

# Rocket Performance

- The rocket equation
- Mass ratio and performance
- Structural and payload mass fractions
- Multistaging
- Optimal  $\Delta V$  distribution between stages
- Trade-off ratios
- Parallel staging
- Modular staging



# Derivation of the Rocket Equation

- Momentum at time  $t$ :

$$M = mv$$

- Momentum at time  $t + \Delta t$ :

$$M = (m - \Delta m)(v + \Delta v) + \Delta m(v - V_e)$$

- Some algebraic manipulation gives:

$$m\Delta v = -\Delta m V_e$$

- Take to limits and integrate:

$$\int_{m_{initial}}^{m_{final}} \left( \frac{dm}{m} \right) = - \int_{V_{initial}}^{V_{final}} \left( \frac{dv}{V_e} \right)$$



# The Rocket Equation

- Alternate forms

$$r \equiv \frac{m_{final}}{m_{initial}} = e^{-\frac{\Delta V}{V_e}} \quad \Delta V = -V_e \ln\left(\frac{m_{final}}{m_{initial}}\right) = -V_e \ln(r)$$

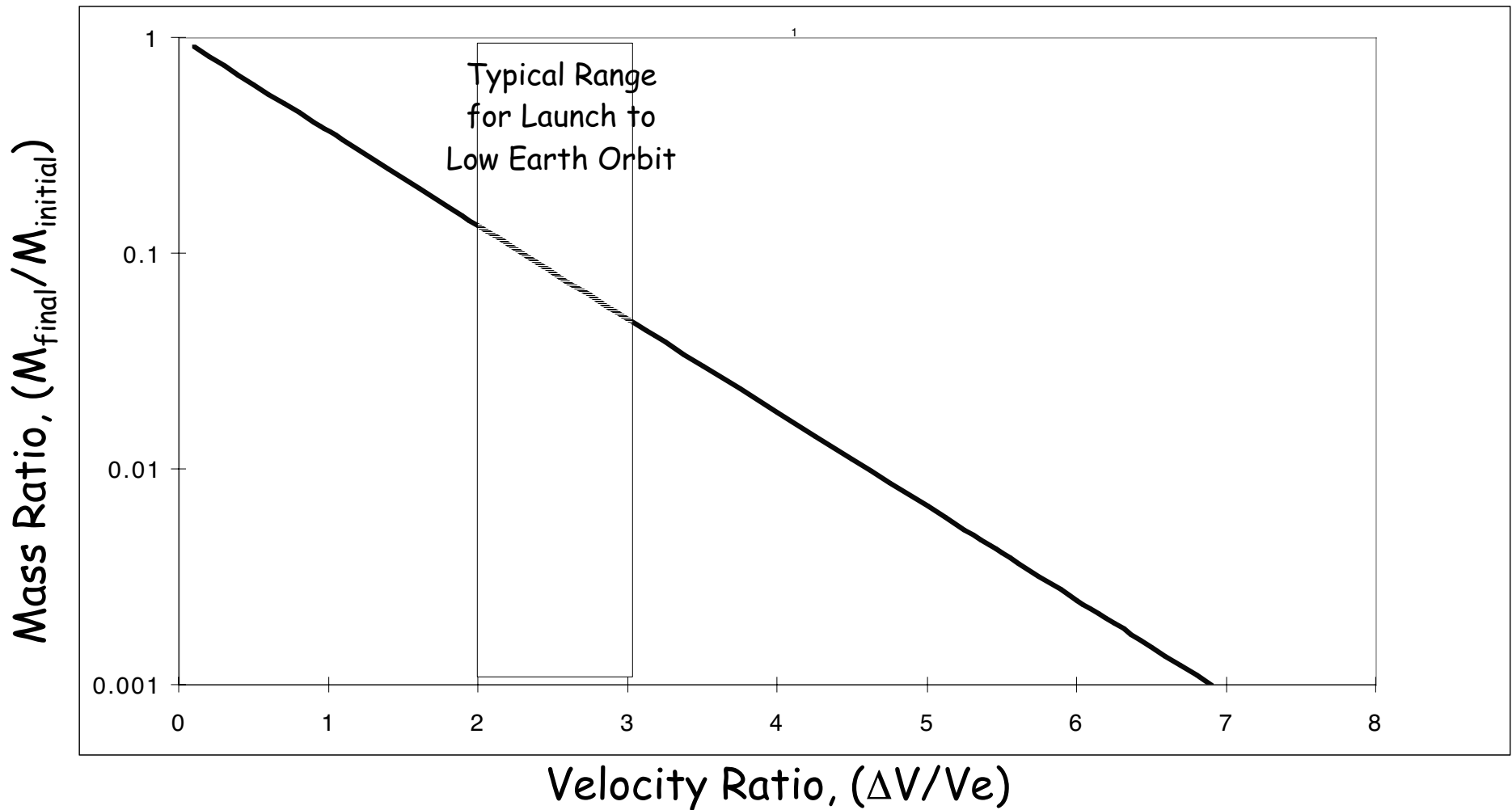
- Basic definitions/concepts

- Mass ratio  $r \equiv \frac{m_{final}}{m_{initial}}$  or  $\mathfrak{R} = \frac{m_{initial}}{m_{final}}$

- Nondimensional velocity change  
"Velocity ratio"  $\frac{\Delta V}{V_e}$



# Rocket Equation (First Look)



# Sources and Categories of Vehicle Mass



- Payload
- Propellants
- Inert Mass
  - Structure
  - Propulsion
  - Avionics
  - Mechanisms
  - Thermal
  - Etc.



# Basic Vehicle Parameters

- Basic mass summary

$$m_0 = m_L + m_p + m_i$$

$m_0$  = initial mass

- Inert mass fraction

$$\delta = \frac{m_i}{m_0} = \frac{m_i}{m_L + m_p + m_i}$$

$m_L$  = payload mass

$m_p$  = propellant mass

$m_i$  = inert mass

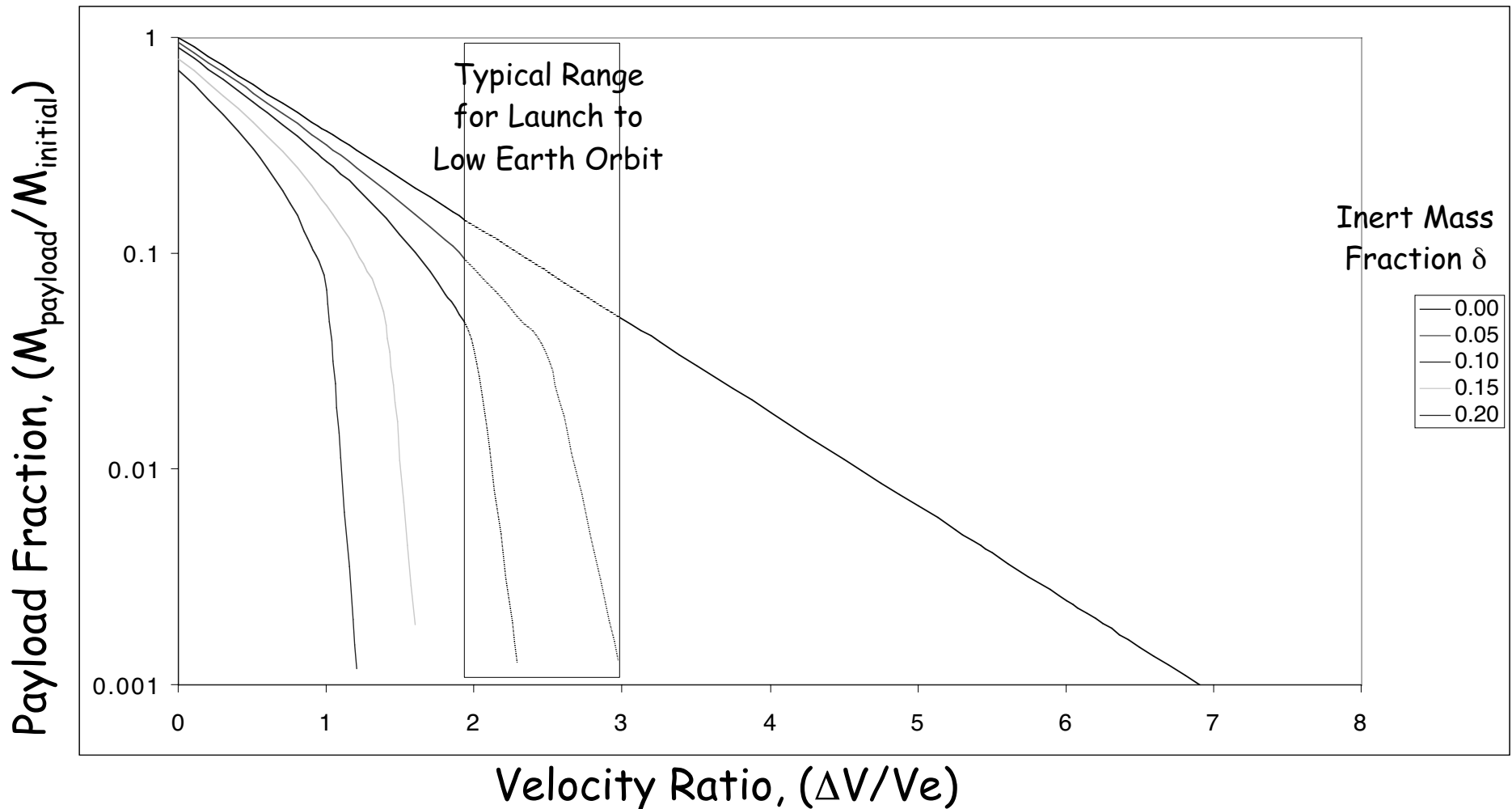
- Payload fraction

$$\lambda = \frac{m_L}{m_0} = \frac{m_L}{m_L + m_p + m_i}$$

- Parametric mass ratio  $r = \lambda + \delta$



# Rocket Equation (including Inert Mass)



# The Rocket Equation for Multiple Stages

- Assume two stages

$$\Delta V_1 = -V_{e,1} \ln\left(\frac{m_{final,1}}{m_{initial,1}}\right) = -V_{e,1} \ln(r_1)$$

$$\Delta V_2 = -V_{e,2} \ln\left(\frac{m_{final,2}}{m_{initial,2}}\right) = -V_{e,2} \ln(r_2)$$

- Assume  $V_{e,1} = V_{e,2} = V_e$

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1) - V_e \ln(r_2) = -V_e \ln(r_1 r_2)$$





## Continued Look at Multistaging

- Converting to masses

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1 r_2) = -V_e \ln\left(\frac{m_{final,1}}{m_{initial,1}} \frac{m_{final,2}}{m_{initial,2}}\right)$$

- Keep in mind that  $m_{final,1} \sim m_{initial,2}$

$$\Delta V_1 + \Delta V_2 = -V_e \ln(r_1 r_2) = -V_e \ln\left(\frac{m_{final,2}}{m_{initial,1}}\right) = -V_e \ln(r_0)$$

- $r_0$  has *no* physical significance!



# Multistage Vehicle Parameters

- Inert mass fraction

$$\delta_0 = \frac{\sum m_{i,j}}{m_0} = \sum_{j=1}^{n \text{ stages}} \left( \delta_j \prod_{\ell=1}^{j-1} \lambda_{\ell} \right)$$

- Payload fraction

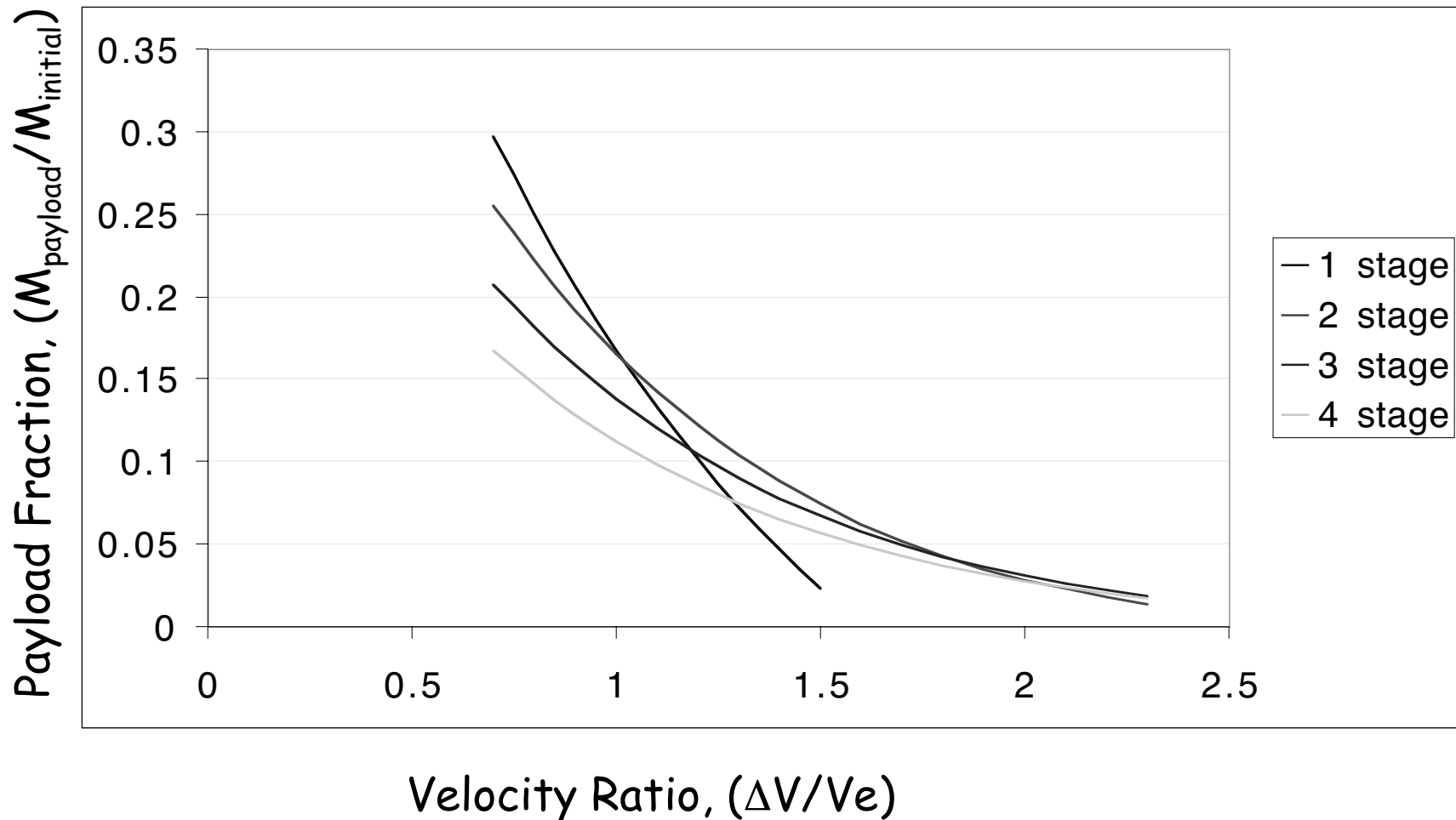
$$\lambda_0 = \frac{m_L}{m_0} = \prod_{i=1}^{n \text{ stages}} \lambda_i$$

- Payload mass/inert mass ratio  $\frac{\lambda_0}{\delta_0}$



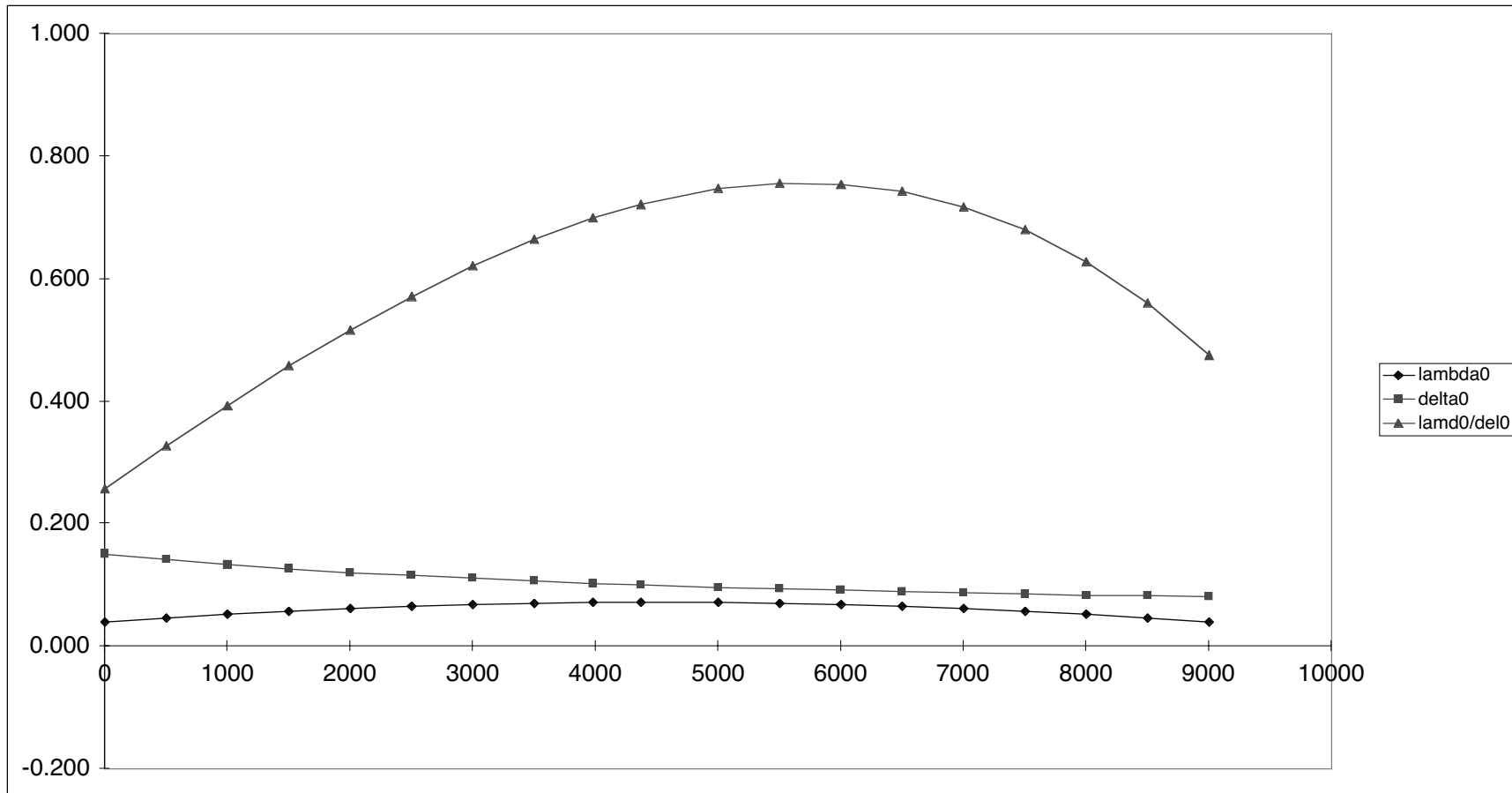
# Effect of Staging

Inert Mass Fraction  $\delta=0.2$



# Effect of $\Delta V$ Distribution

1st Stage: LOX/LH2 2nd Stage: LOX/LH2



1st Stage Delta-V (m/sec)



# Lagrange Multipliers

- Given an objective function

$$y = f(x)$$

subject to constraint function

$$z = g(x)$$

- Create a new objective function

$$y = f(x) + \lambda[g(x) - z]$$

- Solve simultaneous equations

$$\frac{\partial y}{\partial x} = 0 \quad \frac{\partial y}{\partial \lambda} = 0$$



# Optimum $\Delta V$ Distribution Between Stages

- Maximize payload fraction (2 stage case)

$$\lambda_0 = \lambda_1 \lambda_2 = (r_1 - \delta_1)(r_2 - \delta_2)$$

subject to constraint function

$$\Delta V_{total} = \Delta V_1 + \Delta V_2$$

- Create a new objective function

$$\lambda_0 = \left( e^{-\frac{\Delta V_1}{V_{e1}}} - \delta_1 \right) \left( e^{-\frac{\Delta V_2}{V_{e2}}} - \delta_2 \right) + K \left[ \Delta V_1 + \Delta V_2 - \Delta V_{Total} \right]$$

↳ Very messy for partial derivatives!



## Optimum $\Delta V$ Distribution (continued)

- Use substitute objective function

$$\max(\lambda_0) \Leftrightarrow \max[\ln(\lambda_0)]$$

- Create a new constrained objective function

$$\ln(\lambda_0) = \ln(r_1 - \delta_1) + \ln(r_2 - \delta_2)$$

$$+ K[\Delta V_{Total} + V_{e1} \ln(r_1) + V_{e2} \ln(r_2)]$$

- Take partials and set equal to zero

$$\frac{\partial \ln(\lambda_0)}{\partial r_1} = 0 \quad \frac{\partial \ln(\lambda_0)}{\partial r_2} = 0 \quad \frac{\partial \ln(\lambda_0)}{\partial K} = 0$$



# Optimum $\Delta V$ Special Cases

- "Generic" partial of objective function

$$\frac{\partial \ln(\lambda_0)}{\partial r_i} = \frac{1}{r_i - \delta_i} + K \frac{V_{ei}}{r_i} = 0$$

- Case 1:  $\delta_1 = \delta_2$   $V_{e1} = V_{e2}$

$$r_1 = r_2 \Rightarrow \Delta V_1 = \Delta V_2$$

- Case 2:  $\delta_1 \neq \delta_2$   $V_{e1} = V_{e2}$

$$\frac{r_1}{\delta_1} = \frac{r_2}{\delta_2}$$

- More complex cases have to be done numerically





# Sensitivity to Inert Mass

$\Delta V$  for multistaged rocket

$$\Delta V_{tot} = \sum_{k=1}^{n \text{ stages}} \Delta V_k = \sum_{k=1}^n V_{e,k} \ln \left( \frac{m_{o,k}}{m_{f,k}} \right)$$

$$m_{o,k} = m_L + m_{p,k} + m_{i,k} + \sum_{j=k+1}^n (m_{p,j} + m_{i,j})$$

$$m_{f,k} = m_L + m_{i,k} + \sum_{j=k+1}^n (m_{p,j} + m_{i,j})$$

$$\frac{\partial \Delta V_{tot}}{\partial m_L} dm_L + \frac{\partial \Delta V_{tot}}{\partial m_{i,j}} dm_{i,j} = 0$$



# Trade-off Ratio: Payload $\leftrightarrow$ Inert Mass

$$\frac{\partial m_L}{\partial m_{i,k}} \bigg|_{\partial \Delta V_{Total} = 0} = \frac{-\sum_{j=1}^k v_{e,j} \left( \frac{1}{m_{o,j}} - \frac{1}{m_{f,j}} \right)}{\sum_{l=1}^N v_{e,l} \left( \frac{1}{m_{o,l}} - \frac{1}{m_{f,l}} \right)}$$



# Trade-off Ratio : Payload $\leftrightarrow$ Propellant

$$\frac{\partial m_L}{\partial m_{p,k}} \bigg|_{\partial \Delta V_{Total} = 0} = \frac{-\sum_{j=1}^k V_{e,j} \left( \frac{1}{m_{o,j}} \right)}{\sum_{l=1}^N V_{e,l} \left( \frac{1}{m_{o,l}} - \frac{1}{m_{f,l}} \right)}$$



# Trade-off Ratio: Payload $\leftrightarrow$ Exhaust Velocity

$$\left. \frac{\partial m_L}{\partial V_{e,k}} \right|_{\partial \Delta V_{Total} = 0} = \frac{\sum_{j=1}^k \ln \left( \frac{m_{o,k}}{m_{o,k}} \right)}{\sum_{l=1}^N V_{e,l} \left( \frac{1}{m_{o,l}} - \frac{1}{m_{f,l}} \right)}$$



# Trade-off Ratio Examples: Saturn IB

| Stage<br>( $V_e$ ,<br>M/sec) | Stage<br>Initial<br>Mass (kg) | Stage<br>Final<br>Mass (kg) | $\frac{\partial m_L}{\partial m_{i,k}}$ | $\frac{\partial m_L}{\partial m_{p,k}}$ | $\frac{\partial m_L}{\partial V_{e,k}}$ |
|------------------------------|-------------------------------|-----------------------------|---|---|---|
| 1<br>(2568)                  | 588,000                       | 180,000                     | -0.1102                                 | 0.0486                                  | 13.18                                   |
| 2<br>(4126)                  | 144,000                       | 38,000                      | -1                                      | 0.3676                                  | 28.01                                   |

Note that  $\frac{\partial m_L}{\partial V_{e,k}}$  has units of  $\frac{\text{kg}}{\text{m/sec}}$





# Parallel-Staging Rocket Equation

- Momentum at time  $t$ :

$$M = mv$$

- Momentum at time  $t+\Delta t$ :  
(subscript "b"=boosters; "c"=core vehicle)

$$M = (m - \Delta m_b - \Delta m_c)(v + \Delta v) \\ + \Delta m_b (v - V_{e,b}) + \Delta m_c (v - V_{e,c})$$

- Assume thrust (and mass flow rates) constant



# Parallel-Staging Rocket Equation

Rocket equation during booster burn

$$\Delta V = -\bar{V}_e \ln\left(\frac{m_{final}}{m_{initial}}\right) = -\bar{V}_e \ln\left(\frac{m_{i,b} + m_{i,c} + \chi m_{p,c} + m_{o,2}}{m_{i,b} + m_{p,b} + m_{i,c} + m_{p,c} + m_{o,2}}\right)$$

where

$$\bar{V}_e = \frac{V_{e,b}\dot{m}_b + V_{e,c}\dot{m}_c}{\dot{m}_b + \dot{m}_c} = \frac{V_{e,b}m_{p,b} + V_{e,c}(1-\chi)m_{p,c}}{m_{p,b} + (1-\chi)m_{p,c}}$$

and  $\chi$  = fraction of core propellant left when booster propellant is depleted





# Analyzing Parallel-Staging Performance

Parallel stages break down into pseudo-serial stages:

- Stage "0" (boosters and core)

$$\Delta V_0 = -\bar{V}_e \ln \left( \frac{m_{i,b} + m_{i,c} + \chi m_{p,c} + m_{o,2}}{m_{i,b} + m_{p,b} + m_{i,c} + m_{p,c} + m_{o,2}} \right)$$

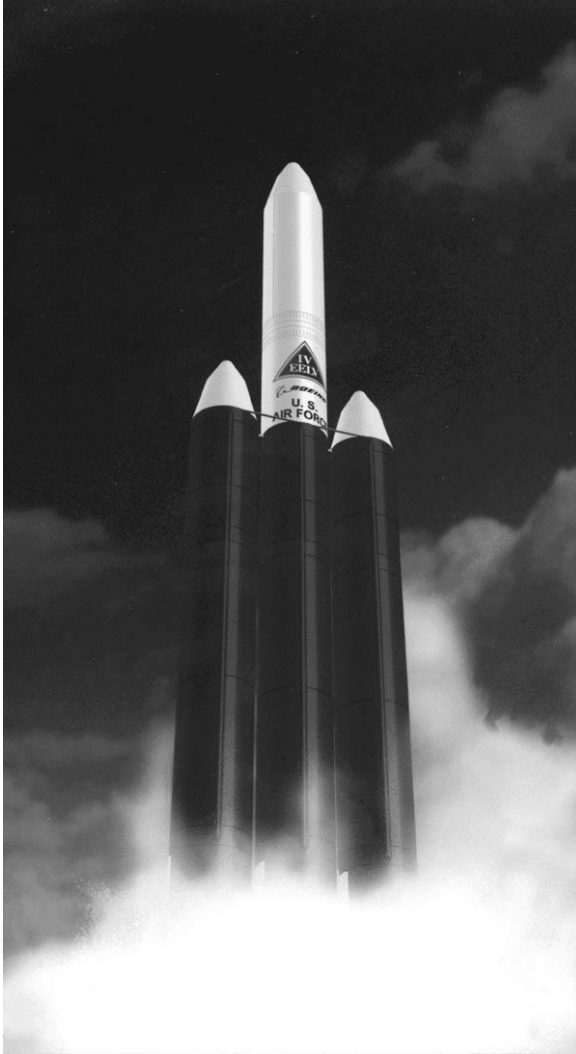
- Stage "1" (core alone)

$$\Delta V_1 = -V_{e,c} \ln \left( \frac{m_{i,c} + m_{o,2}}{m_{i,c} + \chi m_{p,c} + m_{o,2}} \right)$$

- Subsequent stages are as before



# Modular Staging



- Use identical modules to form multiple stages
- Have to cluster modules on lower stages to make up for nonideal  $\Delta V$  distributions
- Advantageous from production and development cost standpoints



# Module Analysis

- All modules have the same inert mass and propellant mass
- Because  $\delta$  varies with payload mass, not all modules have the same  $\delta$ !
- Introduce two new parameters

$$\varepsilon = \frac{m_i}{m_i + m_p} \quad \sigma = \frac{m_i}{m_p}$$

- Conversions  $\varepsilon = \frac{\delta}{1 - \lambda} \quad \sigma = \frac{\delta}{1 - \delta - \lambda}$



# Rocket Equation for Modular Boosters

- Assuming  $n$  modules in stage 1,

$$r_1 = \frac{n(m_i) + m_{o,2}}{n(m_i + m_p) + m_{o,2}} = \frac{n\varepsilon + \frac{m_{o,2}}{m_{o,\text{mod}}}}{n + \frac{m_{o,2}}{m_{o,\text{mod}}}}$$

- If all 3 stages use same modules,  $n_j$  for stage  $j$ ,

$$r_1 = \frac{n_1\varepsilon + n_2 + n_3 + \rho_L}{n_1 + n_2 + n_3 + \rho_L}$$

- where  $\rho_L = \frac{m_L}{m_{\text{mod}}}$ ;  $m_{\text{mod}} = m_i + m_p$

