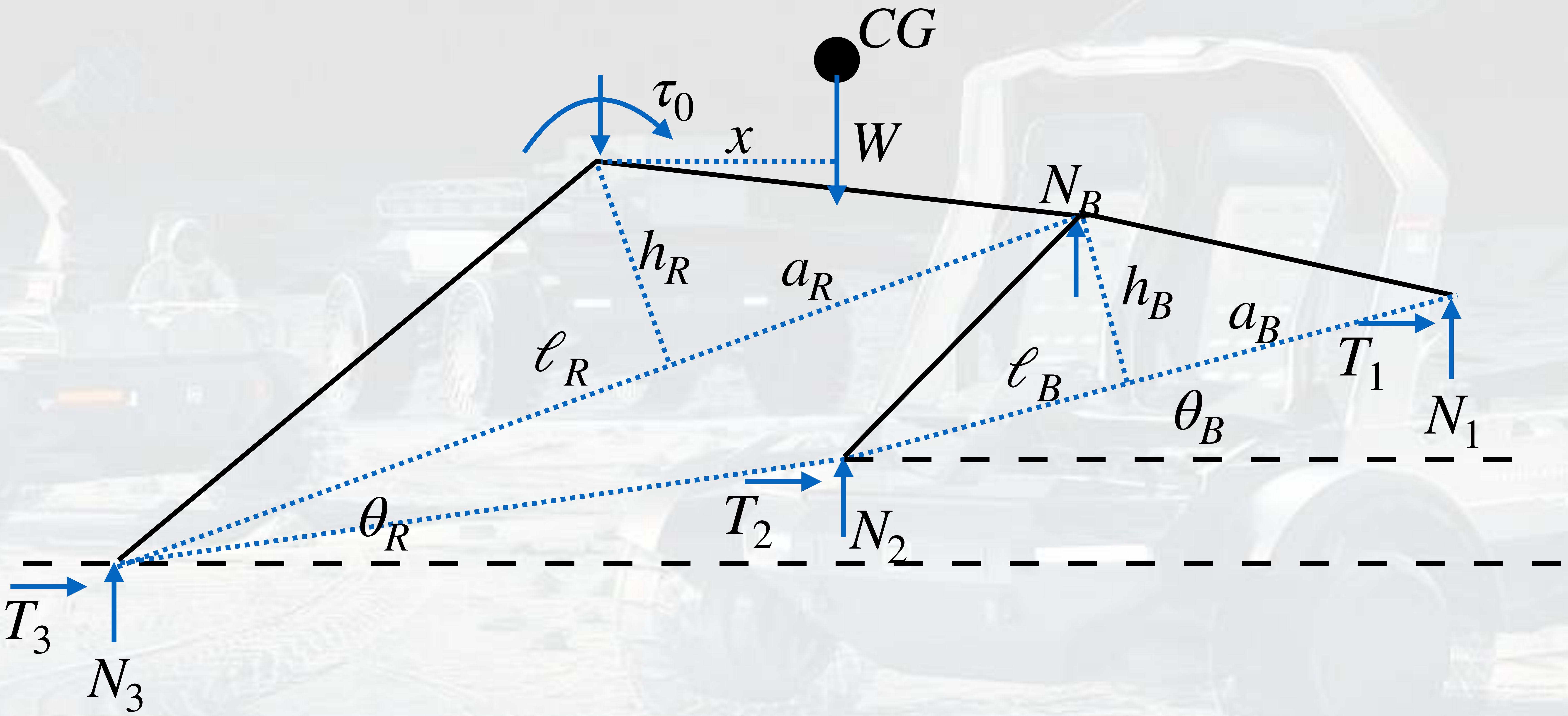
$$\tau_{wheel} = Tr \Rightarrow \frac{\tau_{wheel}}{Wl} = \frac{T}{W}\frac{r}{l}$$

$$\frac{N_1}{W} = 1 - \frac{a}{l} - \frac{h}{l}\tan\theta + \left(\frac{x}{l} - \frac{T_1}{W}\frac{r_1}{l} - \frac{T_2}{W}\frac{r_2}{l}\right)\frac{1}{\cos\theta}$$

$$\frac{N_2}{W} = \frac{a}{l} + \frac{h}{l}\tan\theta + \left(\frac{T_1}{W}\frac{r_1}{l} + \frac{T_2}{W}\frac{r_2}{l} - \frac{x}{l}\right)\frac{1}{\cos\theta}$$

Six-Wheel Rocker-Bogey Suspension

Kinematics of Planar Rocker-Bogey



Rocker-Bogey Wheel Force Distributions

$$\frac{N_3}{W} = \frac{a_R}{l_R} + \frac{h_R}{l_R}\tan\theta_R + \left(\frac{T_3}{W}\frac{r_z}{l_B}\frac{l_B}{l_R} - \frac{x}{l_B}\frac{l_B}{l_R}\right)\frac{1}{\cos\theta_R}$$

$$\frac{N_B}{W} = 1 - \frac{a_R}{l_R} - \frac{h_R}{l_R}\tan\theta_R + \left(\frac{x}{l_B}\frac{l_B}{l_R} - \frac{T_3}{W}\frac{r_3}{l_B}\frac{l_B}{l_R}\right)\frac{1}{\cos\theta_R}$$

$$\frac{N_1}{W} = \frac{N_B}{W}\left(1 - \frac{a_B}{l_B} - \frac{h_B}{l_B}\tan\theta_B\right) - \left(\frac{T_1}{W}\frac{r_1}{l_B} + \frac{T_2}{W}\frac{r_2}{l_B}\right)\frac{1}{\cos\theta_B}$$

$$\frac{N_2}{W} = \frac{N_B}{W}\left(\frac{a_B}{l_B} + \frac{h_B}{l_B}\tan\theta_B\right) + \left(\frac{T_1}{W}\frac{r_1}{l_B} + \frac{T_2}{W}\frac{r_2}{l_B}\right)\frac{1}{\cos\theta_B}$$

Kinematics of Planar Rocker-Bogey

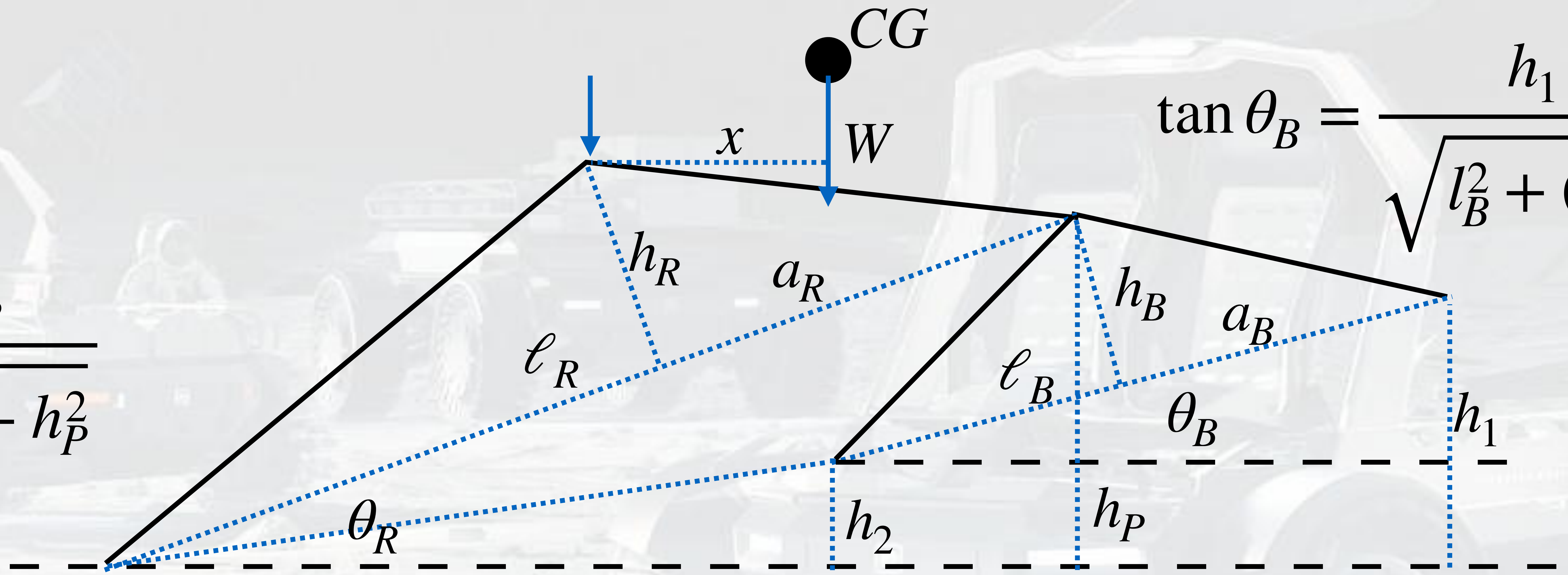
$$h_P = h_2 + (l_B - a_B)\sin\theta_B + h_B\cos\theta_B$$

$$\sin\theta_B = \frac{h_1 - h_2}{l_B}$$

$$\sin\theta_R = \frac{h_P}{l_R}$$

$$\tan\theta_B = \frac{h_1 - h_2}{\sqrt{l_B^2 + (h_1 - h_2)^2}}$$

$$\tan\theta_R = \frac{h_P}{\sqrt{l_R^2 + h_P^2}}$$



$$\frac{1}{\cos\theta_R} = \frac{l_R}{\sqrt{l_R^2 + h_P^2}}$$

$$\frac{1}{\cos\theta_B} = \frac{l_B}{\sqrt{l_B^2 + (h_1 - h_2)^2}}$$

Level Ground: $\theta_R = \theta_B = 0$

$$\frac{N_3}{W} = \frac{a_R}{l_R} + \frac{T_3}{W}\frac{r_z}{l_B}\frac{l_B}{l_R} - \frac{x}{l_B}\frac{l_B}{l_R}$$

$$\frac{N_B}{W} = 1 - \frac{a_R}{l_R} + \frac{x}{l_B}\frac{l_B}{l_R} - \frac{T_3}{W}\frac{r_3}{l_B}\frac{l_B}{l_R}$$

$$\frac{N_1}{W} = \frac{N_B}{W}\left(1 - \frac{a_B}{l_B}\right) - \left(\frac{T_1}{W}\frac{r_1}{l_B} + \frac{T_2}{W}\frac{r_2}{l_B}\right)$$

$$\frac{N_2}{W} = \frac{N_B}{W}\frac{a_B}{l_B} + \frac{T_1}{W}\frac{r_1}{l_B} + \frac{T_2}{W}\frac{r_2}{l_B}$$

Level Ground, Static: $\theta_R = \theta_B = 0, T_1 = T_2 = T_3 = 0$

$$\frac{N_3}{W} = \frac{a_R}{l_R} - \frac{x}{l_B}\frac{l_B}{l_R}$$

$$\frac{N_B}{W} = 1 - \frac{a_R}{l_R} + \frac{x}{l_B}\frac{l_B}{l_R}$$

$$\frac{N_1}{W} = \frac{N_B}{W}\left(1 - \frac{a_B}{l_B}\right) = \left(1 - \frac{a_R}{l_R} + \frac{x}{l_B}\frac{l_B}{l_R}\right)\left(1 - \frac{a_B}{l_B}\right)$$

$$\frac{N_2}{W} = \frac{N_B}{W}\frac{a_B}{l_B} = \left(1 - \frac{a_R}{l_R} + \frac{x}{l_B}\frac{l_B}{l_R}\right)\frac{a_B}{l_B}$$

Level Ground, Static: $x = 0, \frac{a_B}{l_B} = \frac{a_R}{l_R} = 0.5$

$$\frac{N_3}{W} = \frac{a_R}{l_R} = 0.5$$

$$\frac{N_B}{W} = 1 - \frac{a_R}{l_R} = 0.5$$

$$\frac{N_1}{W} = \left(1 - \frac{a_R}{l_R}\right)\left(1 - \frac{a_B}{l_B}\right) = 0.25$$

$$\frac{N_2}{W} = \left(1 - \frac{a_R}{l_R}\right)\frac{a_B}{l_B} = 0.25$$

Level Ground, Static: $x = 0, \frac{a_B}{l_B} = 0.5, \frac{a_R}{l_R} = 0.333$

$$\frac{N_3}{W} = \frac{a_R}{l_R} = 0.333$$

$$\frac{N_B}{W} = 1 - \frac{a_R}{l_R} = 0.667$$

$$\frac{N_1}{W} = \left(1 - \frac{a_R}{l_R}\right)\left(1 - \frac{a_B}{l_B}\right) = 0.333$$

$$\frac{N_2}{W} = \left(1 - \frac{a_R}{l_R}\right)\frac{a_B}{l_B} = 0.333$$

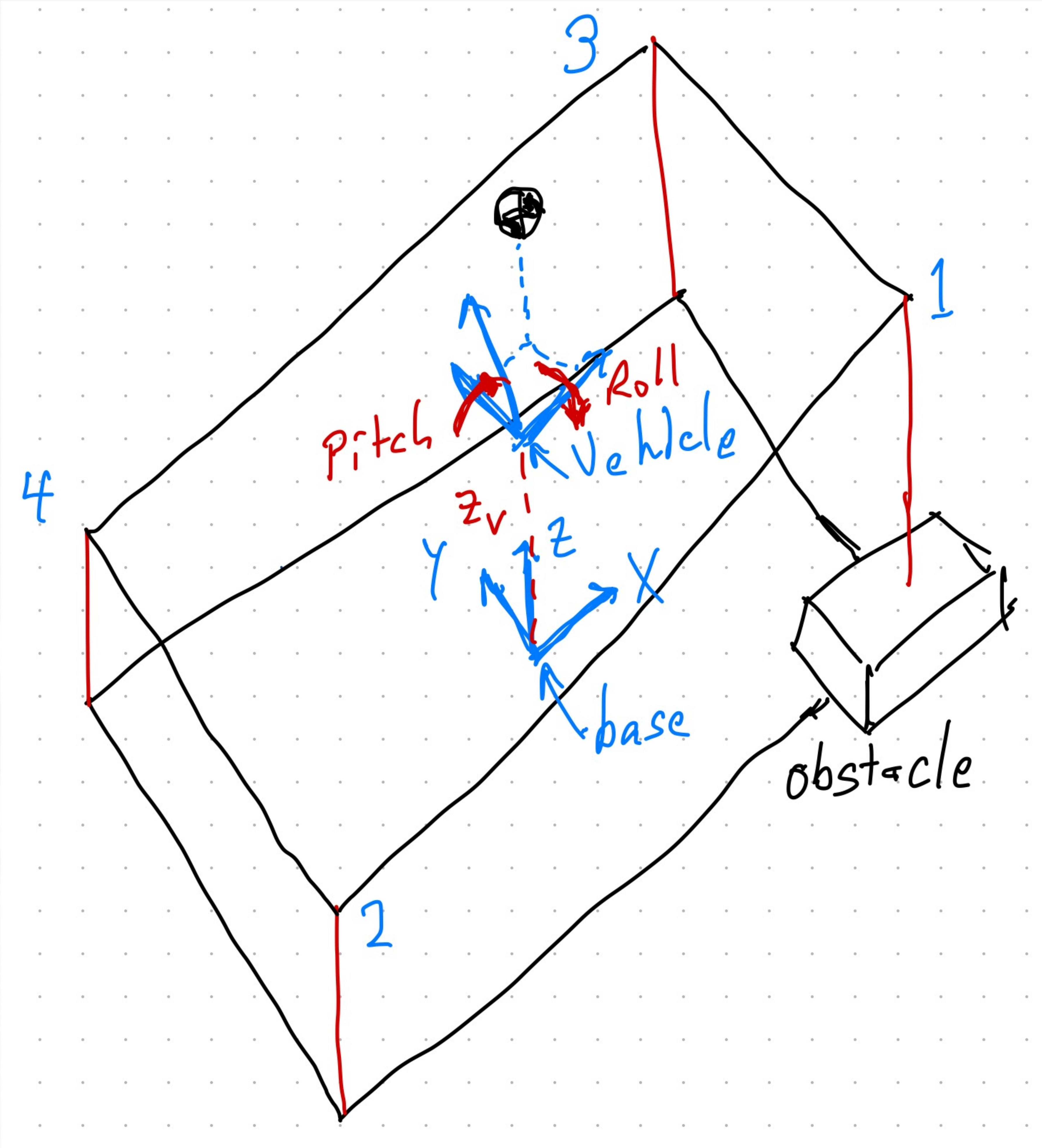
Static Weight Distribution Revisited

- Previous analysis approached static weight distribution of vehicles with no suspensions or purely kinematic suspension systems (e.g., rocker and rocker-bogey systems)
- Four-wheel fixed suspension “cheated” by having one wheel off the ground
  - Three conservation equations: $\sum$wheel forces=weight, $\sum$roll torques=0; $\sum$pitch torques=0
  - Three unknowns (weight on three wheels on ground)
  - $\Rightarrow$Statically determinate system

N-Wheeled Independent Suspension

- Force on wheel dependent on deflection of suspension spring
- Explicit solution only available for three-wheeled vehicle (only three constraint equations)
- Result of suspension and terrain is the height, pitch angle, and roll angle of vehicle chassis (three unknowns)
- Parallel actuator forward kinematics problem
- Solve via assumed body pose and use of gradient search techniques for brute force solution

Vehicle Definition in Vector Form



Location of wheels (vehicle frame)

$$[\mathbf{X}_w]_v = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & \\ y_1 & y_2 & y_3 & y_4 & \\ z_1 & z_2 & z_3 & z_4 & \cdots \\ 1 & 1 & 1 & 1 & \end{bmatrix}$$

$$[\mathbf{X}_{cg}]_v = \begin{bmatrix} x_{cg} \\ y_{cg} \\ z_{cg} \\ 1 \end{bmatrix}$$

Locating Vehicle in Global Coords

$$\{c\theta = \cos\theta\}$$
$$\{s\theta = \sin\theta\}$$

$$\text{Roll Transform } \mathbf{R}_\theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & s\theta & 0 \\ 0 & -s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Pitch Transform } \mathbf{R}_\phi = \begin{bmatrix} c\phi & 0 & s\phi & 0 \\ 0 & 1 & 0 & 0 \\ -s\phi & 0 & c\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Translational Transform } \mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & x_v \\ 0 & 1 & 0 & y_v \\ 0 & 0 & 1 & z_v \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}\mathbf{R}_\theta\mathbf{R}_\phi = \begin{bmatrix} 1 & 0 & 0 & x_v \\ 0 & 1 & 0 & y_v \\ 0 & 0 & 1 & z_v \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & s\theta & 0 \\ 0 & -s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\phi & 0 & s\phi & 0 \\ 0 & 1 & 0 & 0 \\ -s\phi & 0 & c\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Conversion to Base Coordinates

$$[\mathbf{T}]_o^v = \begin{bmatrix} c\phi & 0 & s\phi & x_v \\ -s\phi s\theta & c\theta & c\phi s\theta & y_v \\ -s\phi c\theta & -s\theta & c\phi c\theta & z_v \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[\mathbf{X}_w]_o = [\mathbf{T}]_o^v [\mathbf{X}_w]_v$$

$$[\mathbf{X}_{cg}]_o = [\mathbf{T}]_o^v [\mathbf{X}_{cg}]_v$$

Choose a translation vector that keeps vehicle origin directly above base frame origin

$$\begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z_c \end{bmatrix}$$

Solution Algorithm

- Assumed θ, ϕ, and z_c produces wheel heights z_w
- Spring compression $x_{spr} = \ell_{spr} - z_w + z_{obstacle}$
- Spring force $F_{spr(i)} = K_{spr(i)} x_{spr(i)}$
- Constraint equations

$$\sum_{1}^{n\ wheels} F_{spr(i)} = W_v$$

$$\sum_{1}^{n\ wheels} F_{spr(i)} x_{w(i)} + W_v x_{cg} = 0$$

$$\sum_{1}^{n\ wheels} F_{spr(i)} y_{w(i)} + W_v y_{cg} = 0$$

- Iterate for θ, ϕ, and z_c to meet constraints

Apollo LRV Example Data

$$[\mathbf{X}_w]_v = \begin{bmatrix} 1.143 & -1.143 & 1.143 & -1.143 \\ 0.914 & 0.914 & -0.914 & -0.914 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$W = 1069\ N\ (\text{lunar})$$

$$\ell_{spring} = 0.5\ m\ (\text{unloaded})$$

$$K_{spring} = 2000\ N/m$$

$$[\mathbf{X}_{cg}]_v = \begin{bmatrix} -0.2 \\ .1 \\ 0.5 \\ 1 \end{bmatrix}$$

Apollo LRV on Level Ground

C.G. at LRV geometric center — Pitch=0°; Roll=0°

$$[\mathbf{X}_w]_v = \begin{bmatrix} 1.143 & -1.143 & 1.143 & -1.143 \\ 0.914 & 0.914 & -0.914 & -0.914 \\ 0.366 & 0.366 & 0.366 & 0.366 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[\mathbf{F}_w] = [267 \quad 267 \quad 267 \quad 267] \ (N)$$

C.G. at LRV nominal location — Pitch=−1.24°; Roll=0°

$$[\mathbf{X}_w]_v = \begin{bmatrix} 1.143 & -1.143 & 1.143 & -1.143 \\ 0.914 & 0.914 & -0.914 & -0.914 \\ 0.391 & 0.342 & 0.391 & 0.342 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[\mathbf{F}_w] = [218 \quad 317 \quad 218 \quad 317] \ (N)$$

Apollo LRV on Obstacles

Right front wheel on 30cm obstacle Pitch=−5.2°; Roll=−5.1°

$$[\mathbf{X}_w]_v = \begin{bmatrix} 1.138 & -1.138 & 1.138 & -1.138 \\ 0.901 & 0.920 & -0.920 & -0.901 \\ 0.627 & 0.419 & 0.464 & 0.256 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[\mathbf{F}_w] = [347 \quad 162 \quad 73 \quad 488] \ \ (N)$$

RF wheel in 10cm hole; RR wheel on 30cm obstacle

Pitch=−4.1°; Roll=−3.4°

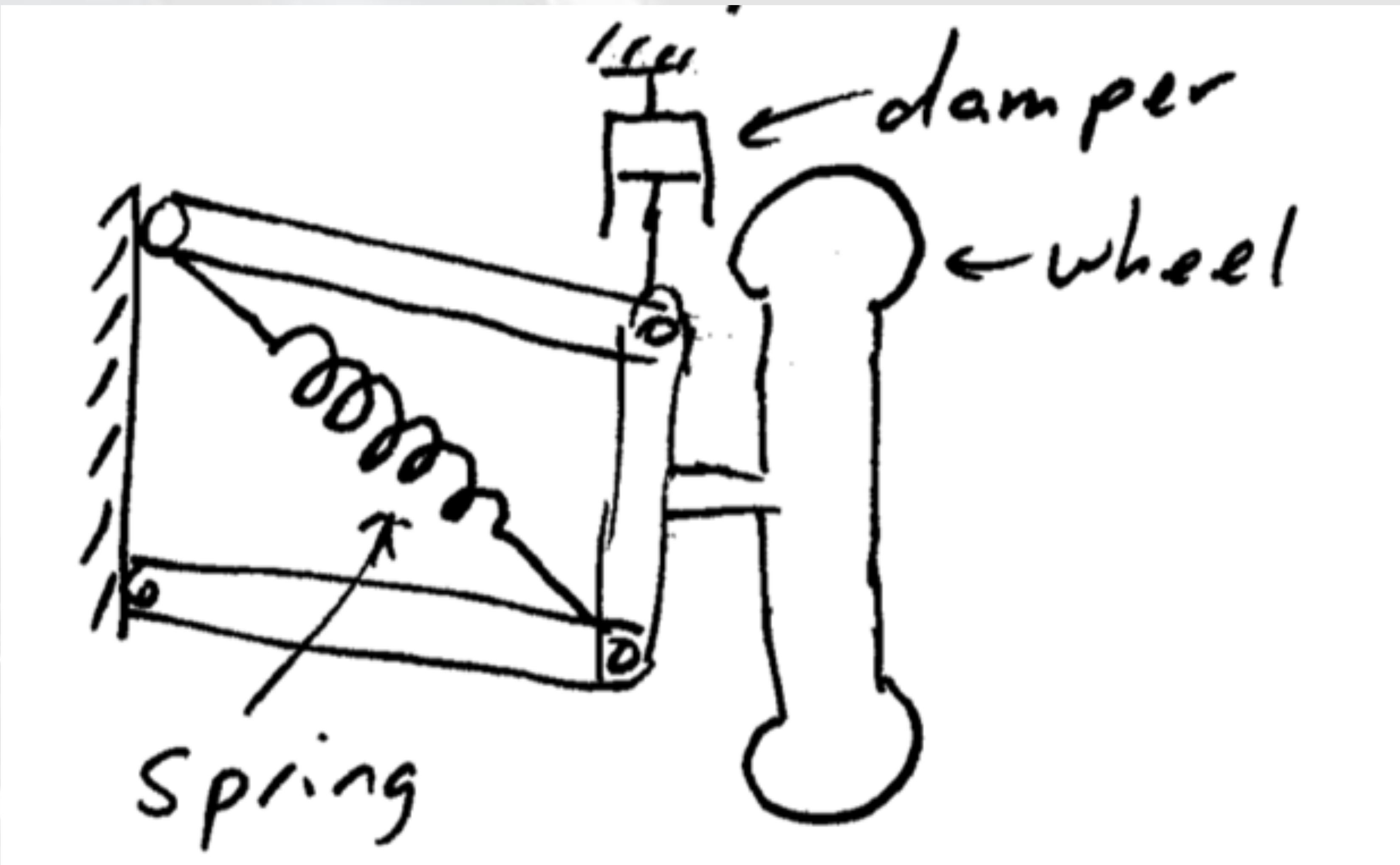
$$[\mathbf{X}_w]_v = \begin{bmatrix} 1.140 & -1.140 & 1.140 & -1.140 \\ 0.918 & 0.908 & -0.908 & -0.918 \\ 0.390 & 0.552 & 0.281 & 0.443 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[\mathbf{F}_w] = [20 \quad 497 \quad 438 \quad 114] \ \ (N)$$

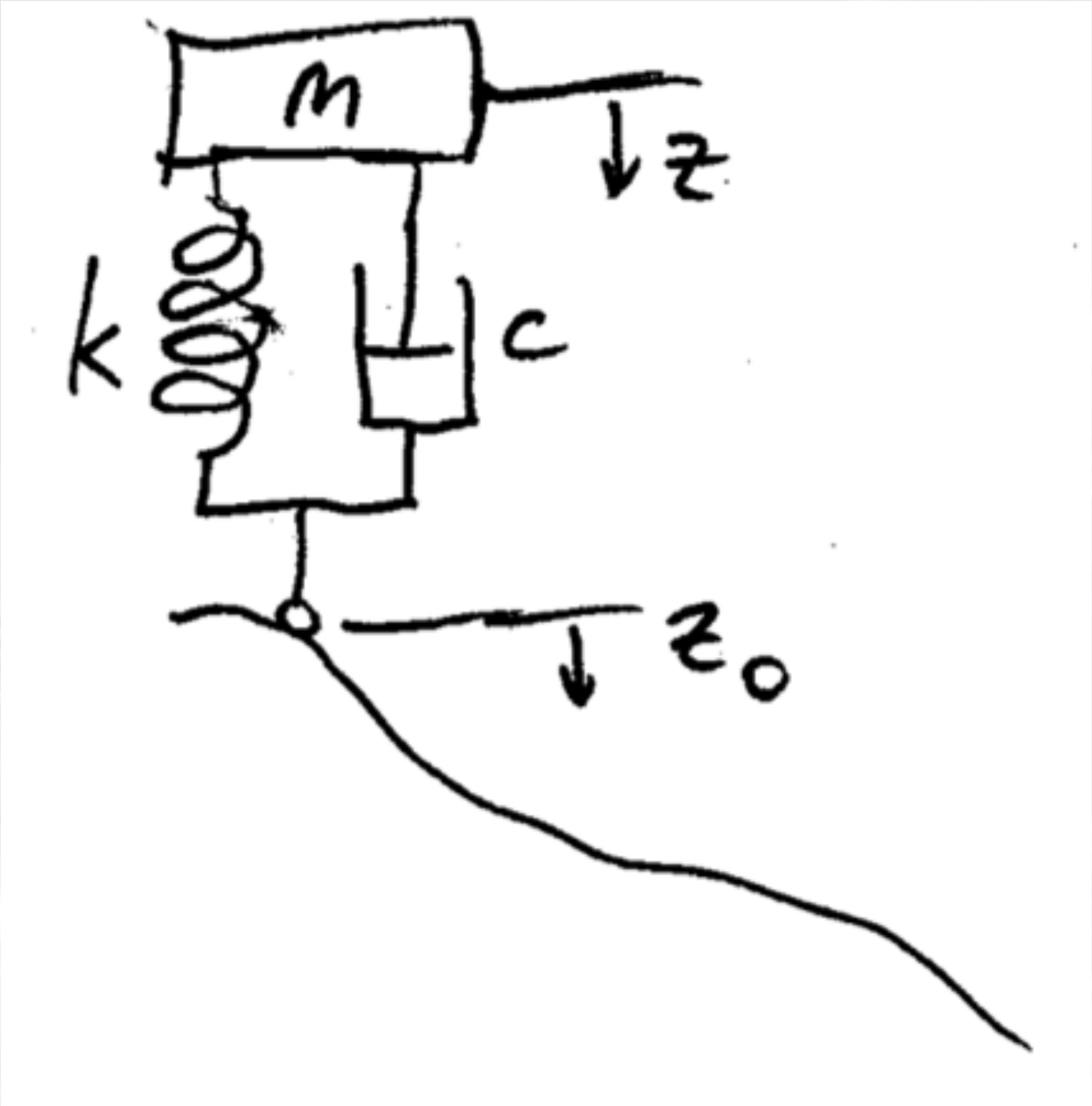
Shortcomings and Extensions

- Made the simplifying assumption that suspensions are always vertical
  - Vehicle deck angles are generally small
  - Can add geometric specifications with more vectors
- Ignored the wheels
  - "Body height" is really suspension height
  - Could add in wheel radius for height off ground
  - Subtract wheel weight and use "sprung weight"
- Assumed independent spring suspensions
- Neglected wheel torques
- Anything is possible with more math

Suspension Systems



Single-Wheel Dynamic Suspension Model



$$m\ddot{z} + c\dot{z} + kz = c\dot{z}_0 + kz_0$$

Undamped force-free equation: $m\ddot{z} + kz = 0$

Assume solution of the form $z = Z\cos\omega_n t$

$$\ddot{z} = -Z\omega_n^2\cos\omega_n t$$

$$-m\omega_n^2 + k = 0$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

Single-Wheel Suspension Example

Mars Rover: $m_{tot} = 500\ kg \implies$ for each wheel, $m = 125\ kg$

$d =$ deflection of suspension (at rest) $\sim 0.1\ m$

$$k = \frac{F}{d} = \frac{mg}{d}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$f_n = \frac{\omega_n}{2\pi}$$

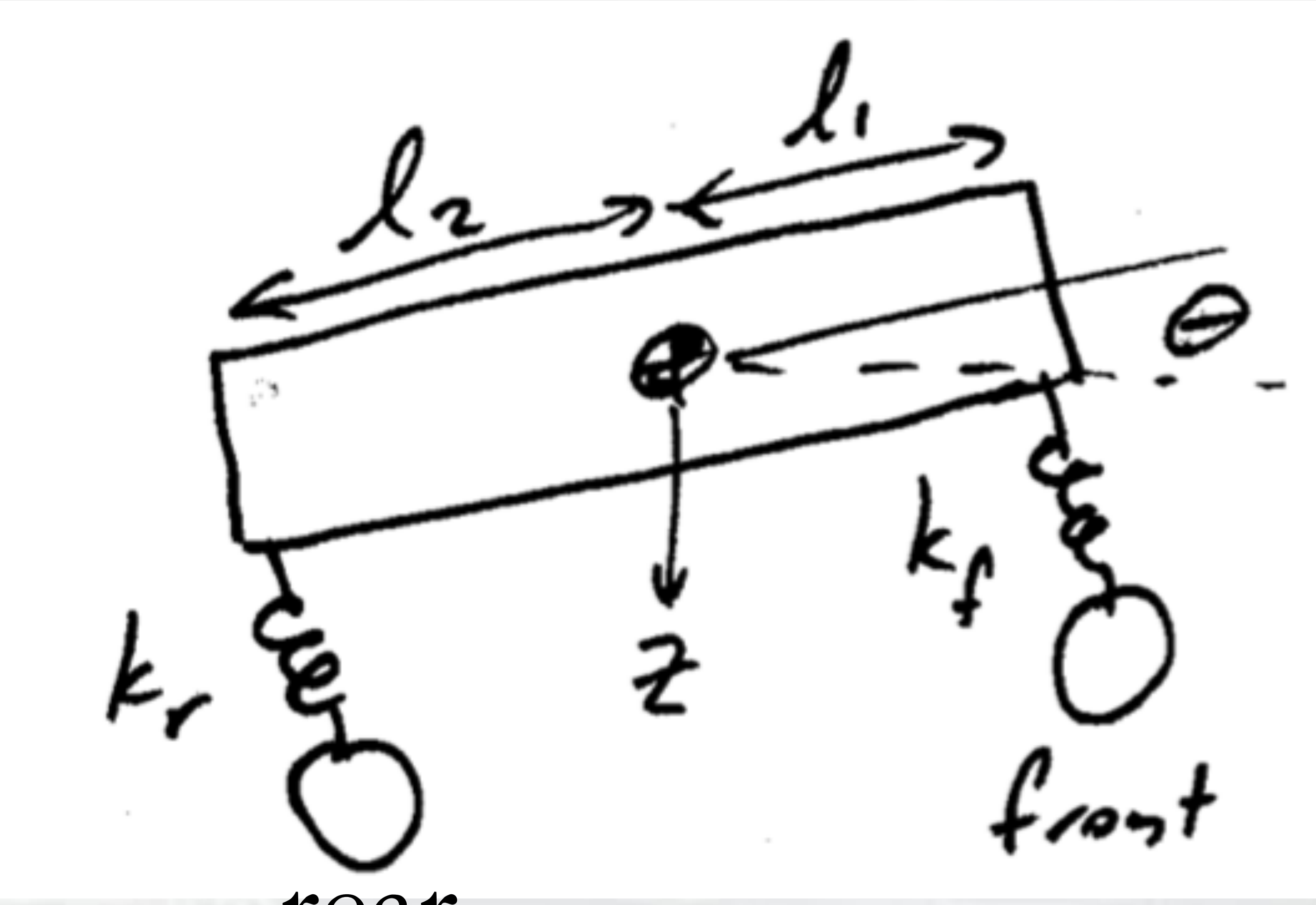
$$\ell_{crit} = \frac{f_n}{v}$$

	Earth	Mars
$k\ \left(\frac{N}{m}\right)$	12,500	2000
$\omega_n\ \left(\frac{rad}{sec}\right)$	9.9	4
$f_n\ (Hz)$	1.6	0.64
$\ell_{crit}\ (m)$	1.8	4.3

@$2.8\ m/sec$ ($10\ km/hr$)

Multiwheel Suspension Dynamic Analysis

Two possible responses to hitting a bump $\Rightarrow$



rear

Equations of motion (assuming small angles)

Bounce: $m\ddot{z} + k_f\left(z - l_1\theta\right) + k_r\left(z + l_2\theta\right) = 0$

Pitch: $I_y\ddot{\theta} + k_f l_1\left(z - l_1\theta\right) + k_r l_2\left(z + l_2\theta\right) = 0$

Solve as a set of coupled differential equations

let $D_1 \equiv \dfrac{k_f + k_r}{m}$ $D_2 \equiv \dfrac{k_r l_2 - k_f l_1}{m}$ $D_3 \equiv \dfrac{k_f l_1^2 + k_r l_2^2}{I_y}$

Rewrite in terms of D variables

$$\ddot{z} + D_1 z + D_2 \theta = 0 \qquad \ddot{\theta} + D_3 \theta + \frac{D_2}{{r_y}^2} z = 0$$

D_2 is the coupling coefficient - equations are independent if $D_2 = 0 \;\Rightarrow\; k_f l_1 = k_r l_2$

If $D_2 = 0$, force @ CG only produces bounce $\omega_{n_z} = \sqrt{D_1}$

force elsewhere produces pitch $\omega_{n_\theta} = \sqrt{D_3}$

Assume $D_2 \neq 0$

$$z = Z \cos \omega_n t \qquad \theta = \Theta \cos \omega_n t$$

$$\left.\begin{aligned}\left(D_1-\omega_n{}^2\right)Z+D_2\theta=0\\ \frac{D_2}{r_y^2}Z+\left(D_3-\omega_n^2\right)\theta=0\end{aligned}\right\}\quad\begin{vmatrix}D_1-\omega_n^2 & D_2\\ \frac{D_2}{r_y^2} & D_3-\omega_n{}^2\end{vmatrix}=0$$

$$\omega_n^4-\left(D_1+D_3\right)\omega_n^2+\left(D_1D_3-\frac{D_2^2}{r_y^2}\right)=0$$

$$\omega_n^2=\frac{D_1+D_3}{2}\pm\frac{1}{2}\sqrt{\left(D_1+D_3\right)^2-4\left(D_1D_3-\frac{D_2^2}{r_\gamma^2}\right)}$$

$$\omega_{n_1}^2 = \frac{D_1 + D_3}{2} + \sqrt{\frac{1}{4}\left(D_1 - D_3\right)^2 + \frac{D_2^2}{r_\gamma^2}}$$

$$\omega_{n_2}^2 = \frac{D_1 + D_3}{2} - \sqrt{\frac{1}{4}\left(D_1 - D_3\right)^2 + \frac{D_2^2}{r_\gamma^2}}$$

Example: $k_f = k_r = 2000\ N/m$ (Moon)

$$l_1 = 1\ m \qquad l_2 = 2\ m$$

$$I = \frac{ml^2}{12} \Rightarrow I_y = 375\ kg - m^2 \Rightarrow r + y = 0.75\ m$$

$$D_1 = \frac{4000}{500} = 8\frac{\text{N/m}}{\text{kg}} \left\langle \frac{1}{\text{sec}^2} \right\rangle$$

$$D_2 = \frac{2000\ N/m\ (2\ m) - 2000\ N/m\ (1\ m)}{500\ kg} = 4\ \frac{N}{kg} = 4\ m/sec^2$$

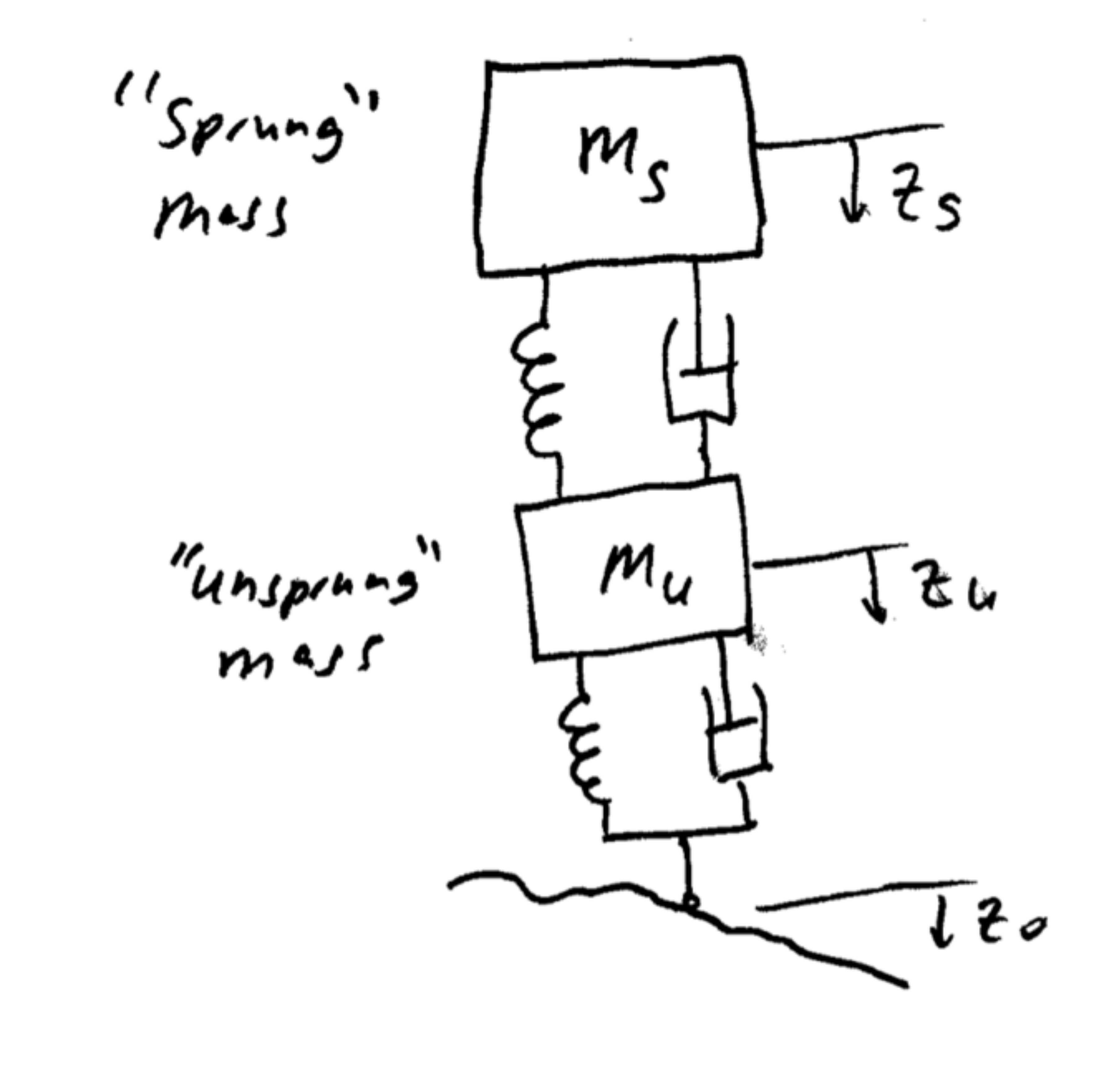
$$D_3 = \frac{2000\ N/m(1\ m)^2 + 2000\ N/m(2m)^2}{375\ kg\ m^2} = 26.7\frac{\frac{N}{m}m^2}{kgm^2} = \left\langle \frac{1}{sec^2} \right\rangle$$

$$\omega_n^2 = 17.33 \pm 10.43$$

$$\omega_{n_1} = 2.63\ rad/sec \Rightarrow 0.42\ Hz$$

$$\omega_{n_2} = 5.67\ rad/sec \Rightarrow 0.84\ Hz$$

Adding in Tire Mass and Stiffness



Sprung mass: $m_s\ddot{z}_s + C_s\left(\dot{z}_s - \dot{z}_u\right) + k_s\left(z_s - z_u\right) = 0$

Unsprung mass: $m_u\ddot{z}_u + c_s\left(\dot{z}_u - \dot{z}_s\right) + k_s\left(z_u - z_s\right)$

$$+c_u\dot{z}_u + k_u z_u = F(t) = c_u\dot{z}_0 + k_u z_0$$

Undamped force-free solutions

$$m_s\ddot{z}_s + k_s\left(z_s - z_u\right) = 0$$

$$m_u\ddot{z}_u + k_s\left(z_u - z_s\right) + k_u z_u = 0$$

$$z_s = Z_s \cos\omega_n t \qquad z_u = Z_u \cos\omega_n t$$

$$\begin{vmatrix} k_s - m_s\omega_n^2 & -k_u \\ -k_s & k_s + k_u - m_u\omega_n^2 \end{vmatrix} = 0$$

$$\omega_n^4 \left(m_u m_s\right) + \omega_n^2 \left(-m_s k_s - m_s k_u - m_u k_s\right) + k_s k_u = 0$$

$$A = m_u m_s \qquad B = m_s\left(k_s + k_u\right) + m_u k_s \qquad C = k_s k_u$$

$$\omega_{n_1} = \frac{B - \sqrt{B^2 - 4AC}}{2A} \qquad \omega_{n_2} = \frac{B + \sqrt{B^2 - 4AC}}{2A}$$

Example using Unsprung Mass

Example:

$$m_s = 100\ kg \qquad m_n = 25\ kg$$

$$k_s = 2000\ N/m \quad k_n = 10{,}000\ N/m$$

$$A = 2500\ kg^2 \qquad B = 1.25 \times 10^6\ kg^2/sec^2 \qquad C = 2 \times 10^7\ N^2/m^2$$

$$\omega_{n_1} = 4.76\ \frac{rad}{sec} = 0.8\ Hz \Longleftarrow \text{suspension frequency}$$

$$\omega_{n_1} = 21.8\ \frac{rad}{sec} = 3.5\ Hz \Longleftarrow \text{wheel stiffness frequency}$$