Rocket Performance

- The rocket equation
- Mass ratio and performance
- Structural and payload mass fractions
- Multistaging
- Optimal ΔV distribution between stages
- Trade-off ratios
- Parallel staging
- Modular staging

Derivation of the Rocket Equation

- Momentum at time t:

$$M = mv$$

- Momentum at time t+Δt:

$$M = (m - \Delta m)(v + \Delta v) + \Delta m(v - V_e)$$

- Some algebraic manipulation gives:

$$m\Delta v = -\Delta m V_e$$

- Take to limits and integrate:

$$\int_{m_{initial}}^{m_{final}} \left(\frac{dm}{m}\right) = -\int_{V_{initial}}^{V_{final}} \left(\frac{dv}{V_e}\right)$$

The Rocket Equation

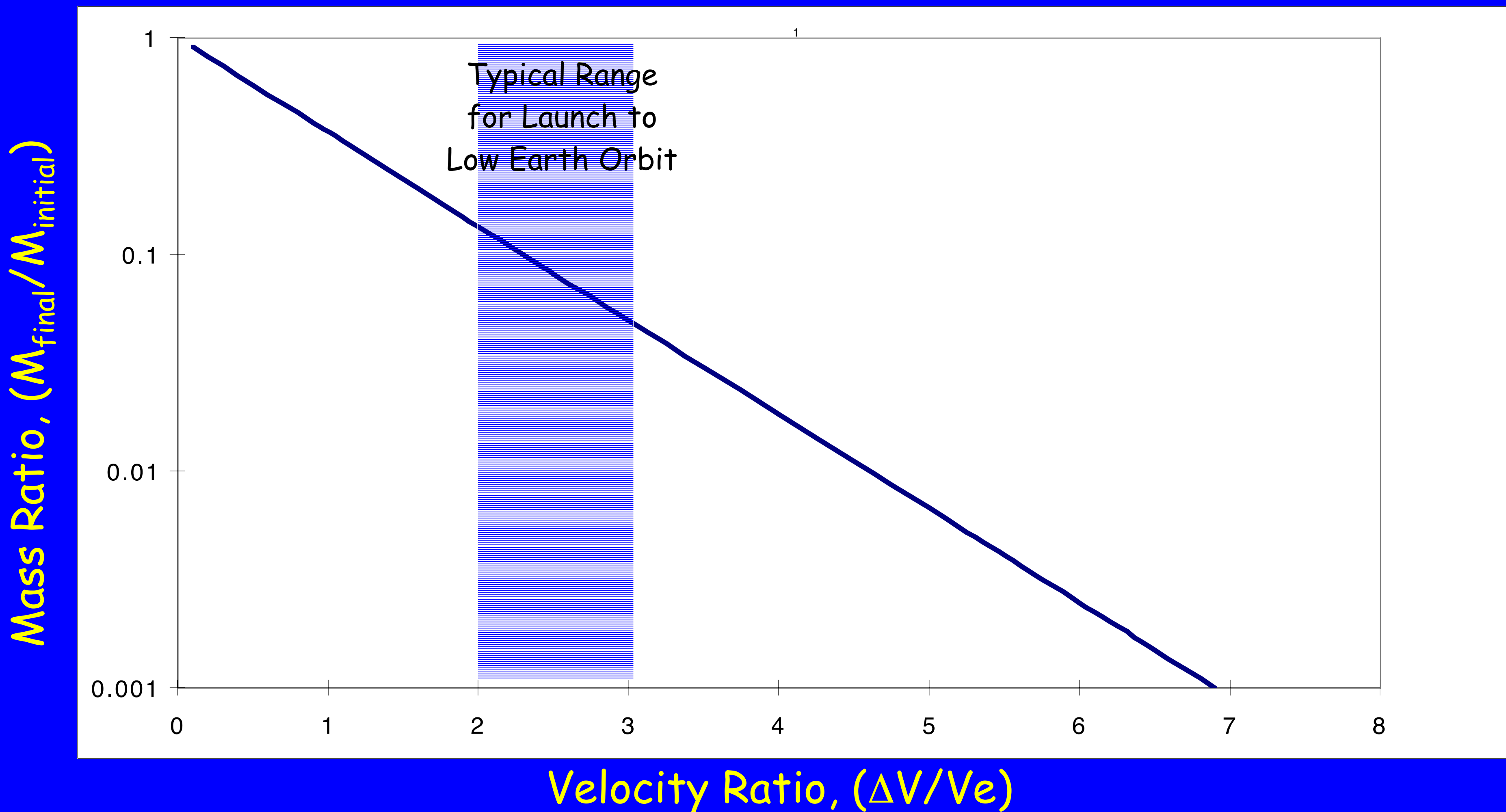
- Alternate forms

$$r \equiv \frac{m_{final}}{m_{initial}} = e^{-\frac{\Delta V}{V_e}}$$

$$\Delta V = -V_e \ln\left(\frac{m_{final}}{m_{initial}}\right) = -V_e \ln(r)$$

- Basic definitions/concepts
  - Mass ratio $r \equiv \frac{m_{final}}{m_{initial}} \text{ or } \Re = \frac{m_{initial}}{m_{final}}$
  - Nondimensional velocity change "Velocity ratio" $\frac{\Delta V}{V_e}$

Rocket Equation (First Look)



Sources and Categories of Vehicle Mass

- Payload
- Propellants
- Inert Mass
  - Structure
  - Propulsion
  - Avionics
  - Mechanisms
  - Thermal
  - Etc.

Basic Vehicle Parameters

- Basic mass summary

$$m_0 = m_L + m_p + m_i$$

m_0 = initial mass

m_L = payload mass

m_p = propellant mass

m_i = inert mass

- Inert mass fraction

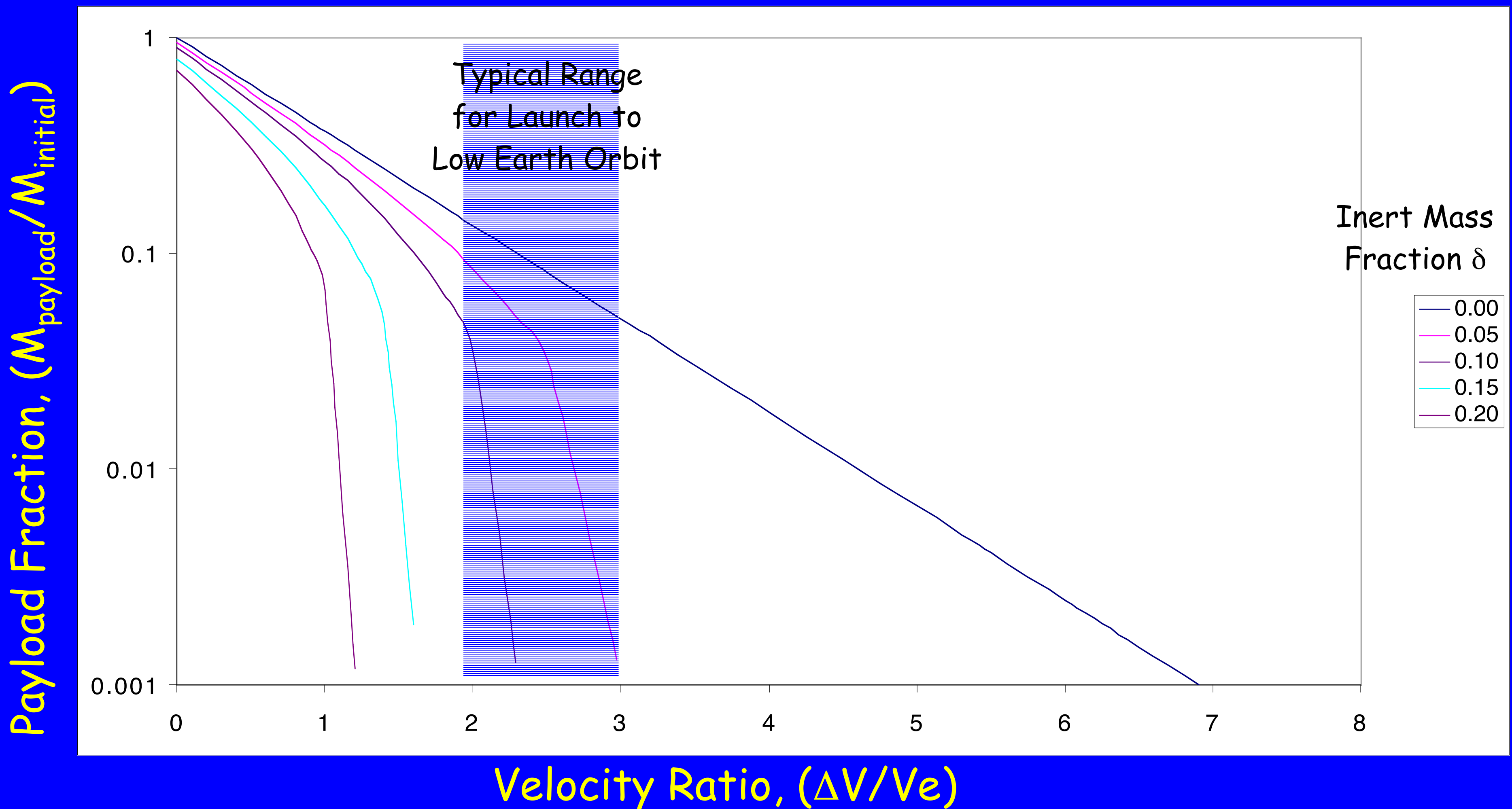
$$\delta = \frac{m_i}{m_0} = \frac{m_i}{m_L + m_p + m_i}$$

- Payload fraction

$$\lambda = \frac{m_L}{m_0} = \frac{m_L}{m_L + m_p + m_i}$$

- Parametric mass ratio $r = \lambda + \delta$

Rocket Equation (including Inert Mass)



The Rocket Equation for Multiple Stages

- Assume two stages

$$\Delta V_1 = -V_{e,1} \ln\left(\frac{m_{final,1}}{m_{initial,1}}\right) = -V_{e,1} \ln(r_1)$$

$$\Delta V_2 = -V_{e,2} \ln\left(\frac{m_{final,2}}{m_{initial,2}}\right) = -V_{e,2} \ln(r_2)$$

- Assume $V_{e,1}=V_{e,2}=V_e$

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1) - V_e \ln(r_2) = -V_e \ln(r_1 r_2)$$

Continued Look at Multistaging

- Converting to masses

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1 r_2) = -V_e \ln\left(\frac{m_{final,1}}{m_{initial,1}} \frac{m_{final,2}}{m_{initial,2}}\right)$$

- Keep in mind that $m_{final,1} \sim m_{initial,2}$

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1 r_2) = -V_e \ln\left(\frac{m_{final,2}}{m_{initial,1}}\right) = -V_e \ln(r_0)$$

- r_0 has *no* physical significance!

Multistage Vehicle Parameters

- Inert mass fraction

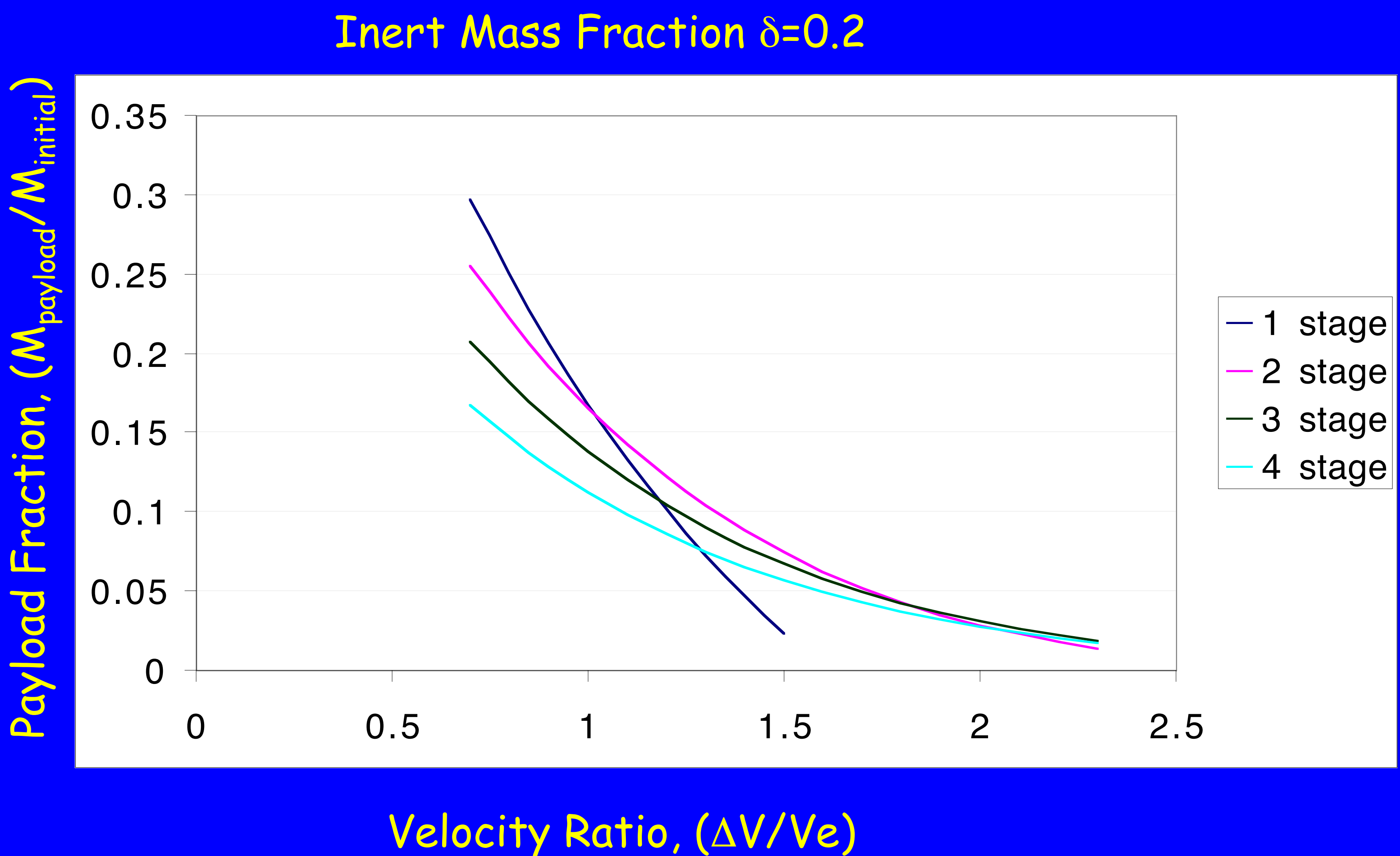
$$\delta_0 = \frac{\sum m_{i,j}}{m_0} = \sum_{j=1}^{n\ stages} \left(\delta_j \prod_{\ell=1}^{j-1} \lambda_\ell \right)$$

- Payload fraction

$$\lambda_0 = \frac{m_L}{m_0} = \prod_{i=1}^{n\ stages} \lambda_i$$

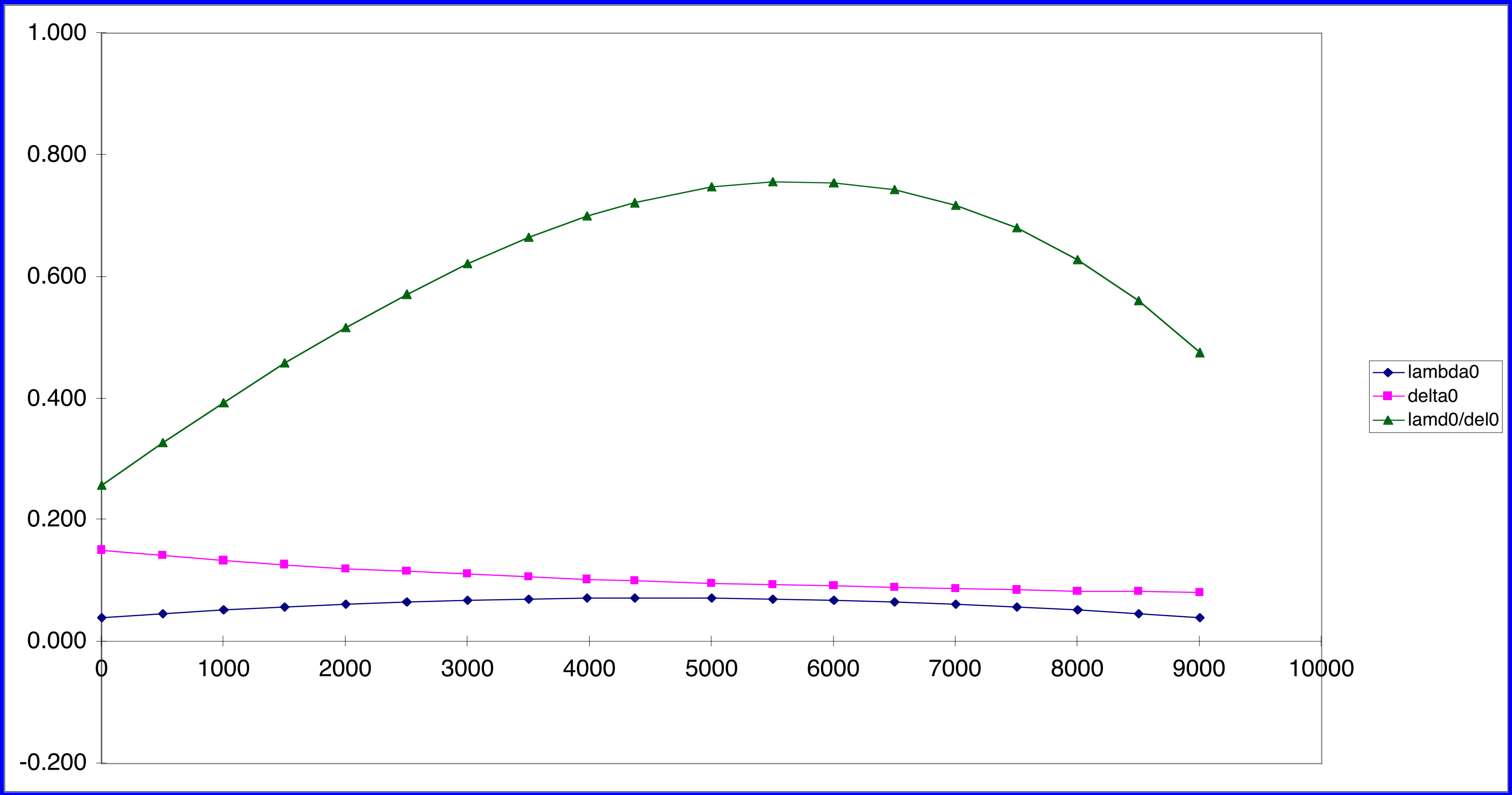
- Payload mass/inert mass ratio $\dfrac{\lambda_0}{\delta_0}$

Effect of Staging



Effect of ΔV Distribution

1st Stage: LOX/LH2   2nd Stage: LOX/LH2



1st Stage Delta-V (m/sec)

Lagrange Multipliers

- Given an objective function

$$y = f(x)$$

subject to constraint function

$$z = g(x)$$

- Create a new objective function

$$y = f(x) + \lambda\left[g(x) - z\right]$$

- Solve simultaneous equations

$$\frac{\partial y}{\partial x} = 0 \qquad \frac{\partial y}{\partial \lambda} = 0$$

Optimum ΔV Distribution Between Stages

- Maximize payload fraction (2 stage case)

$$\lambda_0 = \lambda_1 \lambda_2 = (r_1 - \delta_1)(r_2 - \delta_2)$$

subject to constraint function

$$\Delta V_{total} = \Delta V_1 + \Delta V_2$$

- Create a new objective function

$$\lambda_0 = \left(e^{-\frac{\Delta V_1}{V_{e1}}} - \delta_1\right)\left(e^{-\frac{\Delta V_2}{V_{e2}}} - \delta_2\right) + K\left[\Delta V_1 + \Delta V_2 - \Delta V_{Total}\right]$$

➥ Very messy for partial derivatives!

Optimum ΔV Distribution (continued)

- Use substitute objective function

$$\max(\lambda_0) \Leftrightarrow \max\left[\ln(\lambda_0)\right]$$

- Create a new constrained objective function

$$\ln(\lambda_0) = \ln(r_1 - \delta_1) + \ln(r_2 - \delta_2)$$
$$+K\left[\Delta V_{Total} + V_{e1}\ln(r_1) + V_{e2}\ln(r_2)\right]$$

- Take partials and set equal to zero

$$\frac{\partial \ln(\lambda_0)}{\partial r_1} = 0 \qquad \frac{\partial \ln(\lambda_0)}{\partial r_2} = 0 \qquad \frac{\partial \ln(\lambda_0)}{\partial K} = 0$$

Optimum ΔV Special Cases

- "Generic" partial of objective function

$$\frac{\partial \ln(\lambda_0)}{\partial r_i} = \frac{1}{r_i - \delta_i} + K\frac{V_{ei}}{r_i} = 0$$

- Case 1: $\delta_1 = \delta_2$ $V_{e1} = V_{e2}$

$$r_1 = r_2 \Rightarrow \Delta V_1 = \Delta V_2$$

- Case 2: $\delta_1 \neq \delta_2$ $V_{e1} = V_{e2}$

$$\frac{r_1}{\delta_1} = \frac{r_2}{\delta_2}$$

- More complex cases have to be done numerically

Sensitivity to Inert Mass

ΔV for multistaged rocket

$$\Delta V_{tot} = \sum_{k=1}^{n\ stages} \Delta V_k = \sum_{k=1}^{n} V_{e,k} \ln\left(\frac{m_{o,k}}{m_{f,k}}\right)$$

$$m_{o,k} = m_L + m_{p,k} + m_{i,k} + \sum_{j=k+1}^{n} \left(m_{p,j} + m_{i,j}\right)$$

$$m_{f,k} = m_L + m_{i,k} + \sum_{j=k+1}^{n} \left(m_{p,j} + m_{i,j}\right)$$

$$\frac{\partial \Delta V_{tot}}{\partial m_L} dm_L + \frac{\partial \Delta V_{tot}}{\partial m_{i,j}} dm_{i,j} = 0$$

Trade-off Ratio: Payload<->Inert Mass

$$\left.\frac{\partial m_L}{\partial m_{i,k}}\right|_{\partial \Delta V_{Total}=0} = \frac{-\sum_{j=1}^{k} V_{e,j}\left(\frac{1}{m_{o,j}} - \frac{1}{m_{f,j}}\right)}{\sum_{\ell=1}^{N} V_{e,\ell}\left(\frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}}\right)}$$

Trade-off Ratio : Payload<->Propellant

$$\left.\frac{\partial m_L}{\partial m_{p,k}}\right|_{\partial \Delta V_{Total}=0} = \frac{-\sum_{j=1}^{k} V_{e,j}\left(\frac{1}{m_{o,j}}\right)}{\sum_{\ell=1}^{N} V_{e,\ell}\left(\frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}}\right)}$$

Trade-off Ratio: Payload<->Exhaust Velocity

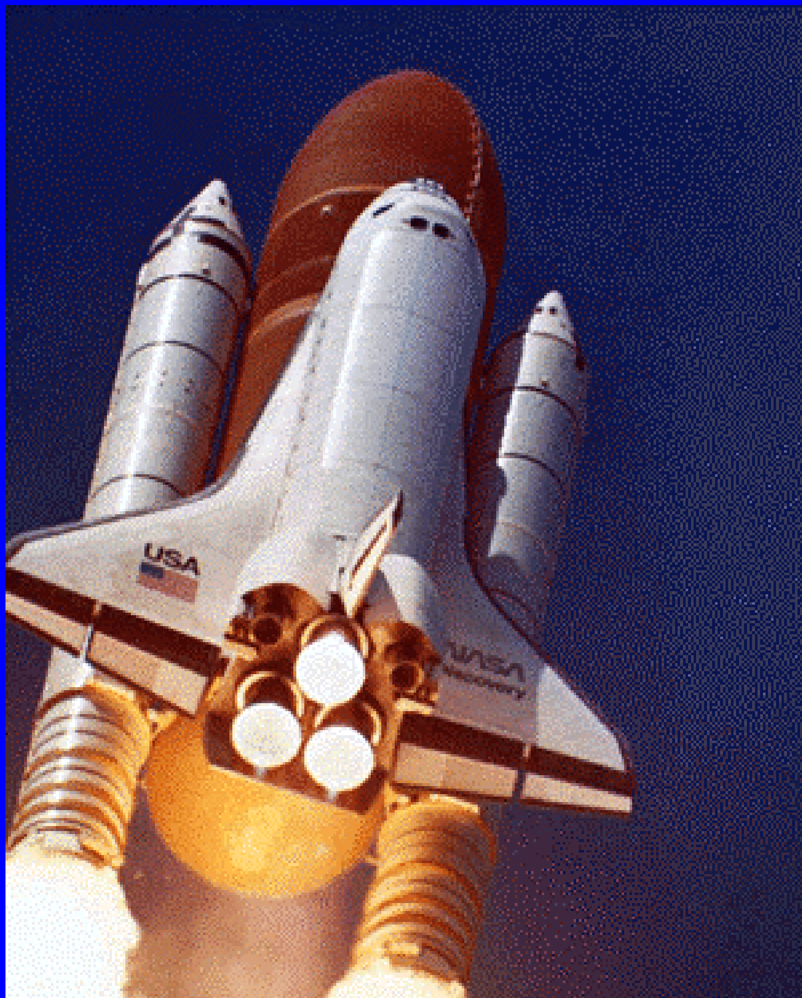
$$\left.\frac{\partial m_L}{\partial V_{e,k}}\right|_{\partial \Delta V_{Total}=0} = \frac{\displaystyle\sum_{j=1}^{k} \ln\left(\frac{m_{o,k}}{m_{o,k}}\right)}{\displaystyle\sum_{\ell=1}^{N} V_{e,\ell}\left(\frac{1}{m_{o,\ell}} - \frac{1}{m_{f,\ell}}\right)}$$

Trade-off Ratio Examples: Saturn IB

Stage (Ve, M/sec)	Stage Initial Mass (kg)	Stage Final Mass (kg)	$\frac{\partial m_L}{\partial m_{i,k}}$	$\frac{\partial m_L}{\partial m_{p,k}}$	$\frac{\partial m_L}{\partial V_{e,k}}$
1 (2568)	588,000	180,000	-0.1102	0.0486	13.18
2 (4126)	144,000	38,000	-1	0.3676	28.01

Note that $\frac{\partial m_L}{\partial V_{e,k}}$ has units of $\frac{kg}{m/\sec}$

Parallel Staging



- Multiple dissimilar engines burning simultaneously
- Frequently a result of upgrades to operational systems
- General case requires "brute force" numerical performance analysis

Parallel-Staging Rocket Equation

- Momentum at time t:

$$M = mv$$

- Momentum at time t+Δt:
(subscript "b"=boosters; "c"=core vehicle)

$$M = (m - \Delta m_b - \Delta m_c)(v + \Delta v) + \Delta m_b (v - V_{e,b}) + \Delta m_c (v - V_{e,c})$$

- Assume thrust (and mass flow rates) constant

Parallel-Staging Rocket Equation

Rocket equation during booster burn

$$\Delta V = -\overline{V}_e \ln\left(\frac{m_{final}}{m_{initial}}\right) = -\overline{V}_e \ln\left(\frac{m_{i,b} + m_{i,c} + \chi m_{p,c} + m_{o,2}}{m_{i,b} + m_{p,b} + m_{i,c} + m_{p,c} + m_{o,2}}\right)$$

where

$$\overline{V}_e = \frac{V_{e,b}\dot{m}_b + V_{e,c}\dot{m}_c}{\dot{m}_b + \dot{m}_c} = \frac{V_{e,b}m_{p,b} + V_{e,c}(1-\chi)m_{p,c}}{m_{p,b} + (1-\chi)m_{p,c}}$$

and χ = fraction of core propellant left when booster propellant is depleted

Analyzing Parallel-Staging Performance

Parallel stages break down into pseudo-serial stages:

- Stage "0" (boosters and core)

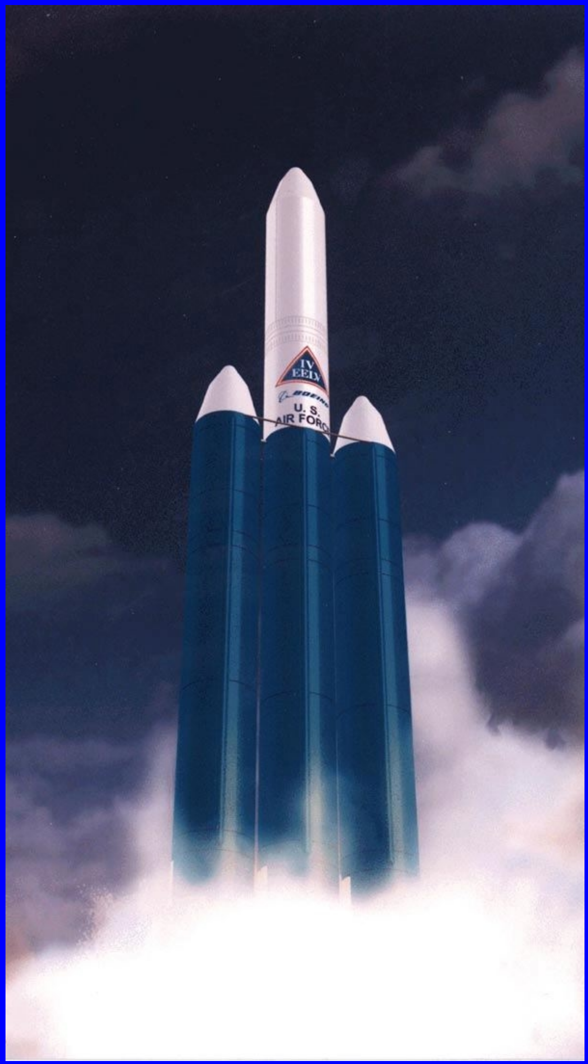
$$\Delta V_0 = -\bar{V}_e \ln\left(\frac{m_{i,b} + m_{i,c} + \chi m_{p,c} + m_{o,2}}{m_{i,b} + m_{p,b} + m_{i,c} + m_{p,c} + m_{o,2}}\right)$$

- Stage "1" (core alone)

$$\Delta V_1 = -V_{e,c} \ln\left(\frac{m_{i,c} + m_{o,2}}{m_{i,c} + \chi m_{p,c} + m_{o,2}}\right)$$

- Subsequent stages are as before

Modular Staging



- Use identical modules to form multiple stages
- Have to cluster modules on lower stages to make up for nonideal ΔV distributions
- Advantageous from production and development cost standpoints

Module Analysis

- All modules have the same inert mass and propellant mass
- Because δ varies with payload mass, not all modules have the same δ!
- Introduce two new parameters

$$\varepsilon = \frac{m_i}{m_i + m_p} \qquad \sigma = \frac{m_i}{m_p}$$

- Conversions $$\varepsilon = \frac{\delta}{1-\lambda} \qquad \sigma = \frac{\delta}{1-\delta-\lambda}$$

Rocket Equation for Modular Boosters

- Assuming n modules in stage 1,

$$r_1 = \frac{n(m_i) + m_{o,2}}{n(m_i + m_p) + m_{o,2}} = \frac{n\varepsilon + \dfrac{m_{o,2}}{m_{o,\mathrm{mod}}}}{n + \dfrac{m_{o,2}}{m_{o,\mathrm{mod}}}}$$

- If all 3 stages use same modules, n_j for stage j,

$$r_1 = \frac{n_1\varepsilon + n_2 + n_3 + \rho_L}{n_1 + n_2 + n_3 + \rho_L}$$

- where $\rho_L = \dfrac{m_L}{m_{\mathrm{mod}}}; \quad m_{\mathrm{mod}} = m_i + m_p$